

# Solving boussinesq equation by bilinear bäcklund transformation

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**Abstract:** In this paper Hirota bilinear method is applied to constructing Backlund transformation of the Boussinesq equation. The bilinear Backlund form are used to obtain the soliton solution of the Boussinesq equation. Also as an application for the bilinear Bäcklund transformation, new classes of wave solutions to the Boussinesq Equation are computed.

**Keywords:** Boussinesq Equation, Backlund Transformation, Hirota Bilinear Form, Travelling Wave

## 1. Introduction

Searching for exact solution to nonlinear equation of mathematical physics are significantly important. The transformed rational function method [1] and multiple exp-function method [2] provide some generic approaches for constructing travelling wave solutions and multiple wave solutions respectively. Perturbation expansion method proposed by Hirota [3] often generates a specific class of multiple wave solution including the N-soliton solutions, provided that the equation possesses a Hirota bilinear form. Also linear superposition principle may apply Hirota bilinear and in particular linear subspaces of solutions for nonlinear equations [4]. Another powerful approach to solution of nonlinear equation are the Bäcklund Transformations, they can be written as Hirota bilinear form when the equation under consideration has a bilinear form [5,6]. The Bilinear Bäcklund Transformation is essentially defined as pair of partial differential relation involving two independent variable and their derivatives, which together imply that each one of the independent variable satisfy a partial differential equation [7,8]. Thus, for example, the transformation

$$\psi_x = F(\phi, \psi, \phi_x, x)$$

would imply that  $\phi$  and  $\psi$  satisfy partial differential equations of the operational form,

$$P(\phi) = 0$$

## 2. Bilinear Form

We consider the following (1+1)-dimensional nonlinear equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{1}$$

Under the dependent variable transformation

$$u = 6 \frac{\partial}{\partial x} (\ln \tau) \tag{2}$$

equation (1) can be mapped into the Hirota bilinear equation

$$(D_t^2 + D_x^4)\tau \cdot \tau = 0 \tag{3}$$

where the bilinear differential operator D is defined by

$$D_t^m D_x^n a(t, x)b(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y)b(t - s, x - y) |_{s=0, y=0} \tag{4}$$

We can re-write equation (1) in terms of  $\tau$  as follows:

$$\left( \frac{\partial^4 \tau}{\partial x^4} + \frac{\partial^2 \tau}{\partial t^2} \right) \tau - 4 \frac{\partial^3 \tau}{\partial x^3} \frac{\partial \tau}{\partial x} + 3 \left( \frac{\partial^2 \tau}{\partial x^2} \right)^2 - \frac{\partial \tau}{\partial t} = 0 \tag{5}$$

In this work, we would like to present a bilinear Bäcklund transformation for equation (1) as proposed in [10].

Let us suppose that we have another solution  $\tau'$  to the equation

$$(D_t^2 + D_x^4) \tau' \cdot \tau' = 0 \tag{6}$$

and we will produce the key function

$$P = [(D_t^2 + D_x^4) \tau \cdot \tau] \tau'^2 - [(D_t^2 + D_x^4) \tau' \cdot \tau'] \tau^2 \tag{7}$$

If  $P = 0$  then

$$\tau \text{ is a solution of (3)} \Leftrightarrow \tau' \text{ is a solution of (3)}$$

Therefore, if we can obtain from  $P = 0$  by interchanging the dependant variable  $\tau$  and  $\tau'$ , a system of bilinear equation that guarantees  $P = 0$

$$B_i (D_t, D_x) \tau \cdot \tau' = 0, 1 \leq i \leq M$$

where the  $B_i$ 's are polynomial in the indicated variables and  $M$  is natural number depending on the complexity of the equation. A question now arises; is  $P = 0$  has unique solution? Otherwise, how to get all the solutions?

The answer to the first part of the equation may be No. for the second part of the equation, it is known that Hirota's bilinear operator identities are the key to break  $P$  into a system of polynomials  $B_i$ 's the following the lemmas are useful in generating all the nontrivial identities. Lemma: Let  $D = \mathbb{R}[f^{(i)}, g^{(j)}], (i, j \in J)$  be the polynomial algebra, where  $J$  denotes the space of multi-indices  $((i_1, i_2, \dots, i_n))$ .

In this algebra, we have the derivations

$$\partial_m (m = 1, \dots, n) \text{ Which act in the very direct way,}$$

$$\begin{aligned} \partial_m (f^{(i)}) &= f^{(i+1_m)} \text{ and } \partial_m (g^{(j)}) = g^{(j+1_m)} \\ \text{where } 1_m &= (0, \dots, 0, 1, 0, \dots, 0), \end{aligned} \tag{7}$$

Where  $1_m = (0, \dots, 0, 1, 0, \dots, 0)$ , the  $m^{th}$  basis vector in the  $n$ -dimensional Euclidean space.

Lemma. Let  $D_m : D \otimes D \rightarrow D \otimes D$  be the linear map:

$$D_m (a \otimes b) = \partial_m (a) \otimes b - a \otimes \partial_m (b)$$

Then

$$\exp(D_m)(a \otimes b) = \exp(\partial_m) a \otimes \exp(-\partial_m)$$

Observe that the image of the map  $D_m$  is again in  $D \otimes D$  and not in  $D$  which makes this map not similar to the usual one. To get the usual Hirota derivatives [6]  $D_m$  we have to project the image of  $D_m$  to  $D$ :

Letting  $D \otimes D \rightarrow D, \prod(\sum a_i \otimes b_i) = \sum a_i b_i$ , be the projection map, then  $D_m^k(a \otimes b) = \prod D_m^k(a \otimes b), k = 1, 2, 3, \dots$

Lemma: let  $\partial_\alpha = \sum a_i \partial_i, \partial_\beta = \sum \beta_i \partial_i$  and  $\partial_\gamma = \sum \gamma_i \partial_i$  and let  $D_\alpha, D_\beta$  and  $D_\gamma$  be the corresponding projected Hirota derivatives then we have all  $a, b, c, d \in D$

$$\begin{aligned} &\exp(D_\alpha)[\exp(D_\beta)(a \otimes b) \otimes \exp(D_\alpha)(c \otimes d)] = \\ &\exp(\frac{1}{2}(D_\beta - D_\alpha)) \\ &[\exp(\frac{1}{2}(D_\beta - D_\alpha) + D_\alpha)(a \otimes b) \\ &\otimes \exp(\frac{1}{2}(D_\beta + D_\alpha) - D_\alpha)(a \otimes b)] \end{aligned} \tag{9}$$

Note that the above identity gives all possibility for re-mitting  $P = 0$  in the form

$$\sum \beta_i \tau \cdot \tau' = 0 \tag{10}$$

through the relation  $\exp(\sum \epsilon_i D_i) = \prod \exp(\epsilon_i D_i)$  for example, letting  $k, l, m \in J$  by taking  $a = b = \tau$  and  $c = d = \tau'$  the coefficient of  $\alpha^k \beta^l \lambda^m$  in equation... (9) gives the following identity

$$\begin{aligned} &D^k [D^l(\tau \cdot \tau) \cdot D^m(\tau' \cdot \tau')] = \\ &\sum D^{\dots} [D^{\dots}(\tau \cdot \tau') \cdot D^{\dots}(\tau \cdot \tau')] \end{aligned} \tag{11}$$

it is easy to check that an expression of such type is not unique if we take  $a = b = c = \tau$  and  $c = d = \tau'$

The coefficients of  $\alpha^k \beta^l \lambda^m$  in the equation... (9) give the following identity

$$\begin{aligned} &D^k [D^l(\tau \cdot \tau') \cdot D^m(\tau \cdot \tau')] = \\ &\sum D^{\dots} [D^{\dots}(\tau \cdot \tau') \cdot D^{\dots}(\tau \cdot \tau')] \end{aligned} \tag{12}$$

Let us now introduce the following identities which are useful for Hirota's bilinear operators:

$$D_\eta (D_\zeta a \cdot b) \cdot ab = D_\zeta (D_\eta a \cdot b) \cdot ba, \tag{13}$$

$$D_\eta ab \cdot cd = (D_\eta a \cdot b) \cdot cb - ad (D_\eta c \cdot b), \tag{14}$$

$$b^2 (D_\zeta^2 a \cdot a) - (D_\zeta^2 b \cdot b) a^2 = 2D_\zeta (D_\zeta a \cdot b) \cdot ba, \tag{15}$$

$$b^2 (D_\eta D_\zeta a \cdot a) - (D_\eta D_\zeta b \cdot b) a^2 = 2D_\zeta (D_\eta a \cdot b) \cdot ba \tag{16}$$

$$\begin{aligned} &b^2 (D_\eta^3 D_\zeta a \cdot a) - (D_\eta^3 D_\zeta b \cdot b) a^2 \\ &= 2D_\zeta (D_\eta^3 a \cdot b) \cdot ba + 6D_\eta (D_\eta D_\zeta a \cdot b) \cdot (D_\eta b \cdot a) \end{aligned} \tag{17}$$

Note that the above identity can be obtained from the identity in lemma 7 for more identities and general exchange formulae, you are referred to see [9] Applying the above identities to equation (5) We can obtain

$$\tau'^2 (D_t \tau \cdot \tau) - (D_t^2 \tau' \cdot \tau') \tau^2 = 2D_t (D_t \tau \cdot \tau') \cdot \tau' \tau \tag{18}$$

and

$$\begin{aligned} &\tau'^2 (D_x^4 \tau \cdot \tau) - (D_x^4 \tau' \cdot \tau') \tau^2 = \\ &2D_x (D_x^3 \tau \cdot \tau') \cdot \tau' \tau + 6D_x (D_x^2 \tau \cdot \tau') \cdot (D_x \tau' \cdot \tau) \end{aligned} \tag{19}$$

Substituting the above result into the right hand side of equation (5), we can obtain

$$P = 2D_t(D_t\tau\tau')\tau'\tau + 2D_x(D_x^3\tau\tau')\tau'\tau + 6D_x(D_x^2\tau\tau')(D_x\tau'\tau) \quad (20)$$

One can observe that  $P = 0$  has more than one unique solution, and the Hirota's bilinear operators identities of the type (13)-(17) was not able to provide general bilinear systems. We will then use to resulting transformation to obtain multiple solution to equation (1) let us now introduce new arbitrary parameters  $\eta$  are  $\square_i$  ( $i=1,2,3$ ) into equation (20) to obtain

$$P_{\pm}^a = 2D_t[(D_t \pm 3\eta D_x \mp \varepsilon_1)\tau\tau']\tau'\tau + 2D_x[(D_x^3 \mp 3\eta D_t \pm \varepsilon_2)\tau\tau']\tau'\tau + 6D_x[(D_x^2 \pm \eta D_x)\tau\tau'](D_x\tau'\tau) \quad (21)$$

this is possible because the coefficient of  $\eta, \varepsilon_1$  and  $\varepsilon_2$

$$\eta : \pm 6D_t(D_x\tau\tau')\tau'\tau \mp 6D_x(D_t\tau\tau')\tau'\tau$$

$$\varepsilon_1 : \mp 2(D_t\tau\tau')\tau'\tau$$

$$\varepsilon_2 : \pm 2(D_x\tau\tau')\tau'\tau$$

$$\mu : \pm 6D_x[D_x\tau\tau'](D_x\tau'\tau)$$

are all equal to zero due to the following properties

$$D_{\zeta}f \cdot f = 0 \quad (22)$$

$$D_{\zeta}f \cdot g = -D_{\zeta}g \cdot f \quad (23)$$

$$D_{\eta}(D_{\zeta}f \cdot g) \cdot gf = D_{\zeta}(D_{\eta}f \cdot g) \cdot gf \quad (24)$$

Then  $p_{\pm}^a = 0$  if  $B_i\tau\tau' = 0, 1 \leq i \leq 3$  where  $B_i$ 's can be found from equation (21) as follows

$$\begin{cases} B_1\tau\tau' \equiv (D_t \pm 3\eta D_x \mp \varepsilon_1)\tau\tau' \\ B_2\tau\tau' \equiv (D_x^3 \mp 3\eta D_t \pm \varepsilon_2)\tau\tau' \\ B_3\tau\tau' \equiv (D_x^2 \pm \mu D_x)\tau\tau' \end{cases} \quad (25)$$

the coefficients of  $\eta, \mu, \varepsilon_1$  and  $\varepsilon_2$  are equal to zero because of the equation (22)-(24) this shows that the system (25) presents a bilinear bäcklund transformation for (1).

Travelling Wave Solution

We now present an application of the bilinear Bäcklund transformation (25) in order to construct a new class of solution to equation (1). For this reason, we begin with  $\tau = 1$ , which is the trivial solution of the equation (3), obviously noting that

$$D_{\eta}^n \phi \cdot 1 = \frac{\partial^n \phi}{\partial \eta^n}, \quad n \geq 1, \quad (26)$$

the bilinear Bäcklund transformation (25) associated with  $\tau = 1$  become a system of linear partial differential equations:

$$\begin{cases} \frac{\partial \tau'}{\partial t} \pm 3\eta \frac{\partial \tau'}{\partial x} \mp \varepsilon_1 \tau' = 0 \\ \frac{\partial^3 \tau'}{\partial x^3} \mp 3\eta \frac{\partial \tau'}{\partial t} \pm \varepsilon_2 \tau' = 0 \\ \frac{\partial^3 \tau'}{\partial x^3} \mp 3\eta \frac{\partial \tau'}{\partial t} \pm \varepsilon_2 \tau' = 0 \\ \frac{\partial^2 \tau'}{\partial x^2} \pm \mu \frac{\partial \tau'}{\partial x} = 0 \end{cases} \quad (27)$$

Let us consider a class of exponential wave solution of the form

$$\tau^1 = 1 + \varepsilon e^{kx - wt + \zeta^0}, \quad \zeta^0 = \cos t \quad (28)$$

where  $\varepsilon, k$  and  $w$  are constants to be determine. On selecting

$$\varepsilon_1 = 0, \varepsilon_2 = 0 \quad (29)$$

and after some straightforward calculation, we can set

$$w = \pm 3\eta k, \mu = \mp k \quad (30)$$

therefore, we obtain a class of exponential wave solution to the equation (3)

$$\tau_{\pm}^1 = 1 + \varepsilon \exp(kx \mp 3\eta kt + \zeta^0) \quad (31)$$

where  $\varepsilon, k, \eta$  and  $\mu$  are arbitrary constants; and

$$U = 6(\ln \tau_{\pm}^1)_x$$

solves equation (1).

Let us now consider a second class of first order polynomial solution

$$\tau' = kx - wt,$$

where  $k$  and  $w$  are constants to determined again on selecting

$$\varepsilon_1 = 0, \varepsilon_2 = 0$$

a direct computation shows that the system (27) becomes

$$w = \pm 3\eta k$$

$$\pm 3\eta w = 0$$

$$\pm \mu k = 0$$

which gives  $\tau'_{\pm} = 0$

### 3. Conclusions

The bilinear Backlund Transformation for the nonlinear Boussinesq equation was constructed base on the existence of exchange identities for Hirota bilinear operators. Also and application for the bilinear Backlund transformation was used to construct a new class of exact wave solutions for the Boussinesq equation.

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