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# Some relationships between left (right) semi-uninorms and implications on a complete lattice

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**Abstract:** In this paper, we study the relationships between left (right) semi-uninorms and implications on a complete lattice. We firstly discuss the residual operations of left and right semi-uninorms and show that the right (left) residual operator of a conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorm is a right  $\wedge$ -distributive implication that satisfies the neutrality principle. Then, we investigate the left and right semi-uninorms induced by an implication, give some conditions such that two operations induced by an implication constitute left or right semi-uninorms, and demonstrate that the operations induced by a right  $\wedge$ -distributive implication, which satisfies the order property or neutrality principle, are left (right)  $\vee$ -distributive left (right) semi-uninorms or right (left) semi-uninorms. Finally, we reveal the relationships between conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorms and right  $\wedge$ -distributive implications which satisfy the neutrality principle.

**Keywords:** Fuzzy Logic, Fuzzy Connective, Left (Right) Semi-Uninorm, Implication, Neutrality Principle, Order Property

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## 1. Introduction

In fuzzy logic, the truth function for a conjunction connective is usually taken as a triangular norm [1] ( $t$ -norm for short), which is monotone, associative, commutative and has neutral element 1. However, the  $t$ -norms cannot deal with natural interpretations of linguistic words because the axioms of  $t$ -norms are strong. To interpret the non-commutative conjunctions, Flondor et al. [2] introduced non-commutative  $t$ -norms by throwing away the axiom of commutativity of  $t$ -norms and using them to construct pseudo- $BL$ -algebras. About another axiom of  $t$ -norms, associativity, Fodor and Keresztfalvi [3] underlined that “if one works with binary conjunctions and there is no need to extend them for three or more arguments, as happens, for example, in the inference pattern called generalized modus ponens, associativity of the conjunction is an unnecessarily restrictive condition”. By removing the associativity and commutativity from the axioms of  $t$ -norms, weak  $t$ -norms [4] and pseudo- $t$ -norms [5] were introduced and discussed.

Uninorms, introduced by Yager and Rybalov [6] and studied by Fodor et al. [7], are special aggregation oper-

ators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling [8-11]. This kind of operation is an important generalization of both  $t$ -norms and  $t$ -conorms and a special combination of  $t$ -norms and  $t$ -conorms [7]. However, there are real-life situations when truth functions cannot be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. introduced the concepts of left and right uninorms on  $[0, 1]$  in [12] and later in a finite chain in [13], Wang and Fang [14-15] studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [16] introduced the concept of semi-uninorms and Su et al. [17] discussed the notion of left and right semi-uninorms on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm, left and right uninorms)  $U$  can be conjunctive or disjunctive whenever  $U(0, 1) = 0$  or  $1$ , respectively. This fact allows to use uninorms in defining fuzzy implications and coimplications [14-16, 18-20].

In this paper, based on [14, 16, 18-20], we study left (right) semi-uninorms and implications on a complete lattice. After recalling some necessary definitions and

examples about the left and right semi-uninorms on a complete lattice in Section 2, we discuss the residual operations of left and right semi-uninorms in Section 3 and show that the right (left) residual operator of a conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorm is a right  $\wedge$ -distributive implication that satisfies the neutrality principle. In Section 4, we investigate the left and right semi-uninorms induced by an implication and give some conditions such that the operations induced by an implication become either left or right semi-uninorms. In Section 5, we reveal the relationships between conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorms and right  $\wedge$ -distributive implications which satisfy the neutrality principle.

The knowledge about lattices required in this paper can be found in [21].

Throughout this paper, unless otherwise stated,  $L$  always represents any given complete lattice with maximal element 1 and minimal element 0;  $J$  stands for any index set.

## 2. Left and Right Semi-Uninorms

Noting that the commutativity and associativity are not desired for aggregation operators in a number of cases, Liu [16] introduced the concept of semi-uninorms and Su *et al.* [17] studied the notions of left and right semi-uninorms. Here, we recall some necessary definitions and examples of the left and right semi-uninorms on a complete lattice.

*Definition 2.1* (Su *et al.* [17]). *A binary operation  $U$  on  $L$  is called a left (right) semi-uninorm if it satisfies the following two conditions:*

(U1) *there exists a left (right) neutral element, i.e., an element  $e_L \in L$  ( $e_R \in L$ ) satisfying  $U(e_L, x) = x$  ( $U(e_R, x) = x$  for all  $x \in L$ ,*

(U2)  *$U$  is non-decreasing in each variable.*

For any left (right) semi-uninorm  $U$  on  $L$ ,  $U$  is said to be left-conjunctive and right-conjunctive if  $U(0, 1) = 0$  and  $U(1, 0) = 0$ , respectively.  $U$  is said to be conjunctive if both  $U(1, 0) = 0$  and  $U(0, 1) = 0$  since it satisfies the classical boundary conditions of AND. If  $U(1, 0) = 1$  ( $U(0, 1) = 1$ ), then we call  $U$  left-disjunctive (right-disjunctive). We call  $U$  disjunctive if both  $U(1, 0) = 1$  and  $U(0, 1) = 1$  by a similar reason.

If a left (right) semi-uninorm  $U$  is associative, then  $U$  is the left (right) uninorm [14-15] on  $L$ .

If a left (right) semi-uninorm  $U$  with the left (right) neutral element  $e_L \in L$  ( $e_R \in L$ ) has a right (left) neutral element  $e_R \in L$  ( $e_L \in L$ ), then  $e_L = U(e_L, e_R) = e_R$ . Let  $e = e_L = e_R$ . Here,  $U$  is the semi-uninorm [16]. In particular, if the neutral element  $e = 1$ , then the semi-uninorm  $U$  becomes a  $t$ -seminorm [22] or a semi-copula [23-24]; if the neutral element  $e = 0$ , then the semi-

uninorm  $U$  becomes a  $t$ -seminorm [25].

Clearly,  $U(0, 0) = 0$  and  $U(1, 1) = 1$  hold for any left (right) semi-uninorm  $U$  on  $L$ . Moreover, the left (right) neutral elements need not to be unique. In fact, the projection operator given by  $U(x, y) = x$  for all  $x, y \in L$  is such that any element in  $L$  is a right neutral element. But, left (right) neutral elements are all idempotent [26] because  $U(e_L, e_L) = e_L$  ( $U(e_R, e_R) = e_R$ ) for any left (right) neutral element  $e_L$  ( $e_R$ ) of  $U$ .

*Definition 2.2* (Wang and Fang [15]). *A binary operation  $U$  on  $L$  is called left (right)  $\vee$ -distributive if*

$$U(\vee_{j \in J} x_j, y) = \vee_{j \in J} U(x_j, y) \quad \forall x_j, y \in L$$

$$(U(x, \vee_{j \in J} y_j) = \vee_{j \in J} U(x, y_j) \quad \forall x, y_j \in L);$$

*left (right)  $\wedge$ -distributive if*

$$U(\wedge_{j \in J} x_j, y) = \wedge_{j \in J} U(x_j, y) \quad \forall x_j, y \in L$$

$$(U(x, \wedge_{j \in J} y_j) = \wedge_{j \in J} U(x, y_j) \quad \forall x, y_j \in L).$$

*If a binary operation  $U$  is left  $\vee$ -distributive ( $\wedge$ -distributive) and also right  $\vee$ -distributive ( $\wedge$ -distributive), then  $U$  is said to be  $\vee$ -distributive ( $\wedge$ -distributive).*

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1, we have

$$U(0, y) = U(\vee_{j \in \emptyset} x_j, y) = \vee_{j \in \emptyset} U(x_j, y) = 0$$

$$(U(x, 0) = U(x, \vee_{j \in \emptyset} y_j) = \vee_{j \in \emptyset} U(x, y_j) = 0)$$

for any  $x, y \in L$  when  $U$  is left (right)  $\vee$ -distributive,

$$U(1, y) = U(\wedge_{j \in \emptyset} x_j, y) = \wedge_{j \in \emptyset} U(x_j, y) = 1$$

$$(U(x, 1) = U(x, \wedge_{j \in \emptyset} y_j) = \wedge_{j \in \emptyset} U(x, y_j) = 1)$$

for any  $x, y \in L$  when  $U$  is left (right)  $\wedge$ -distributive.

For the sake of convenience, we introduce the following symbols:

$U_{s^L}^{e_L}(L)$  ( $U_{s^R}^{e_R}(L)$ ): the set of all left (right) semi-uninorms with the left (right) neutral element  $e_L$  ( $e_R$ ) on  $L$ ;

$U_{s\vee}^{e_L}(L)$  ( $U_{s\vee}^{e_R}(L)$ ): the set of all right  $\vee$ -distributive left (right) semi-uninorms with the left (right) neutral element  $e_L$  ( $e_R$ ) on  $L$ ;

$U_{s\wedge}^{e_L}(L)$  ( $U_{s\wedge}^{e_R}(L)$ ): the set of all left  $\wedge$ -distributive left (right) semi-uninorms with the left (right) neutral element  $e_L$  ( $e_R$ ) on  $L$ .

Now, we present two examples of left and right semi-uninorms on  $L$ .

*Example 2.1* (Su *et al.* [17]). Let  $e_L \in L$ ,

$$U_{sW}^{e_L}(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_L}(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sM}^{e_L^*}(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ y & \text{if } x \leq e_L, y \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where  $x$  and  $y$  are elements of  $L$ . Then  $U_{sW}^{e_L}$  and  $U_{sM}^{e_L}$  are, respectively, the smallest and greatest elements of  $U_s^{e_L}(L)$ ;  $U_{sW}^{e_L}$  and  $U_{sM}^{e_L^*}$  are, respectively, the smallest and greatest elements of  $U_{s\vee}^{e_L}(L)$ .

*Example 2.2* (Su et al. [17]). Let  $e_R \in L$ ,

$$U_{sW}^{e_R}(x, y) = \begin{cases} x & \text{if } y \geq e_R, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_R}(x, y) = \begin{cases} x & \text{if } y \geq e_R, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sM}^{e_R^*}(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ x & \text{if } y \leq e_L, x \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where  $x$  and  $y$  are elements of  $L$ . Then  $U_{sW}^{e_R}$  and  $U_{sM}^{e_R}$  are, respectively, the smallest and greatest elements of  $U_s^{e_R}(L)$ ;  $U_{sW}^{e_R}$  and  $U_{sM}^{e_R^*}$  are, respectively, the smallest and greatest elements of  $U_{s\vee}^{e_R}(L)$ .

### 3. The Residual Operations of Left and Right Semi-Uninorms

Recently, De Baets and Fodor [18] investigated the residual operators of uninorms on  $[0, 1]$ , Torrens et al. [19-20] studied the implications and coimplications derived from uninorms on  $[0, 1]$ , Wang and Fang [14] discussed the residual implications of left and right uninorms on a complete lattice, and Liu [16] researched semi-uninorms and implications on a complete lattice. In this section, based on [14, 16, 18-20], we consider the residual implications of left and right semi-uninorms on a complete lattice.

First of all, we recall the definition of implications.

*Definition 3.1* (De Baets and Fodor [18], Baczynski and Jayaram [27], De Baets [28]). *An implication  $I$  on  $L$  is a hybrid monotonous (with non-increasing first and non-decreasing second partial mappings) binary operation that satisfies the corner conditions  $I(0, 0) = I(1, 1) = 1$  and  $I(1, 0) = 0$ .*

Implications are extensions of the Boolean implication  $\rightarrow$  ( $P \rightarrow Q$  meaning that  $P$  is sufficient for  $Q$ ).

Note that for any implication  $I$  on  $L$ , due to the monotonicity, the absorption principle holds, i.e.,  $I(0, x) = I(x, 1) = 1$  for any  $x \in L$ .

We denote the set of all implications and the set of all right  $\wedge$ -distributive implications on  $L$  by  $I(L)$  and  $I_{\wedge}(L)$ , respectively.

*Example 3.1.* Let

$$I_W(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_M(x, y) = \begin{cases} 0 & \text{if } (x, y) = (1, 0), \\ 1 & \text{otherwise,} \end{cases}$$

where  $x$  and  $y$  are elements of  $L$ . It is easy to see that  $I_W$  and  $I_M$  are, respectively, the smallest and greatest elements of  $I(L)$  and  $I_W$  is also the smallest element of  $I_{\wedge}(L)$ .

*Definition 3.1.* Let  $U$  be a binary operation on  $L$ .

Define  $I_U^L, I_U^R \in L^{L \times L}$  as follows:

$$I_U^L(x, y) = \vee \{z \in L \mid U(z, x) \leq y\} \quad \forall x, y \in L,$$

$$I_U^R(x, y) = \vee \{z \in L \mid U(x, z) \leq y\} \quad \forall x, y \in L.$$

Here,  $I_U^L$  and  $I_U^R$  are, respectively, called the left and right residual operators of  $U$ .

For any operation  $U$  on  $L$  and  $x, y \in L$ , it is straightforward to verify that

$$(1) \quad I_U^L(x, 1) = I_U^R(x, 1) = 1.$$

$$(2) \quad x \leq I_U^L(y, U(x, y)) \quad \text{and} \quad y \leq I_U^R(x, U(x, y)).$$

$$(3) \quad \text{If } U(1, 0) = 0, \text{ then } I_U^L(0, y) = 1 \text{ and if } U(0, 1) = 0, \text{ then } I_U^R(0, y) = 1.$$

When  $U$  is a left (right) semi-uninorm on  $L$ , it is easy to see that  $I_U^L$  and  $I_U^R$  are all non-increasing in the first variable and non-decreasing in the second one.

*Example 3.2.* For some left and right semi-uninorms in Examples 2.1 and 2.2, a simple computation shows that

$$I_{U_{sW}^{e_R}}^L(x, y) = \begin{cases} y & \text{if } x \geq e_R, \\ 1 & \text{otherwise,} \end{cases} \quad I_{U_{sW}^{e_L}}^R(x, y) = \begin{cases} y & \text{if } x \geq e_L, \\ 1 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^{e_R}}^L(x, y) = I_{U_{sM}^{e_L}}^R(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x \geq e_L, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^{e_L}}^L(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ e_L & \text{if } x \leq y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^{e_L}}^L(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ e_L & \text{if } 0 < x \leq y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^R}(x, y) = I_{U_{sM}^{e_L}}^R(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x \leq e_L, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^{e_R}}^R(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ e_R & \text{if } x \leq y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{U_{sM}^{e_L}}^R(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ e_R & \text{if } 0 < x \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

When  $e_L, e_R \in L \setminus \{0, 1\}$ , we see that  $I_{U_{sM}^L}$  and  $I_{U_{sM}^R}$  are two implications,  $I_{U_{sM}^{e_R}}^L$  and  $I_{U_{sM}^{e_L}}^R$  are two right  $\wedge$ -distributive implications, but  $I_{U_{sM}^L}, I_{U_{sM}^{e_R}}^L, I_{U_{sM}^{e_L}}^R, I_{U_{sM}^R}$ ,  $I_{U_{sM}^L}^R$  and  $I_{U_{sM}^R}$  are not implications.

*Theorem 3.1.* Let  $U \in U_s^{e_L}(L)$ .

- (1) For any  $x, y \in L$ ,  $x \leq y \Rightarrow I_U^L(x, y) \geq e_L$ .
- (2)  $I_U^R$  satisfies the neutrality principle with  $e_L$ , i.e.,  $I_U^R(e_L, y) = y$  for any  $y \in L$ .
- (3) If  $U$  is left-conjunctive, then  $I_U^R \in I(L)$ .
- (4) If  $U \in U_{s\vee}^{e_L}(L)$  is left-conjunctive, then

$$I_U^R \in I_{\wedge}(L) \text{ and } I_U^R(x, y) = \max\{z \in L \mid U(x, z) \leq y\}.$$

Here,  $I_U^R$  is called the right residual implication of the left semi-uninorm  $U$ .

Proof. Clearly, statements (1) and (2) hold.

- (3) If  $U$  is left-conjunctive, then  $U(0, 1) = 0$  and

$$I_U^R(0, 0) = \vee\{z \in L \mid U(0, z) = 0\} = 1.$$

By the non-decreasingness of  $U$ , we see that

$$\begin{aligned} I_U^R(1, 0) &= \vee\{z \in L \mid U(1, z) = 0\} \\ &\leq \vee\{z \in L \mid z = U(e_L, z) \leq U(1, z) = 0\} = 0. \end{aligned}$$

Moreover, it follows from the statements before Example 3.2 that  $I_U^R(1, 1) = 1$  and  $I_U^R$  is non-increasing in its first and non-decreasing in its second variable. Therefore,  $I_U^R$  is an implication on  $L$ .

- (4) Assume that  $U$  is left-conjunctive right  $\vee$ -distributive. Then,  $I_U^R \in I(L)$  by statement (3). Let  $x, y, z \in L$ . If  $U(x, z) \leq y$ , then  $z \leq I_U^R(x, y)$ ; if  $z \leq I_U^R(x, y)$ , then it follows from the non-decreasingness of  $U$  that

$$\begin{aligned} U(x, z) &\leq U(x, I_U^R(x, y)) = U(x, \vee\{z \in L \mid U(x, z) \leq y\}) \\ &= \vee\{U(x, z) \mid z \in L, U(x, z) \leq y\} \leq y \end{aligned}$$

Noting that  $U(x, I_U^R(x, y)) \leq y$ , we know that

$$I_U^R(x, y) = \max\{z \in L \mid U(x, z) \leq y\}.$$

Moreover, when  $J \neq \Phi$ , for any  $x, y_j \in L (j \in J)$ , we have that

$$\begin{aligned} I_U^R(x, \wedge_{j \in J} y_j) &= \vee\{z \in L \mid U(x, z) \leq \wedge_{j \in J} y_j\} \\ &= \vee\{z \in L \mid U(x, z) \leq y_j \forall j \in J\} \\ &= \vee\{z \in L \mid z \leq I_U^R(x, y_j) \forall j \in J\} \\ &= \vee\{z \in L \mid z \leq \wedge_{j \in J} I_U^R(x, y_j)\} = \wedge_{j \in J} I_U^R(x, y_j). \end{aligned}$$

When  $J = \Phi$ , we see that

$$I_U^R(x, \wedge_{j \in \Phi} y_j) = I_U^R(x, 1) = 1 = \wedge_{j \in \Phi} I_U^R(x, y_j).$$

Therefore,  $I_U^R$  is right  $\wedge$ -distributive, i.e.,  $I_U^R \in I_{\wedge}(L)$ . The theorem is proved.

When  $e_L < 1$ , for the right  $\vee$ -distributive left semi-uninorm  $U_{sM}^{e_L}$ , we see that  $I_{U_{sM}^{e_L}}^L$  by Example 3.2, but  $I_{U_{sM}^{e_L}}^L(e_L, y) = e_L \neq y$  when  $e_L < y < 1$ , i.e.,  $I_{U_{sM}^{e_L}}^L$  does not satisfy the neutrality principle with  $e_L$ . This illustrates Theorem 3.1 doesn't hold for the left residual operator of a left semi-uninorm.

If  $P$  and  $Q$  are two propositions, then the generalized modus ponens (GMP) [18] gives a lower bound for the truth value of  $Q$  when the truth values of propositions  $P$  and  $P \rightarrow Q$  are known. By the proof of Theorem 3.1(4), we know that  $U$  and  $I_U^R$  satisfy the GMP rule:

$$U(x, I_U^R(x, y)) \leq y \quad \forall x, y \in L$$

and the following right residual principle:

$$U(x, z) \leq y \Leftrightarrow z \leq I_U^R(x, y) \quad \forall x, y, z \in L$$

when a binary operation  $U$  is right  $\vee$ -distributive.

Similarly,  $U$  and  $I_U^L$  satisfy GMP rule in the form:

$$U(I_U^L(x, y), x) \leq y \quad \forall x, y \in L$$

and the following left residual principle:

$$U(z, x) \leq y \Leftrightarrow z \leq I_U^L(x, y) \quad \forall x, y, z \in L$$

when  $U$  is left  $\vee$ -distributive. Thus, for right semi-uniforms on  $L$ , we have a similar result.

*Theorem 3.2.* Let  $U \in U_s^{e_R}(L)$ .

- (1) For any  $x, y \in L$ ,  $x \leq y \Rightarrow I_U^R(x, y) \geq e_R$ .
- (2)  $I_U^L$  satisfies the neutrality principle with  $e_R$ , i.e.,  
 $I_U^L(e_R, y) = y$  for any  $y \in L$ .
- (3) If  $U$  is right-conjunctive, then  $I_U^L \in I(L)$ .
- (4) If  $U \in U_{\vee_s}^{e_R}(L)$  is right-conjunctive, then

$$I_U^L \in I_{\wedge}(L) \quad \text{and} \quad I_U^L(x, y) = \max\{z \in L \mid U(z, x) \leq y\}.$$

Here,  $I_U^L$  is called the left residual implication of the left semi-uniform  $U$ .

Combining Theorems 3.1 and 3.2, we know that both  $I_U^L$  and  $I_U^R$  are all right  $\vee$ -distributive implications when  $U$  is a  $\vee$ -distributive left (right) semi-uniform.

*Theorem 3.3.* (1) If  $U \in U_{\vee_s}^{e_L}(L)$ , then  $I_U^L$  is right  $\wedge$ -distributive and satisfies the left residual principle and the order property with  $e_L$ :

$$x \leq y \Leftrightarrow I_U^L(x, y) \geq e_L \quad \forall x, y \in L.$$

(2) If  $U \in U_{\vee_s}^{e_R}(L)$ , then  $I_U^R$  is right  $\wedge$ -distributive and satisfies the right residual principle and the order property with  $e_R$ :

$$x \leq y \Leftrightarrow I_U^R(x, y) \geq e_R \quad \forall x, y \in L.$$

*Proof.* Assume that  $U$  is a left  $\wedge$ -distributive left semi-uniforms with the left neutral element  $e_L$ . By virtue of the proof of Theorem 3.1(4), we can see that  $I_U^L$  is right  $\wedge$ -distributive and satisfies the left residual principle. Moreover, if  $x, y \in L$  and  $x \leq y$ , then it follows from Theorem 3.1(1) that  $I_U^L(x, y) \geq e_L$ ; if  $I_U^L(x, y) \geq e_L$ , then

$$x = U(e_L, x) \leq U(I_U^L(x, y), x) \leq y.$$

Thus,  $I_U^L$  satisfies the order property with  $e_L$ .

Similarly, we can show that  $I_U^R$  is right  $\wedge$ -distributive and satisfies the right residual principle and the order property with  $e_R$  when  $U$  is a right  $\vee$ -distributive right semi-uniforms with the right neutral element

$e_R$ .

The theorem is proved.

In particular, if  $U$  is a  $\vee$ -distributive semi-uniform with the neutral element  $e$ , then  $I_U^L$  and  $I_U^R$  satisfy the the residual principle (RP) and the order property (OP) and are all right  $\wedge$ -distributive implications (see Theorem 3.6 in [16]).

## 4. The Left and Right Semi-Uniforms Induced by Implications

Liu [16] discussed the semi-uniforms induced by implications and Su and Wang [29] studied the pseudo-uniforms induced by coimplications. In this section, based on these works, we investigate the left and right semi-uniforms induced by implications on a complete lattice.

*Definition 4.1.* Let  $I$  be a binary operation on  $L$ . Define two induced operators  $U_I^L$  and  $U_I^R$  of  $I$  as follows:

$$U_I^L(x, y) = \wedge\{z \in L \mid x \leq I(y, z)\} \quad \forall x, y \in L,$$

$$U_I^R(x, y) = \wedge\{z \in L \mid y \leq I(y, z)\} \quad \forall x, y \in L.$$

Clearly,  $U_I^L(0, x) = U_I^R(x, 0) = 0$ ,  $U_I^L(1, x) = U_I^R(x, 1)$  for any  $x \in L$ , and  $U_I^L = U_I^R$  if  $I$  satisfies the condition:

$$x \leq I(y, z) \Leftrightarrow y \leq I(x, z) \quad \forall x, y, z \in L.$$

When  $I$  is hybrid monotonous, it is easy to see that  $U_I^L$  and  $U_I^R$  are all non-decreasing in its each variable. Moreover, for any binary operation  $I$ , it follows from Definition 4.1 that

$$U_I^L(I(x, y), x) \leq y, U_I^R(x, I(x, y)) \leq y \quad \forall x, y \in L.$$

These explain that  $U_I^L$  and  $I$ ,  $U_I^R$  and  $I$  satisfy the GMP rule.

*Example 4.1.* For two implications  $I_W$  and  $I_M$  in Example 3.1, we have that

$$U_{I_W}^L(x, y) = U_{I_W}^R(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{I_M}^L(x, y) = \begin{cases} \bigwedge_{a \in L \setminus \{0\}} a & \text{if } x > 0 \text{ and } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{I_M}^R(x, y) = \begin{cases} \bigwedge_{a \in L \setminus \{0\}} a & \text{if } x = 1 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, four operations induced by implications  $I_W$  and  $I_M$  are neither left semi-uninorms nor right semi-uninorms on  $L$ .

Below, we find some conditions such that these operations induced by implications are left or right semi-uninorms.

*Theorem 4.1. Let  $I$  be an implication on  $L$ .*

- (1) *If  $I$  satisfies the order property with  $e_L$ , then  $U_I^L \in U_s^{e_L}(L)$ ; if  $I$  satisfies the neutrality principle with  $e_L$ , then  $U_I^R \in U_s^{e_L}(L)$ . Here,  $U_I^L$  and  $U_I^R$  are called the left semi-uninorms induced by the implication  $I$ .*
- (2) *If  $I$  satisfies the order property with  $e_R$ , then  $U_I^R \in U_s^{e_R}(L)$ ; if  $I$  satisfies the neutrality principle with  $e_R$ , then  $U_I^L \in U_s^{e_R}(L)$ . Here,  $U_I^L$  and  $U_I^R$  are called the right semi-uninorms induced by the implication  $I$ .*
- (3) *If  $I$  satisfies the order property with  $e_L$  and the neutrality principle with  $e_R$ , then  $U_I^L$  is a semi-uninorm on  $L$ .*
- (4) *If  $I$  satisfies the order property with  $e_R$  and the neutrality principle with  $e_L$ , then  $U_I^R$  is also a semi-uninorm on  $L$ .*

*Proof.* Assume that  $I \in I(L)$ . Then  $U_I^L$  is non-decreasing in each variable. If  $I$  satisfies the order property with  $e_L$ , then

$$\begin{aligned} U_I^L(e_L, y) &= \wedge \{z \in L \mid e_L \leq I(y, z)\} \\ &= \wedge \{z \in L \mid y \leq z\} = y \quad \forall y \in L. \end{aligned}$$

Thus,  $U_I^L \in U_s^{e_L}(L)$ . If  $I$  satisfies the neutrality principle with  $e_L$ , then

$$\begin{aligned} U_I^R(e_L, y) &= \wedge \{z \in L \mid y \leq I(e_L, z)\} \\ &= \wedge \{z \in L \mid y \leq z\} = y \quad \forall y \in L. \end{aligned}$$

So,  $U_I^R \in U_s^{e_L}(L)$ .

Similarly, we can show that  $U_I^R \in U_s^{e_R}(L)$  when the implication  $I$  satisfies the order property with  $e_R$  and  $U_I^L \in U_s^{e_R}(L)$  when  $I$  satisfies the neutrality principle with  $e_R$ .

If  $I$  satisfies the order property with  $e_L$  and the neutrality principle with  $e_R$ , then

$$U_I^L(e_L, x) = U_I^L(x, e_R) = x \quad \forall x \in L.$$

Thus,  $e_L = e_R$ , i.e.,  $U_I^L$  is a semi-uninorm with the neutral element  $e_L (= e_R)$  on  $L$ .

In a similar way, we can see that  $U_I^R$  is also a semi-uninorm on  $L$  when  $I$  satisfies the order property with  $e_R$  and the neutrality principle with  $e_L$ .

The theorem is proved.

When  $I \in I(L)$ ,  $I(0, x) = 1$  for any  $x \in L$  and hence it follows from Definition 4.1 that

$$U_I^L(1, 0) = U_I^R(0, 1).$$

Thus,  $U_I^L$  and  $U_I^R$  in Theorem 4.1 are all conjunctive left or right semi-uninorms induced by the implication  $I$ .

*Theorem 4.2. Let  $I \in I_{\wedge}(L)$ .*

- (1) *If  $I$  satisfies the order property with  $e_L$ , then  $U_I^L \in U_{\vee s}^{e_L}(L)$ ; if  $I$  satisfies the neutrality principle with  $e_L$ , then  $U_I^R \in U_{\vee s}^{e_L}(L)$ .*
- (2) *If  $I$  satisfies the order property with  $e_R$ , then  $U_I^R \in U_{\vee s}^{e_R}(L)$ ; if  $I$  satisfies the neutrality principle with  $e_R$ , then  $U_I^L \in U_{\vee s}^{e_R}(L)$ .*
- (3) *If  $I$  satisfies the order property with  $e_L$  and the neutrality principle with  $e_R$ , then  $U_I^L$  is a left  $\vee$ -distributive semi-uninorm on  $L$ .*
- (4) *If  $I$  satisfies the order property with  $e_R$  and the neutrality principle with  $e_L$ , then  $U_I^R$  is also a right  $\vee$ -distributive semi-uninorm on  $L$ .*

*Proof.* Assume that  $I$  is a right  $\wedge$ -distributive implication. Let  $x, y, z \in L$ . If  $x \leq I(y, z)$ , then it follows from Definition 4.1 that  $U_I^L(x, y) \leq z$ ; if  $U_I^L(x, y) \leq z$ , then

$$\begin{aligned} I(y, z) &\geq I(y, U_I^L(x, y)) \\ &= I(y, \wedge \{z \in L \mid x \leq I(y, z)\}) \\ &= \wedge \{I(y, z) \mid z \in L, x \leq I(y, z)\} \geq x. \end{aligned}$$

Noting that  $x \leq I(y, U_I^L(x, y))$ , we know that

$$U_I^L(x, y) = \min \{z \in L \mid x \leq I(y, z)\}.$$

Moreover, when  $J \neq \Phi$ , for any  $x_j, y \in L (j \in J)$ , we have that

$$\begin{aligned}
U_I^L(\bigvee_{j \in J} x_j, y) &= \wedge \{z \in L \mid \bigvee_{j \in J} x_j \leq I(y, z)\} \\
&= \wedge \{z \in L \mid x_j \leq I(y, z) \forall j \in J\} \\
&= \wedge \{z \in L \mid U_I^L(x_j, y) \leq z \forall j \in J\} \\
&= \wedge \{z \in L \mid \bigvee_{j \in J} U_I^L(x_j, y) \leq z\} \\
&= \bigvee_{j \in J} U_I^L(x_j, y).
\end{aligned}$$

When  $J = \Phi$ , we see that

$$U_I^L(\bigvee_{j \in \Phi} x_j, y) = U_I^L(0, y) = 0 = \bigvee_{j \in \Phi} U_I^L(x_j, y).$$

Thus,  $U_I^L$  is left  $\vee$ -distributive. Therefore, by virtue of Theorem 4.1,  $U_I^L \in U_{\vee s}^{e_L}(L)$  when  $I$  satisfies the order property with  $e_L$  and  $U_I^L \in U_{\vee s}^{e_R}(L)$  when  $I$  satisfies the neutrality principle with  $e_R$ .

Similarly, we can show that  $U_I^R$  is a right  $\vee$ -distributive right semi-uniform and left semi-uniform when  $I$  satisfies the order property with  $e_R$  and the neutrality principle with  $e_L$ , respectively.

Statements (3) and (4) are the direct consequences of statements (1) and (2) and Theorem 4.1.

The theorem is proved.

By virtue of Theorem 4.2, we see that  $U_I^L$  and  $U_I^R$  are a left  $\vee$ -distributive semi-uniform and a right  $\vee$ -distributive semi-uniform on  $L$  when  $I \in I_{\wedge}(L)$  satisfies the order property (OP) and neutrality principle (NP), respectively. This explains Theorem 4.2 is a generalization of Theorem 4.5 in [16].

When  $I$  is a right  $\wedge$ -distributive implication on  $L$ , by the proof of Theorem 4.2, we know that  $I$ ,  $U_I^L$  and  $U_I^R$  satisfy the following adjunction conditions:

$$\begin{aligned}
x \leq I(y, z) &\Leftrightarrow U_I^L(x, y) \leq z, \\
y \leq I(x, z) &\Leftrightarrow U_I^R(x, y) \leq z \quad \forall x, y, z \in L.
\end{aligned}$$

Moreover, we have

$$U_I^R(x, y) = \min\{z \in L \mid y \leq I(x, z)\}.$$

## 5. The Relationships between Left (Right) Semi-Uniforms and Implications

In the final section, we reveal the relationships between conjunctive right (left)  $\vee$ -distributive left (right) semi-uniforms and right  $\wedge$ -distributive implications which satisfy the neutrality principle on a complete lattice.

*Theorem 5.1.*

- (1) If  $U \in U_{\vee s}^{e_R}(L)$  is right-conjunctive, then  $I_U^L \in I_{\wedge}(L)$  satisfies the neutrality principle with  $e_R$  and  $U_{I_U^L}^L = U$ .
- (2) If  $U \in U_{s\vee}^{e_L}(L)$  is left-conjunctive, then  $I_U^R \in I_{\wedge}(L)$  satisfies the neutrality principle with  $e_L$  and  $U_{I_U^R}^R = U$ .
- (3) If  $I \in I_{\wedge}(L)$  satisfies the neutrality principle with  $e_L$ , then  $U_I^R \in U_{s\vee}^{e_L}(L)$  is conjunctive and  $I_{U_I^R}^R = I$ .
- (4) If  $I \in I_{\wedge}(L)$  satisfies the neutrality principle with  $e_R$ , then  $U_I^L \in U_{\vee s}^{e_R}(L)$  is conjunctive and  $I_{U_I^L}^L = I$ .

*Proof.* We only prove the statements (1) and (3) hold.

(1) If  $U$  is a right-conjunctive left  $\vee$ -distributive right semi-uniform, then  $I_U^L \in I_{\wedge}(L)$  and satisfies the neutrality principle with  $e_R$  by Theorem 3.2. Moreover, it follows from the left residual principle that

$$\begin{aligned}
U_{I_U^L}^L(x, y) &= \wedge \{z \in L \mid x \leq I_U^L(y, z)\} \\
&= \wedge \{z \in L \mid U(x, y) \leq z\} = U(x, y) \quad \forall x, y \in L.
\end{aligned}$$

Thus,  $U_{I_U^L}^L = U$ .

(3) If  $I \in I_{\wedge}(L)$  satisfies the neutrality principle with  $e_L$ , then  $U_I^R$  is a conjunctive right  $\vee$ -distributive left semi-uniform by Theorem 4.2. Moreover, it follows from the adjunction conditions that

$$\begin{aligned}
I_{U_I^R}^R(x, y) &= \bigvee \{z \in L \mid U_I^R(y, z) \leq y\} \\
&= \bigvee \{z \in L \mid z \leq I(x, y)\} = I(x, y) \quad \forall x, y \in L.
\end{aligned}$$

Therefore,  $I_{U_I^R}^R = I$ .

The theorem is proved.

We denote by  $U_{cs\vee}^{e_L}(L)$  and  $U_{\vee cs}^{e_R}(L)$ , respectively, the set of all conjunctive right  $\vee$ -distributive left semi-uniforms and the set of all conjunctive left  $\vee$ -distributive right semi-uniforms; by  $I_{\wedge}^{e_L}(L)$  and  $I_{\wedge}^{e_R}(L)$ , respectively, the set of all right  $\wedge$ -distributive implications which satisfy the neutrality principle with  $e_L$  and  $e_R$  on a complete lattice.

It is easy to verify that  $U_{cs\vee}^{e_L}(L)$  and  $U_{\vee cs}^{e_R}(L)$  are two join-semilattices with the smallest elements  $U_{sW}^{e_L}$  and

$U_{sW}^{eR}$ , respectively; and  $I_{\wedge}^{eL}(L)$  and  $I_{\wedge}^{eR}(L)$  are two meet-semilattices with the greatest elements  $I_{U_{sW}^{eL}}^R$  and  $I_{U_{sW}^{eR}}^L$ , respectively, where

$$(U_1 \vee U_2)(x, y) = U_1(x, y) \vee U_2(x, y),$$

$$(I_1 \wedge I_2)(x, y) = I_1(x, y) \wedge I_2(x, y) \quad \forall x, y \in L.$$

Define two mappings  $\phi: U_{cs\vee}^{eL}(L) \rightarrow I_{\wedge}^{eL}(L)$  and  $\varphi: U_{\vee cs}^{eR}(L) \rightarrow I_{\wedge}^{eR}(L)$  as follows:

$$\phi(U) = I_U^R \quad \forall U \in U_{cs\vee}^{eL}(L), \quad \varphi(U) = I_U^L \quad \forall U \in U_{\vee cs}^{eR}(L).$$

Then it follows from Theorem 5.1 that  $\phi$  and  $\varphi$  are all invertible and

$$\phi^{-1}(I) = U_I^R \quad \forall I \in I_{\wedge}^{eL}(L), \quad \varphi^{-1}(I) = U_I^L \quad \forall I \in I_{\wedge}^{eR}(L).$$

Moreover, we have the following theorem.

*Theorem 5.2.*

- (1) If  $U_1, U_2 \in U_{cs\vee}^{eL}(L)$ , then  $I_{U_1 \vee U_2}^R = I_{U_1}^R \wedge I_{U_2}^R$ .
- (2) If  $U_1, U_2 \in U_{\vee cs}^{eR}(L)$ , then  $I_{U_1 \vee U_2}^L = I_{U_1}^L \wedge I_{U_2}^L$ .
- (3) If  $I_1, I_2 \in I_{\wedge}^{eL}(L)$ , then  $U_{I_1 \wedge I_2}^R = U_{I_1}^R \vee U_{I_2}^R$ .
- (4) If  $I_1, I_2 \in I_{\wedge}^{eR}(L)$ , then  $U_{I_1 \wedge I_2}^L = U_{I_1}^L \vee U_{I_2}^L$ .

*Proof.* We only prove the statements (2) and (4) hold.

(2) If  $U_1, U_2 \in U_{\vee cs}^{eR}(L)$ , then it follows from the left residual principle that

$$\begin{aligned} I_{U_1 \vee U_2}^L(x, y) &= \vee \{z \in L \mid (U_1 \vee U_2)(z, x) \leq y\} \\ &= \vee \{z \in L \mid U_1(z, x) \vee U_2(z, x) \leq y\} \\ &= \vee \{z \in L \mid U_1(z, x) \leq y, U_2(z, x) \leq y\} \\ &= \vee \{z \in L \mid z \leq I_{U_1}^L(x, y), z \leq I_{U_2}^L(x, y)\} \\ &= \vee \{z \in L \mid z \leq I_{U_1}^L(x, y) \wedge I_{U_2}^L(x, y)\} \\ &= (I_{U_1}^L \wedge I_{U_2}^L)(x, y) \quad \forall x, y \in L, \end{aligned}$$

$$\text{i.e., } I_{U_1 \vee U_2}^L = I_{U_1}^L \wedge I_{U_2}^L.$$

(4) If  $I_1, I_2 \in I_{\wedge}^{eR}(L)$ , then it follows from the adjunction condition that

$$\begin{aligned} U_{I_1 \wedge I_2}^L(x, y) &= \wedge \{z \in L \mid x \leq (I_1 \wedge I_2)(y, z)\} \\ &= \wedge \{z \in L \mid x \leq I_1(y, z) \wedge I_2(y, z)\} \\ &= \wedge \{z \in L \mid x \leq I_1(y, z), x \leq I_2(y, z)\} \\ &= \wedge \{z \in L \mid U_{I_1}^L(x, y) \leq z, U_{I_2}^L(x, y) \leq z\} \\ &= \wedge \{z \in L \mid U_{I_1}^L(x, y) \vee U_{I_2}^L(x, y) \leq z\} \\ &= (U_{I_1}^L \vee U_{I_2}^L)(x, y) \quad \forall x, y \in L, \end{aligned}$$

Therefore,  $U_{I_1 \wedge I_2}^L = U_{I_1}^L \vee U_{I_2}^L$ .

The theorem is proved.

By virtue of Theorem 5.2, we know that  $\phi$  and  $\varphi$  are, respectively, anti-order isomorphisms of  $U_{cs\vee}^{eL}(L)$  onto  $I_{\wedge}^{eL}(L)$  and  $U_{\vee cs}^{eR}(L)$  onto  $I_{\wedge}^{eR}(L)$ ;  $\phi^{-1}$  and  $\varphi^{-1}$  are, respectively, anti-order isomorphisms of  $I_{\wedge}^{eL}(L)$  onto  $U_{cs\vee}^{eL}(L)$  and  $I_{\wedge}^{eR}(L)$  onto  $U_{\vee cs}^{eR}(L)$ .

## 6. Conclusions and Future Works

In this paper, we discuss the residual operations of left and right semi-uninorms and the left and right semi-uninorms induced by implications, show that the right (left) residual operator of a conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorm is a right  $\wedge$ -distributive implication which satisfies the neutrality principle, give some conditions such that the operations induced by an implication constitute left or right semi-uninorms, demonstrate that the operations induced by a right  $\wedge$ -distributive implication, which satisfies the order property or the neutrality principle, are left (right)  $\vee$ -distributive left (right) semi-uninorms or right (left) semi-uninorms, and reveal the relationships between conjunctive right (left)  $\vee$ -distributive left (right) semi-uninorms and right  $\wedge$ -distributive implications which satisfy the neutrality principle.

In a forthcoming paper, we will investigate the relationships between left (right) semi-uninorms and coimplications on a complete lattice.

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