Research/Technical Note

Notes on “Some Properties of L -fuzzy Approximation Spaces on Bounded Integral Residuated Lattices”

Yuan Wang^{1,*}, Keming Tang¹, Zhudeng Wang²¹College of Information Engineering, Yancheng Teachers University, Yancheng, People's Republic of China²School of Mathematics and Statistics, Yancheng Teachers University, Jiangsu, People's Republic of China**Email address:**

yctuwangyuan@163.com (Yuan Wang), tkmchina@126.com (Keming Tang), zhudengwang2004@163.com (Zhudeng Wang)

*Corresponding author

To cite this article:Yuan Wang, Keming Tang, Zhudeng Wang. Notes on “Some properties of L -fuzzy Approximation Spaces on Bounded Integral Residuated Lattices”. *Automation, Control and Intelligent Systems*. Vol. 4, No. 2, 2016, pp. 10-14. doi: 10.11648/j.acis.20160402.11**Received:** February 29, 2016; **Accepted:** March 9, 2016; **Published:** March 23, 2016

Abstract: In this note, we continue the works in the paper [Some properties of L -fuzzy approximation spaces on bounded integral residuated lattices", Information Sciences, 278, 110-126, 2014]. For a complete involutive residuated lattice, we show that the L -fuzzy topologies generated by a reflexive and transitive L -relation satisfy $(TC)_L$ or $(TC)_R$ axioms and the L -relations induced by two L -fuzzy topologies, which are generated by a reflexive and transitive L -relation, are all the original L -relation; and give out some conditions such that the L -fuzzy topologies generated by two L -relations, which are induced by an L -fuzzy topology, are all the original L -fuzzy topology.

Keywords: Involutive Residuated Lattice, L -relation, L -fuzzy Topology, L -fuzzy Approximation Space

1. Introduction

A residuated lattice (see [1, 10]) is an algebra $L = (L, \wedge, \vee, \cdot, \rightarrow, \leftarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(L1) $(L, \wedge, \vee)$ is a lattice,(L2) $(L, \cdot, 1)$ is a monoid, i.e., is associative and $x \cdot 1 = 1 \cdot x = x$ for any $x \in L$,(L3) $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \leftarrow z$ for any $x, y, z \in L$.

A residuated lattice with a constant 0 is called an FL -algebra. If $x \leq 1$ for all $x \in L$, then L is called integral residuated lattice. An FL -algebra L , which satisfies the condition $0 \leq x \leq 1$ for all $x \in L$, is called an FL_w -algebra or a bounded integral residuated lattice (see [1]).

We adopt the usual convention of representing the monoid operation by juxtaposition, writing ab for $a \cdot b$.

Let L be a bounded integral residuated lattice. Define two negations on L , $\neg^L$ and $\neg^R$:

$$\neg^L x = x \rightarrow 0, \quad \neg^R x = x \leftarrow 0 \quad \forall x \in L.$$

A bounded residuated lattice L is called an involutive residuated lattice (see [3]) if

$$\neg^L \neg^R x = \neg^R \neg^L x \quad \forall x \in L.$$

In the sequel, unless otherwise stated, L always represents any given complete involutive residuated lattice with maximal element 1 and minimal element 0.

Definition 1.1 (see Liu and Luo [5]). Let $\tau \subseteq L^X$ and J be an index set. If τ satisfies the following three conditions:

(LFT1) $0_X, 1_X \in \tau$,(LFT2) $\mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau$,(LFT3) $\mu_j \in \tau (j \in J) \Rightarrow \bigvee_{j \in J} \mu_j \in \tau$,

then τ is called an L -fuzzy topology on X and (L^X, τ) L -fuzzy topological space. Every element in τ is called an open subset in L^X .

When $L = [0, 1]$, an L -fuzzy topological space (L^X, τ) is

also called an F -topological space.

Let $\tau'_L = \{\neg^L \mu \mid \mu \in \tau\}$ and $\tau'_R = \{\neg^R \mu \mid \mu \in \tau\}$. The elements of τ'_L and τ'_R are, respectively, called left closed subsets and right closed subsets in L^X (see Wang et al. [12]).

Definition 1.2 (Wang and Liu [11], Wang et al. [12]). Let τ be an L -fuzzy topology on X and μ L -fuzzy subset of X . The interior, left closure and right closure of μ w.r.t τ are, respectively, defined by

$$\text{int}(\mu) = \bigvee \{\eta \mid \eta \leq \mu, \eta \in \tau\},$$

$$cl_L(\mu) = \bigwedge \{\xi \mid \mu \leq \xi, \xi \in \tau'_L\},$$

$$cl_R(\mu) = \bigwedge \{\zeta \mid \mu \leq \zeta, \zeta \in \tau'_R\}.$$

int , cl_L and cl_R are, respectively, called the interior, left closure and right closure operators.

For the sake of convenience, we denote $\text{int}(\mu)$, $cl_L(\mu)$ and $cl_R(\mu)$ by μ° , μ_L^- and μ_R^- , respectively.

Zhang et al. [14, 15] studied some properties of rough sets and rough approximation operators, Ouyang et al. [6, 7] investigated some generalized models of fuzzy rough sets, Liu and Lin [4] considered the intuitionistic fuzzy rough set model, Wu et al. [13] discussed the axiomatic characterizations of fuzzy rough approximation operators, Radzikowska and Kerre [9] studied L -fuzzy rough sets and lower (upper) L -fuzzy approximation based on commutative residuated lattices. Recently, Wang et al. [12] discussed the notion of left (right) lower and left (right) upper L -fuzzy rough approximation based on complete bounded integral residuated lattices.

Definition 1.3 (Wang et al. [12]). Let R be an L -relation on X . A pair (X, R) is called an L -fuzzy approximation space. Define the following four mappings $R \downarrow_L$, $R \uparrow_L$, $R \downarrow_R$, $R \uparrow_R$: $L^X \rightarrow L^X$, called a left lower, left upper, right lower, and right upper L -fuzzy rough approximation operators, respectively, as follows: for every $\mu \in L^X$ and $x \in X$,

$$R \downarrow_L(\mu)(x) = \bigwedge_{y \in X} (R(x, y) \rightarrow \mu(y)),$$

$$R \uparrow_L(\mu)(x) = \bigvee_{y \in X} \mu(y) R(y, x),$$

$$R \downarrow_R(\mu)(x) = \bigwedge_{y \in X} (R(y, x) \leftarrow \mu(y)),$$

$$R \uparrow_R(\mu)(x) = \bigvee_{y \in X} R(x, y) \mu(y).$$

$R \downarrow_L(\mu)$, $R \uparrow_L(\mu)$, $R \downarrow_R(\mu)$ and $R \uparrow_R(\mu)$ are called left lower, left upper, right lower, and right upper L -fuzzy rough approximations of μ , respectively.

A pair $(\lambda, \xi) \in L^X \times L^X$ such that $\lambda = R \downarrow_L(\mu)$ ($\lambda = R \downarrow_R(\mu)$) and $\xi = R \uparrow_L(\mu)$ ($\xi = R \uparrow_R(\mu)$) for some

$\mu \in L^X$, is called a left (right) L -fuzzy rough set in (X, R) .

When $L = [0, 1]$, L -fuzzy rough approximation operators, L -fuzzy approximation space and left (right) L -fuzzy rough sets are, respectively, called fuzzy rough approximation operators, fuzzy approximation space and left (right) fuzzy rough sets.

2. The L -fuzzy Topologies Generated by a Reflexive and Transitive L -relation

In this section, we supplement some properties of the L -fuzzy topologies generated by a reflexive and transitive L -relation.

If R is a reflexive and transitive L -relation on X , then it follows from Theorem 6.1 in [12] that

$$\tau_1 = \{\xi \mid R \downarrow_L(\xi) = \xi, \xi \in L^X\},$$

$$\tau_2 = \{\xi \mid R \downarrow_R(\xi) = \xi, \xi \in L^X\}$$

are all L -fuzzy topologies on X and $R \downarrow_L$ and $R \downarrow_R$ are just the interior operators w.r.t τ_1 and τ_2 , respectively. Here, τ_1 and τ_2 are called the L -fuzzy topologies generated by the L -relation R or by left lower L -fuzzy rough approximation operator $R \downarrow_L$ and right lower L -fuzzy rough approximation operator $R \downarrow_R$, respectively.

Theorem 2.1. If R is a reflexive and transitive L -relation on X , then

$$\tau_1 = \{\neg^L R \uparrow_R(\xi) \mid \xi \in L^X\},$$

$$\tau_2 = \{\neg^R R \uparrow_L(\xi) \mid \xi \in L^X\},$$

$R \uparrow_R$ and $R \uparrow_L$ are, respectively, the right closure operator w.r.t τ_1 and the left closure operator w.r.t τ_2 .

Proof. When L is an involutive residuated lattice, $\neg^R \neg^L \mu = \neg^L \neg^R \mu = \mu$ for any $\mu \in L^X$.

If R is a reflexive and transitive L -relation on X , then it follows from Theorem 4.1(5) and Remark 5.2 in [12] that

$$\begin{aligned} R \downarrow_L(\neg^L R \uparrow_R(\xi)) &= \neg^L (R \uparrow_R(R \uparrow_R(\xi))) \\ &= \neg^L (R \uparrow_R(\xi)) \quad \forall \xi \in L^X, \end{aligned}$$

i.e., $\neg^L R \uparrow_R(\xi) \in \tau_1$ for any $\xi \in L^X$. If $\xi \in \tau_1$ and $\mu \in L^X$, then it follows from Theorems 3.1(3) and 4.1(5) in [12] that

$$\begin{aligned} \xi &= R \downarrow_L(\xi) = R \downarrow_L(\neg^L \neg^R \xi) \\ &= \neg^L (R \uparrow_R(\neg^R \xi)) = \neg^L R \uparrow_R(\eta), \end{aligned}$$

$$\begin{aligned} \mu_R^- &= \neg^R (\neg^L \mu)^\circ \\ &= \neg^R (R \downarrow_L(\neg^L \mu)) = \neg^R (\neg^L R \uparrow_R(\mu)) \\ &= \neg^R \neg^L (R \uparrow_R(\mu)) = R \uparrow_R(\mu), \end{aligned}$$

where $\eta = \neg^R \xi$. So, $\tau_1 = \{\neg^L R \uparrow_R(\eta) | \eta \in L^X\}$ and $R \uparrow_R$ is the right closure operator w.r.t. τ_1 .

Similarly, we can show that $\tau_2 = \{\neg^R R \uparrow_L(\eta) | \eta \in L^X\}$ and $R \uparrow_L$ is the left closure operator w.r.t. τ_2 .

The theorem is proved.

Recently, Qin et al. [2, 8] studied the topological properties of fuzzy rough sets. The following left and right (TC) axioms are generalizations of (TC) axiom in [8].

(TC)_L axiom: for any $x, y \in X$ and $\mu \in \tau$ there exists $\mu^* \in \tau$ such that $\mu^*(x) = 0$ and

$$\mu^*(y) \rightarrow \mu^*(x) \leq \mu(y) \rightarrow \mu(x).$$

(TC)_R axiom: for any $x, y \in X$ and $\nu \in \tau$ there exists $\nu^* \in \tau$ such that $\nu^*(y) = 0$ and

$$\nu^*(x) \leftarrow \nu^*(y) \leq \nu(x) \leftarrow \nu(y).$$

Theorem 2.2. If R is a reflexive and transitive L -relation on X , then the L -fuzzy topologies τ_1 and τ_2 , generated by R , satisfy (TC)_R and (TC)_L axioms, respectively.

Proof. For any $x, y \in X$ and $\mu \in \tau_1$, let

$$\mu^* = \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}})),$$

then

$$\begin{aligned} \mu^*(y) &= \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}}))(y) \\ &= \neg^L R_{\tau_1}^R(y, y) = \neg^L 1 = 0, \\ \mu^*(x) \leftarrow \mu^*(y) &= \neg^L (R_{\tau_1}^R \uparrow_R (1_{\{y\}}))(x) \leftarrow 0 \\ &= \neg^R \neg^L R_{\tau_1}^R(x, y) = R_{\tau_1}^R(x, y) \\ &= \wedge_{\xi \in \tau_1} (\xi(x) \leftarrow \xi(y)) \leq \mu(x) \leftarrow \mu(y), \end{aligned}$$

i.e., τ_1 satisfies (TC)_R axiom; for any $\nu \in \tau_2$, let

$$\nu^* = \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}})),$$

Then

$$\begin{aligned} \nu^*(x) &= \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}}))(x) \\ &= \neg^R R_{\tau_2}^L(x, x) = \neg^R 1 = 0, \\ \nu^*(y) \rightarrow \nu^*(x) &= \neg^R (R_{\tau_2}^L \uparrow_L (1_{\{x\}}))(y) \rightarrow 0 \\ &= \neg^L \neg^R R_{\tau_2}^L(x, y) = R_{\tau_2}^L(x, y) \\ &= \wedge_{\xi \in \tau_2} (\xi(y) \rightarrow \xi(x)) \leq \nu(y) \rightarrow \nu(x), \end{aligned}$$

i.e., τ_2 satisfies (TC)_L axiom.

The theorem is proved.

3. The L -relations Induced by an L -fuzzy Topology

In this section, we supplement some properties of the L -relations induced by an L -fuzzy topology.

Let τ be an L -fuzzy topology on X . For any $x, y \in X$, let

$$\begin{aligned} R_{\tau}^L(x, y) &= \wedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x)), \\ R_{\tau}^R(x, y) &= \wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(y)). \end{aligned}$$

Clearly, R_{τ}^L and R_{τ}^R are reflexive L -relations on X . Moreover, it follows from Theorem 2.1(5) in [12] that

$$\begin{aligned} R_{\tau}^L(x, y) R_{\tau}^L(y, z) &= \{\wedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x))\} \{\wedge_{\nu \in \tau} (\nu(z) \rightarrow \nu(y))\} \\ &\leq \wedge_{\mu \in \tau} \{\mu(y) \rightarrow \mu(x)\} \{\mu(z) \rightarrow \mu(y)\} \\ &\leq \wedge_{\mu \in \tau} (\mu(z) \rightarrow \mu(x)) \\ &= R_{\tau}^L(x, z) \quad \forall x, y, z \in X, \end{aligned}$$

$$\begin{aligned} R_{\tau}^R(x, y) R_{\tau}^R(y, z) &= \{\wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(y))\} \{\wedge_{\nu \in \tau} (\nu(y) \leftarrow \nu(z))\} \\ &\leq \wedge_{\mu \in \tau} \{\mu(x) \leftarrow \mu(y)\} \{\mu(y) \leftarrow \mu(z)\} \\ &\leq \wedge_{\mu \in \tau} (\mu(x) \leftarrow \mu(z)) \\ &= R_{\tau}^R(x, z) \quad \forall x, y, z \in X. \end{aligned}$$

Thus, R_{τ}^L and R_{τ}^R are all transitive L -relations on X . Let

$$\begin{aligned} R_{\tau}(x, y) &= R_{\tau}^R(x, y) \wedge R_{\tau}^L(x, y) \\ &= \wedge_{\mu \in \tau} \{\mu(x) \leftarrow \mu(y)\} \wedge \{\mu(y) \rightarrow \mu(x)\} \quad \forall x, y \in X. \end{aligned}$$

It is easy to see that $R_{\tau} = R_{\tau}^R \wedge R_{\tau}^L$ is also a reflexive and transitive L -relations on X .

Theorem 3.1. If R is a reflexive and transitive L -relation on X , then

$$R = R_{\tau_1}^R = R_{\tau_2}^L.$$

Proof. For any $x, y \in X$, by virtue of Definitions 1.2 and 1.3 and Theorem 2.1, we see that

$$\begin{aligned} R(x, y) &= R \uparrow_R (1_{\{y\}})(x) = (1_{\{y\}})_R^-(x) \\ &= \wedge \{\neg^R \xi(x) | 1_{\{y\}} \leq \neg^R \xi, \xi \in \tau_1\} \\ &= \wedge \{\neg^R \xi(x) | \neg^R \xi(y) = 1, \xi \in \tau_1\} \\ &= \wedge \{\xi(x) \leftarrow 0 | \xi(y) = 0, \xi \in \tau_1\} \\ &= \wedge \{\xi(x) \leftarrow \xi(y) | \xi(y) = 0, \xi \in \tau_1\} \\ &\geq \wedge \{\xi(x) \leftarrow \xi(y) | \xi \in \tau_1\} = R_{\tau_1}^R(x, y); \end{aligned}$$

$$\begin{aligned} R(x, y) &= R \uparrow_L (1_{\{x\}})(y) = (1_{\{x\}})_L^-(y) \\ &= \wedge \{\neg^L \xi(y) | 1_{\{x\}} \leq \neg^L \xi, \xi \in \tau_2\} \\ &= \wedge \{\neg^L \xi(y) | \neg^L \xi(x) = 1, \xi \in \tau_2\} \\ &= \wedge \{\xi(y) \rightarrow 0 | \xi(x) = 0, \xi \in \tau_2\} \\ &= \wedge \{\xi(y) \rightarrow \xi(x) | \xi(x) = 0, \xi \in \tau_2\} \\ &\geq \wedge \{\xi(y) \rightarrow \xi(x) | \xi \in \tau_2\} = R_{\tau_2}^L(x, y). \end{aligned}$$

Thus, $R \geq R_{\tau_1}^R$ and $R \geq R_{\tau_2}^L$.

On the other hand, $R \downarrow_L$ and $R \uparrow_R$ are, respectively, the interior and right closure operators w.r.t. τ_1 and $R \downarrow_R$ and $R \uparrow_L$ are, respectively, the interior and left closure operators w.r.t. τ_2 . Thus, by virtue Theorem 3.1(3) and Remark 5.2 in [12], we can see that

$$\begin{aligned} R \uparrow_R \{\neg^R(R \downarrow_L(\mu))\} &= \{\neg^R(R \downarrow_L(\mu))\}_R^- \\ &= \neg^R\{\neg^L\neg^R(R \downarrow_L(\mu))\}^o = \neg^R\{R \downarrow_L(\mu)\}^o \\ &= \neg^R\{R \downarrow_L(R \downarrow_L(\mu))\} \\ &= \neg^R(R \downarrow_L(\mu)) \forall R \downarrow_L(\mu) \in \tau_1, \\ R \uparrow_L \{\neg^L(R \downarrow_R(\mu))\} &= \{\neg^L(R \downarrow_R(\mu))\}_L^- \\ &= \neg^L\{\neg^R\neg^L(R \downarrow_R(\mu))\}^o = \neg^L\{R \downarrow_R(\mu)\}^o \\ &= \neg^L\{R \downarrow_R(R \downarrow_R(\mu))\} \\ &= \neg^L(R \downarrow_R(\mu)) \forall R \downarrow_R(\mu) \in \tau_2. \end{aligned}$$

So, it follows from the proof of Theorem 7.2 in [12] that $R \leq R_{\tau_1}^R$ and $R \leq R_{\tau_2}^L$.

Therefore, $R = R_{\tau_1}^R = R_{\tau_2}^L$.

The theorem is proved.

This result shows that the reflexive and transitive L -relations $R_{\tau_1}^R$ and $R_{\tau_2}^L$ induced by, respectively, the L -fuzzy topologies τ_1 and τ_2 are all the original reflexive and transitive L -relation.

For any $\mu \in L^X$ and $R \in L^{X \times X}$,

$$\mu = \bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}}).$$

Thus, by Definition 1.3 and Theorem 4.1(3) in [12], we see that

$$\begin{aligned} R \uparrow_L(\mu) &= \bigvee_{x \in X} R \uparrow_L(\mu(x)_X \wedge 1_{\{x\}}) \\ &= \bigvee_{x \in X} \mu(x)_X \cdot R \uparrow_L(1_{\{x\}}), \\ R \uparrow_R(\mu) &= \bigvee_{x \in X} R \uparrow_R(\mu(x)_X \wedge 1_{\{x\}}) \\ &= \bigvee_{x \in X} R \uparrow_L(1_{\{x\}}) \cdot \mu(x)_X. \end{aligned}$$

Theorem 3.2. Let τ be an L -fuzzy topology on X and J index set. Then the following properties hold.

(1) If τ satisfies $(TC)_L$ axiom and the left closure operator w.r.t. τ satisfies the following two conditions:

$$(CL1) \quad (\bigvee_{j \in J} \mu_j)_L^- = \bigvee_{j \in J} (\mu_j)_L^- \quad \forall \mu_j \in L^X,$$

$$(CL2) \quad (a \wedge 1_{\{x\}})_L^- = a \cdot (1_{\{x\}})_L^- \quad \forall a \in L, x \in X,$$

then $R_{\tau}^L \uparrow_L$ and $R_{\tau}^L \downarrow_R$ are, respectively, just the left closure operator and the interior operator w.r.t. τ and

$$\tau = \{\xi \mid R_{\tau}^L \downarrow_R(\xi) = \xi, \xi \in L^X\}.$$

(2) If τ satisfies $(TC)_R$ axiom and the right closure operator w.r.t. τ satisfies the following two conditions:

$$(CR1) \quad (\bigvee_{j \in J} \mu_j)_R^- = \bigvee_{j \in J} (\mu_j)_R^- \quad \forall \mu_j \in L^X,$$

$$(CR2) \quad (a \wedge 1_{\{x\}})_R^- = (1_{\{x\}})_R^- \cdot a \quad \forall a \in L, x \in X,$$

then $R_{\tau}^R \uparrow_R$ and $R_{\tau}^R \downarrow_L$ are, respectively, just the right closure operator and the interior operator w.r.t. τ and

$$\tau = \{\xi \mid R_{\tau}^R \downarrow_L(\xi) = \xi, \xi \in L^X\}$$

Proof. We only prove (1).

If τ satisfies $(TC)_L$ axiom and the left closure operator w.r.t. τ satisfies the conditions (CL1) and (CL2), then it follows from Definition 1.3 and the proof of Theorem 3.1 that

$$\begin{aligned} R_{\tau}^L \uparrow_L(1_{\{x\}})(y) &= R_{\tau}^L(x, y) = \bigwedge_{\mu \in \tau} (\mu(y) \rightarrow \mu(x)) \\ &= \bigwedge \{\mu(y) \rightarrow \mu(x) \mid \mu(x) = 0, \mu \in \tau\} \\ &= (1_{\{x\}})_L^-(y) \quad \forall x, y \in X, \end{aligned}$$

i.e., $(1_{\{x\}})_L^- = R_{\tau}^L \uparrow_L(1_{\{x\}})$ for any $x \in X$. Thus, for any $\mu \in L^X$, we have that

$$\begin{aligned} \mu_L^- &= \{\bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})\}_L^- \\ &= \bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})_L^- = \bigvee_{x \in X} \mu(x)_X \cdot (1_{\{x\}})_L^- \\ &= \bigvee_{x \in X} \mu(x)_X \cdot R_{\tau}^L \uparrow_L(1_{\{x\}}) \\ &= \bigvee_{x \in X} R_{\tau}^L \uparrow_L(\mu(x)_X \wedge 1_{\{x\}}) \\ &= R_{\tau}^L \uparrow_L \{\bigvee_{x \in X} (\mu(x)_X \wedge 1_{\{x\}})\} \\ &= R_{\tau}^L \uparrow_L(\mu), \end{aligned}$$

i.e., $R_{\tau}^L \uparrow_L$ is just the left closure operator w.r.t. τ . By Theorems 3.1(2) and 4.1(5) in [12], we see that

$$\begin{aligned} \mu^o &= \neg^R(\neg^L \mu)_L^- = \neg^R R_{\tau}^L \uparrow_L(\neg^L \mu) \\ &= R_{\tau}^L \downarrow_R(\neg^R \neg^L \mu) = R_{\tau}^L \downarrow_R(\mu) \quad \forall \mu \in L^X, \end{aligned}$$

i.e., $R_{\tau}^L \downarrow_R$ is just the interior closure operator w.r.t. τ . Therefore,

$$\tau = \{\xi \mid R_{\tau}^L \downarrow_R(\xi) = \xi, \xi \in L^X\}.$$

The theorem is proved.

This result shows that the L -topologies generated by two reflexive and transitive L -relations R_{τ}^L and R_{τ}^R , which are induced by an L -topology τ , on X are all the original L -topology τ when τ satisfies some conditions.

Moreover, if τ satisfies (CL1) or (CR1), then it follows from Remark 2.1 and Theorem 3.1(2) in [12] that

$$\begin{aligned} (\bigwedge_{j \in J} \mu_j)^o &= \neg^R(\neg^L(\bigwedge_{j \in J} \mu_j))_L^- \\ &= \neg^R(\bigvee_{j \in J} \neg^L \mu_j)_L^- = \neg^R \{\bigvee_{j \in J} (\neg^L \mu_j)_L^-\} \\ &= \bigwedge_{j \in J} \neg^R(\neg^L \mu_j)_L^- = \bigwedge_{j \in J} (\mu_j)^o \quad \forall \mu_j \in L^X, \end{aligned}$$

i.e., the interior operator int of τ distributes over arbitrary intersection of L -fuzzy sets. Thus, the intersection of arbitrarily many open subsets is still an open subset.

4. Conclusions and Future Work

In this note, we continue the works in [12]. For a complete involutive residuated lattice, we have supplemented some properties of the L -fuzzy topologies generated by a reflexive and transitive L -relation; showed that the L -fuzzy topologies generated by a reflexive and transitive L -relation satisfy $(\text{TC})_L$ or $(\text{TC})_R$ axioms; and given out some conditions such that the L -fuzzy topologies generated by two L -relations, which are induced by an L -fuzzy topology, are all the original L -fuzzy topology.

In a forthcoming paper, we will discuss the relationships between the L -fuzzy topological spaces and the L -fuzzy rough approximation spaces on a complete involutive residuated lattice.

Acknowledgements

The authors wish to thank the anonymous referees for their valuable comments and suggestions.

This work is supported by the National Natural Science Foundation of China (61379064).

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