Notes on “Some Properties of $L$-fuzzy Approximation Spaces on Bounded Integral Residuated Lattices”

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To cite this article:

Received: February 29, 2016; Accepted: March 9, 2016; Published: March 23, 2016

Abstract: In this note, we continue the works in the paper [Some properties of $L$-fuzzy approximation spaces on bounded integral residuated lattices”, Information Sciences, 278, 110-126, 2014]. For a complete involutive residuated lattice, we show that the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation satisfy (TC)$_{L}$ or (TC)$_{R}$ axioms and the $L$-relations induced by two $L$-fuzzy topologies, which are generated by a reflexive and transitive $L$-relation, are all the original $L$-relation; and give out some conditions such that the $L$-fuzzy topologies generated by two $L$-relations, which are induced by an $L$-fuzzy topology, are all the original $L$-fuzzy topology.

Keywords: Involutive Residuated Lattice, $L$-relation, $L$-fuzzy Topology, $L$-fuzzy Approximation Space

1. Introduction

A residuated lattice (see [1, 10]) is an algebra $L=(L, \land, \lor, \cdot, \to, \neg, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(L1) $(L, \land, \lor)$ is a lattice,
(L2) $(L, \cdot, 1)$ is a monoid, i.e., is associative and $x\cdot 1=1\cdot x=x$ for any $x \in L$,
(L3) $x \cdot y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \leftarrow z$ for any $x, y, z \in L$.

A residuated lattice with a constant 0 is called an FL-algebra. If $x \leq 1$ for all $x \in L$, then $L$ is called integral residuated lattice. An $FL_{a}$-algebra $L$, which satisfies the condition $0 \leq x \leq 1$ for all $x \in L$, is called an $FL_{b}$-algebra or a bounded integral residuated lattice (see [1]).

We adopt the usual convention of representing the monoid operation by juxtaposition, writing $ab$ for $a \cdot b$.

Let $L$ be a bounded integral residuated lattice. Define two negations on $L$, $\neg^{\text{i}}$ and $\neg^{\text{r}}$:

$\neg^{\text{i}} x = x \to 0$, $\neg^{\text{r}} x = x \leftarrow 0 \ \forall x \in L$.

A bounded residuated lattice $L$ is called an involutive residuated lattice (see [3]) if

$\neg^{\text{i}} \neg^{\text{r}} x = \neg^{\text{r}} \neg^{\text{i}} x \ \forall x \in L$.

In the sequel, unless otherwise stated, $L$ always represents any given complete involutive residuated lattice with maximal element 1 and minimal element 0.

Definition 1.1 (see Liu and Luo [5]). Let $\tau \subseteq L^{x}$ and $J$ be an index set. If $\tau$ satisfies the following three conditions:

(LFT1) $0_{x}, 1_{x} \in \tau$,
(LFT2) $\mu, \nu \in \tau \Rightarrow \mu \land \nu \in \tau$,
(LFT3) $\mu_{j} \in \tau \land (j \in J) \Rightarrow \vee_{j \in J} \mu_{j} \in \tau$,

then $\tau$ is called an $L$-fuzzy topology on $X$ and $(L^{x}, \tau)$ is an $L$-fuzzy topological space. Every element in $\tau$ is called an open subset in $L^{x}$.

When $L=[0, 1]$, an $L$-fuzzy topological space $(L^{x}, \tau)$ is...
also called an $F$-topological space.

Let $\tau_\lambda = \{ \mu \| \mu \in \tau \}$ and $\tau_\lambda = \{ \mu \| \mu \in \tau \}$.

The elements of $\tau_\lambda$ and $\tau_\lambda$ are, respectively, called left closed subsets and right closed subsets in $L^X$ (see Wang et al. [12]).

**Definition 1.2** (Wang and Liu [11], Wang et al. [12]). Let $\tau$ be an $L$-fuzzy topology on $X$ and $\mu$ an $L$-fuzzy subset of $X$. The interior, left closure and right closure of $\mu$ w.r.t $\tau$ are, respectively, defined by

\[ \text{int}(\mu) = \vee \{ \eta \| \eta \leq \mu, \eta \in \tau \}, \]
\[ cl_{L}(\mu) = \wedge \{ \xi \| \mu \leq \xi, \xi \in \tau_\lambda \}, \]
\[ cl_{R}(\mu) = \wedge \{ \xi \| \mu \leq \xi, \xi \in \tau_\lambda \}. \]

int, $cl_L$ and $cl_R$ are, respectively, called the interior, left closure and right closure operators.

For the sake of convenience, we denote $\text{int}(\mu)$, $cl_L(\mu)$ and $cl_R(\mu)$ by $\mu^\top$, $\mu_\lambda$ and $\mu_\lambda$, respectively.


**Definition 1.3** (Wang et al. [12]). Let $R$ be an $L$-relation on $X$. A pair $(X, R)$ is called an $L$-fuzzy approximation space. Define the following four mappings $R_\lambda$, $R_\lambda$, $R_\lambda$, $R_\lambda$: $L^X \rightarrow L^X$, called a left lower, left upper, right lower, and right upper $L$-fuzzy rough approximation operators, respectively, as follows: for every $\mu \in L^X$ and $x \in X$,

\[ R_\lambda(\mu)(x) = \wedge_{\mu \in L^X} (R(x, y) \rightarrow \mu(y)), \]
\[ R_\lambda(\mu)(x) = \vee_{\mu \in L^X} \mu(y) \wedge R(y, x), \]
\[ R_\lambda(\mu)(x) = \vee_{\mu \in L^X} (R(y, x) \leftarrow \mu(y)), \]
\[ R_\lambda(\mu)(x) = \vee_{\mu \in L^X} R(x, y) \mu(y). \]

$R_\lambda(\mu)$, $R_\lambda(\mu)$, $R_\lambda(\mu)$ and $R_\lambda(\mu)$ are called left lower, left upper, right lower, and right upper $L$-fuzzy rough approximations of $\mu$, respectively.

A pair $(\lambda, \xi) \in L^X \times L^X$ such that $\lambda = R_\lambda(\mu)$ ($\lambda = R_\lambda(\mu)$) and $\xi = R_\lambda(\mu)$ ($\xi = R_\lambda(\mu)$) for some $\mu \in L^X$, is called a left (right) $L$-fuzzy rough set in $(X, R)$.

When $L = [0, 1]$, $L$-fuzzy rough approximation operators, $L$-fuzzy approximation space and left (right) $L$-fuzzy rough sets are, respectively, called fuzzy rough approximation operators, fuzzy approximation space and left (right) fuzzy rough sets.

**2. The $L$-fuzzy Topologies Generated by a Reflexive and Transitive $L$-relation**

In this section, we supplement some properties of the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation.

If $R$ is a reflexive and transitive $L$-relation on $X$, then it follows from Theorem 6.1 in [12] that

\[ \tau_1 = \{ \xi \| R_\lambda(\xi) = \xi, \xi \in L^X \}, \]
\[ \tau_2 = \{ \xi \| R_\lambda(\xi) = \xi, \xi \in L^X \}, \]

are all $L$-fuzzy topologies on $X$ and $R_\lambda$ and $R_\lambda$ are just the interior operators w.r.t $\tau_1$ and $\tau_2$, respectively. Here, $\tau_1$ and $\tau_2$ are called the $L$-fuzzy topologies generated by the $L$-relation $R$ or by left lower $L$-fuzzy rough approximation operator $R_\lambda$ and right lower $L$-fuzzy rough approximation operator $R_\lambda$, respectively.

**Theorem 2.1.** If $R$ is a reflexive and transitive $L$-relation on $X$, then

\[ \tau_1 = \{ \xi \| R_\lambda(\xi) = \xi, \xi \in L^X \}, \]
\[ \tau_2 = \{ \xi \| R_\lambda(\xi) = \xi, \xi \in L^X \}. \]

$R_\lambda$ and $R_\lambda$ are, respectively, the right closure operator w.r.t $\tau_1$ and the left closure operator w.r.t. $\tau_2$.

**Proof.** When $L$ is an involutive residuated lattice, $\lambda^\top - \lambda^\top = \lambda^\top - \lambda^\top$ for any $\mu \in L^X$.

If $R$ is a reflexive and transitive $L$-relation on $X$, then it follows from Theorem 4.1(5) and Remark 5.2 in [12] that

\[ R_\lambda(\lambda^\top R_\lambda(\xi)) = \lambda^\top (R_\lambda R_\lambda(\xi)), \]
\[ \forall \xi \in L^X, \]

i.e., $\lambda^\top R_\lambda(\xi) \in \tau_1$ for any $\xi \in L^X$. If $\xi \in \tau_1$ and $\mu \in L^X$, then it follows from Theorems 3.1(3) and 4.1(5) in [12] that

\[ \xi = R_\lambda(\lambda^\top R_\lambda(\xi)) = R_\lambda(R_\lambda(\xi)), \]
\[ \forall \xi \in L^X, \]

\[ \mu^\top = \lambda^\top, \]
\[ \forall \xi \in L^X, \]

\[ \lambda^\top \mu = \lambda, \]
\[ \forall \xi \in L^X, \]

\[ \mu = \lambda^\top, \]
\[ \forall \xi \in L^X, \]

\[ \lambda = \lambda^\top R_\lambda(\xi), \]
\[ \forall \xi \in L^X, \]

\[ \xi = R_\lambda(\lambda^\top R_\lambda(\xi)) = R_\lambda(R_\lambda(\xi)), \]
\[ \forall \xi \in L^X, \]

\[ \mu = \lambda^\top, \]
\[ \forall \xi \in L^X, \]

\[ \lambda = \lambda^\top R_\lambda(\xi), \]
\[ \forall \xi \in L^X, \]
where η = ¬θ (τ). So, τ1 = \{−I R ↑ L (η) | η ∈ L^x \} and R ↑ L is the right closure operator w.r.t. τ1.

Similarly, we can show that τ2 = \{−R ↑ L (η) | η ∈ L^x \} and R ↑ L is the left closure operator w.r.t. τ2.

The theorem is proved.

Recently, Qin et al. [2, 8] studied the topological properties of fuzzy rough sets. The following left and right (TC) axioms are generalizations of (TC) axiom in [8].

( TC ) L axiom: For any x, y ∈ X and μ ∈ τ there exists μ’ ∈ τ such that μ’(x) = 0 and

μ’(y) → μ’(x) ≤ μ(y) → μ(x).

( TC ) R axiom: For any x, y ∈ X and ν ∈ τ there exists ν’ ∈ τ such that ν’(y) = 0 and

ν’(x) ≤ ν’(y) ≤ ν(x) ≤ ν(y).

Theorem 2.2. If R is a reflexive and transitive L-relation on X, then the L-fuzzy topologies τ1 and τ2, generated by R, satisfy ( TC ) R and ( TC ) L axioms, respectively.

Proof. For any x, y ∈ X and μ ∈ τ1, let

μ’ = η → ( R↑_L (1_1, y)),

then

μ’(y) = η → ( R↑_L (1_1, y))(y)
= η → R↑_L (y, y) = η → 1 = 0,

μ’(x) ≤ μ’(y) = η → ( R↑_L (1_1, y))(x) ≤ 0
= η → R↑_L (x, y) = R↑_L (x, y)
= μ(x) ≤ μ(y),

i.e., τ1 satisfies ( TC ) R axiom; for any ν ∈ τ2, let

ν’ = η → ( R↑_L (1_1, y)),

Then

ν’(y) = η → ( R↑_L (1_1, y))(y)
= η → R↑_L (x, y) = η → 1 = 0,

ν’(y) → ν’(x) = η → ( R↑_L (1_1, y))(y) → 0
= η → R↑_L (x, y) = R↑_L (x, y)
= μ(x) ≤ μ(y),

i.e., τ2 satisfies ( TC ) L axiom.

The theorem is proved.

3. The L-relations Induced by an L-fuzzy Topology

In this section, we supplement some properties of the L-relations induced by an L-fuzzy topology.
Thus, $\mathcal{R} \geq \mathcal{R}^L_\downarrow$ and $\mathcal{R} \geq \mathcal{R}^L_\uparrow$.

On the other hand, $\mathcal{R} \downarrow_L$ and $\mathcal{R} \uparrow_L$ are, respectively, the interior and right closure operators w.r.t. $\tau_1$ and $\mathcal{R} \downarrow_R$ and $\mathcal{R} \uparrow_R$ are, respectively, the interior and left closure operators w.r.t. $\tau_2$. Thus, by virtue Theorem 3.1(3) and Remark 5.2 in [12], we can see that

$$R \uparrow_L \{\mathcal{R}(\mathcal{R} \downarrow_L (\mu))\} = \mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \uparrow_L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

$$R \uparrow_L \{\mathcal{R}(\mathcal{R} \downarrow_R (\mu))\} = \mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \uparrow_R \mathcal{R} \downarrow_L (\mu) \in \tau_2.$$ 

So, it follows from the proof of Theorem 7.2 in [12] that $\mathcal{R} \leq \mathcal{R}^L_\downarrow$ and $\mathcal{R} \leq \mathcal{R}^L_\uparrow$.

Therefore, $\mathcal{R} = \mathcal{R}^L_\downarrow = \mathcal{R}^L_\uparrow$.

The theorem is proved.

This result shows that the reflexive and transitive $L$-relations $\mathcal{R}^L_\downarrow$ and $\mathcal{R}^L_\uparrow$ induced by, respectively, the $L$-fuzzy topologies $\tau_1$ and $\tau_2$ are all the original reflexive and transitive $L$-relation.

For any $\mu \in L^X$ and $\mathcal{R} \in L^{\times X}$,

$$\mu = \mathcal{R}_{\vee}\mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

Thus, by Definition 1.3 and Theorem 4.1(3) in [12], we see that

$$\mathcal{R} \uparrow_L (\mu) = \mathcal{R}_{\vee}\mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

$$\mathcal{R} \uparrow_R (\mu) = \mathcal{R}_{\vee}\mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_R (\mu)) \downarrow L \mathcal{R} \downarrow_R (\mu) \in \tau_2.$$ 

Theorem 3.2. Let $\tau$ be an $L$-fuzzy topology on $X$ and $J$ index set. Then the following properties hold.

(1) If $\tau$ satisfies (TC)_L axiom and the left closure operator w.r.t. $\tau$ satisfies the following two conditions:

(CL1) $(\mathcal{R}_L \mu)_{\vee} = \mathcal{R}_{\vee}\mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$

(CL2) $(\mathcal{R}_L \mu \vee \mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$

then $\mathcal{R}^L_\downarrow$ and $\mathcal{R}^L_\uparrow$ are, respectively, just the left closure operator and the interior operator w.r.t. $\tau$ and $\mathcal{R}$.

(2) If $\tau$ satisfies (TC)_R axiom and the right closure operator w.r.t. $\tau$ satisfies the following two conditions:

(CR1) $(\mathcal{R}_R \mu)_{\vee} = \mathcal{R}_{\vee}\mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_R (\mu)) \downarrow L \mathcal{R} \downarrow_R (\mu) \in \tau_2,$

(CR2) $(\mathcal{R}_R \mu \vee \mathcal{R}(\mathcal{R} \downarrow_R (\mu)) \downarrow L \mathcal{R} \downarrow_R (\mu) \in \tau_2,$

then $\mathcal{R}^L_\downarrow$ and $\mathcal{R}^L_\uparrow$ are, respectively, just the right closure operator and the interior operator w.r.t. $\tau$ and $\mathcal{R}$.

$$\mathcal{R} = \{\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

Proof. We only prove (1).

If $\tau$ satisfies (TC)_L axiom and the left closure operator w.r.t. $\tau$ satisfies the conditions (CL1) and (CL2), then it follows from Definition 1.3 and the proof of Theorem 3.1 that

$$\mathcal{R}^L_\downarrow (\tau_1(x)) \tau_1(x) \mathcal{R}(\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

i.e., $(\tau_1(x)) \mathcal{R}^L_\downarrow (\tau_1(x))$ for any $x \in X$. Thus, for any $\mu \in L^X$, we have that

$$\tau_2 = \{\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_2,$$

$$\mathcal{R}^L_\downarrow = \{\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_2.$$ 

Theorem is proved.

This result shows that the $L$-topologies generated by two reflexive and transitive $L$-relations $\mathcal{R}^L_\downarrow$ and $\mathcal{R}^L_\uparrow$, which are induced by an $L$-topology $\tau$, on $X$ are all the original $L$-topology $\tau$ when $\tau$ satisfies some conditions.

Moreover, if $\tau$ satisfies (CL1) or (CR1), then it follows from Remark 2.1 and Theorem 3.2 and Theorem 4.1(3) in [12] that

$$\mathcal{R}(\mathcal{R} \downarrow_L (\mu)) \downarrow L \mathcal{R} \downarrow_L (\mu) \in \tau_1,$$

$$\mathcal{R}(\mathcal{R} \downarrow_R (\mu)) \downarrow L \mathcal{R} \downarrow_R (\mu) \in \tau_2.$$ 

The theorem is proved.
i.e., the interior operator int of $\tau$ distributes over arbitrary intersection of $L$-fuzzy sets. Thus, the intersection of arbitrarily many open subsets is still an open subset.

4. Conclusions and Future Work

In this note, we continue the works in [12]. For a complete involutive residuated lattice, we have supplemented some properties of the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation; showed that the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation satisfy (TC)$_L$ or (TC)$_R$ axioms; and given out some conditions such that the $L$-fuzzy topologies generated by two $L$-relations, which are induced by an $L$-fuzzy topology, are all the original $L$-fuzzy topology.

In a forthcoming paper, we will discuss the relationships between the $L$-fuzzy topological spaces and the $L$-fuzzy rough approximation spaces on a complete involutive residuated lattice.

Acknowledgements

The authors wish to thank the anonymous referees for their valuable comments and suggestions.

This work is supported by the National Natural Science Foundation of China (61379064).

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