
Theorem on a matrix of right-angled triangles

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Abstract: The following theorem is proved: All primitive right-angled triangles (primitive Pythagorean triples) may be defined by a pair of positive integer indices (i, j) , where i is an uneven number and j is an even number and have no common factor. The sides of every positive integer right angled triangle are then defined by the indices as follows: For hypotenuse h , uneven leg u and even leg e , $h = i^2 + ij + j^2/2$, $e = ij + j^2/2$, $u = i^2 + ij$. This defines an infinite by infinite matrix of right angled triangles with positive integer sides.

Keywords: Primitive Right-Angled Triangles, Pythagorean Triples, Infinite Two-Dimensional Matrix

1. Introduction

Primitive right-angled triangles are a part of number theory that still fascinates both professional and amateur mathematicians [1-3]. These may be defined as right-angled triangles that have the lengths of their sides defined by positive integer multiples of the same units of length where the three respective positive integers are relatively prime. They are commonly referred to as primitive Pythagorean triples. Ways have been developed for generating infinite series of Pythagorean triples. These include the series where the hypotenuse and the even leg differ by 1 only, described *ia* by Fibonacci [4,5] and Stifel [6,7], and the series where the hypotenuse and the odd leg differ by 2 only, described *ia* by Ozanam [8]. These two series are the first of an infinite number of series, forming a two dimensional infinite by infinite array of right-angled triangles. This double infinity is accommodated by Pythagorean triple generating formulae developed by *ia* Euclid [9] and Dickson [10] where two positive integer variables uniquely describe the triples.

This paper presents a theorem in which all Pythagorean triples are described by two positive integer variables, termed indices, of which the first is an uneven number and the second even. The Pythagorean triples are primitive if and only if the associated pair of indices are relatively prime. The indices are developed from the concept that the difference between the hypotenuse and the even leg is a perfect square and the difference between the hypotenuse and the uneven leg is half of a perfect square [1]. Some of the arguments are very basic, but are included for clarity

and comprehensiveness allowing for a rigorous proof of the theorem.

Theorem: All rational right-angled triangles may be raised or reduced to a primitive right-angled triangle (primitive Pythagorean triple) defined by a pair of positive integer indices (i, j) where i is an uneven number, j is an even number and i and j are relatively prime. The even numbered leg (e), the uneven numbered leg (u) and the hypotenuse (h) of the triangle are algebraically defined by the indices (i, j) as follows:

$$u = i^2 + ij$$

$$e = j^2/2 + ij$$

$$h = i^2 + ij + j^2/2$$

2. Definitions

A rational right-angled triangle is one where all three the sides have lengths defined in the same unit by rational numbers.

A primitive right-angled triangle (primitive Pythagorean triple) is one where all three the sides have lengths defined in the same unit by positive integers which are relatively prime, *ie* do not have factors that are common to all three.

Similarly, i and j do not have any common factors.

All variables in the arguments that follow are positive integers.

3. Proof

As first part of the proof (3.1), it is necessary to show that all primitive right-angled triangles are comprised of an even numbered leg (e), an uneven numbered leg (u), and from there it follows that the hypotenuse (h) is an uneven number.

3.1.

There are only three possibilities: The two legs may both be even numbered (3.1.2), both be uneven numbered (3.1.3) or one leg is even and the other uneven numbered (3.1.4).

3.1.1.

Before these possibilities are considered, it is necessary to show the following:

3.1.1.1. An Even Number Squared is Even-

Let the even number

$$e = 2a$$

$$\therefore e^2 = 4a^2$$

$4a^2$ is an even number, implying that the square of all even numbers are even.

3.1.1.2. An Uneven Number Squared is Uneven-

Let the even number

$$u = 2a + 1$$

$$\therefore u^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1$$

The first term is an even number and the second term uneven. The sum of an uneven number and an even number is uneven, implying that the square of all uneven numbers are uneven.

3.1.2. Both Legs Are Even Numbered-

then the squares of both legs are respectively even, and the sum of two even numbers is even. Therefore, by Pythagorus' theorem, the square of the hypotenuse is even, which implies that the hypotenuse is an even number. If all three the sides of the triangle are even, then they are not relatively prime. Therefore a relatively prime right-angled triangle cannot have the two even legs.

3.1.3. Both legs are uneven numbered-

Any uneven number may be defined as $u = 2f + 1$. Let the two uneven numbers u_1 and u_2 be defined by Pythagorus' theorem as follows:

$$h^2 = u_1^2 + u_2^2$$

$$= (2f_1 + 1)^2 + (2f_2 + 1)^2$$

$$= 4f_1^2 + 4f_1 + 1 + 4f_2^2 + 4f_2 + 1$$

$$= 4(f_1^2 + f_1 + f_2^2 + f_2) + 2$$

h^2 is therefore an even number.
Therefore h is an even number.

Let $h = 2g$
then $h^2 = 4g^2$

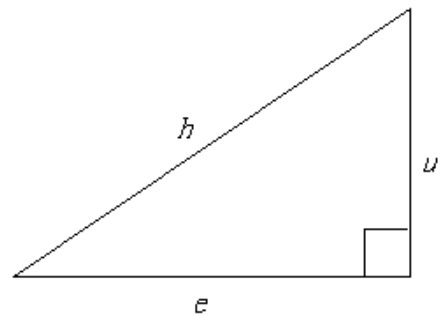
Therefore h^2 is a multiple of 4.
But $u_1^2 + u_2^2$ is not a multiple of 4.

Therefore a primitive right-angled triangle cannot have both the legs uneven numbered.

3.1.4. One Side is Even and the other Uneven Numbered-

By the process of elimination, all primitive right-angled triangles have one leg of even numbered length and the other of uneven numbered length.

Since the square of an even number is even and the square of an uneven number is uneven, by Pythagorus' theorem the square of the hypotenuse is uneven and thus the hypotenuse of a primitive right-angled triangle is always uneven. ie:



where h and u are uneven and e is even.

3.2.

In the next part of the proof, the focus is on the difference in lengths between the longer hypotenuse h , and the shorter legs e and u in primitive right-angled triangles. Emphasis is laid on the fact that the lengths of the sides of the triangle under discussion are relatively prime and even and uneven numbers are ascribed to the respective legs as in the figure above.

$$\text{Let } f = h - e \text{ and } g = h - u.$$

It will now be proven that f is a perfect square (3.2.1) and g is half (or two times) a perfect square (3.2.2).

It follows from the natures of h , e and u , that f is uneven and g is even.

3.2.1. By Pythagorus' Theorem:

$$h^2 - e^2 = u^2$$

$$\therefore (h - e)(h + e) = u^2 \quad (\text{difference of squares})$$

$$\therefore f(h + e) = u^2 \quad (f = h - e)$$

There are two possibilities for u with respect to factors: Either u has factors (3.2.1.1), or it is a prime number (3.2.1.2).

3.2.1.1.

Let us consider the possibility where u has factors:
Let $u = mn$.

Then, without loss of generality, either $f = m$ and $h + e = mn^2$, or $f = m^2$ (perfect square) and $h + e = n^2$. (The third possibility: $f = mn^2$ and $h + e = m$, is not viable since $h + e > f$.)

Let us consider the first possibility:

$$\begin{aligned} mn^2 &= h + e \\ &= e + m + e \quad (h = e + f \text{ and } f = m) \\ &= 2e + m \\ \therefore 2e &= mn^2 - m \\ &= m(n^2 - 1) \\ e &= m(n^2 - 1)/2 \quad (\text{the even side of the primitive} \\ &\quad \text{right angled-triangle}) \end{aligned}$$

Note that $(n^2 - 1)/2$ is a positive integer, since n is an uneven number, being a factor of the uneven number u .

$$\begin{aligned} h &= e + m \\ &= m(n^2 - 1)/2 + m \quad [e = m(n^2 - 1)/2] \\ &= m[(n^2 - 1)/2 + 1] \\ h &= m(n^2 + 1)/2 \quad (\text{the hypotenuse of the primitive} \\ &\quad \text{right-angled triangle}) \end{aligned}$$

Note that $(n^2 + 1)/2$ is a positive integer, since n is an uneven number, being a factor of the uneven number u .

By definition:

$$u = m \cdot n \quad (\text{the uneven side of the primitive right angled-triangle})$$

All three the sides of the supposed primitive right-angled triangle have m as a multiple (m may not be 1, since it is a factor of u), thus proving that the triangle is not primitive. This therefore proves that in the equation $f(h + e) = u^2$, f cannot be a factor of u without being a perfect square (the second possibility).

3.2.1.2.

Alternatively, u may not have factors, then either f or $h + e$ is 1, and the other is equal to u^2 . Quite clearly $h + e$ cannot be 1 since it is the sum of the hypotenuse and the even side of a primitive right-angled triangle. Therefore f must be 1, which is in any case a perfect square.

It has thus been proven that whether u is a prime number or a composite number, f is always a perfect square.

3.2.2.

Now let us consider g where $g = h - u$. It has already been stated that g is an even number.

By Pythagorus' theorem:

$$h^2 - u^2 = e^2$$

$$\therefore (h - u)(h + u) = e^2 \quad (\text{difference of squares})$$

$$\therefore g(h + u) = e^2 \quad (g = h - u)$$

Since g , e , and $(h + u)$ are even numbers, let $g = 2k$, $e = 2p$ and $h + u = 2q$, (Take note, for later use, that $q > k$.)

$$\text{then } 2k \cdot 2q = 4p^2$$

$$\text{and } k \cdot q = p^2$$

There are two possibilities for p with respect to factors: Either p has factors (3.2.2.1), or it is a prime number (3.2.2.2).

3.2.2.1.

Let us consider the possibility where p has factors:

Let $p = rs$.

Then, without loss of generality, either $k = r$ and $q = rs^2$, or $k = r^2$ (perfect square) and $q = s^2$. (The third possibility: $k = rs^2$ and $q = r$, is not viable since $q > k$.)

Let us consider the first possibility:

$$\begin{aligned} 2rs^2 &= h + u \\ &= u + 2r + u \quad (h = u + g, g = 2k \text{ and } k = r) \\ &= 2r + 2u \end{aligned}$$

$$\therefore 2u = 2rs^2 - 2r$$

$$u = r(s^2 - 1) \quad (\text{the uneven side of the primitive} \\ \text{right angled-triangle})$$

$$h = u + 2r \quad (h = u + g, g = 2k \text{ and } k = r)$$

$$= r(s^2 - 1) + 2r \quad [u = r(s^2 - 1)]$$

$$= r[(s^2 - 1) + 2]$$

$$h = r(s^2 + 1) \quad (\text{the hypotenuse of the primitive} \\ \text{right-angled triangle})$$

By definition: $e = 2p$

$$e = 2rs \quad (p = rs) \quad (\text{the even side of the} \\ \text{primitive right angled-triangle})$$

All three the sides of the supposed primitive right-angled triangle have r as a multiple, thus proving that the triangle is not primitive. It therefore proves that in the equation $kq = p^2$, k cannot be a factor of p without being a perfect square (which is the second possibility).

3.2.2.2.

Alternatively, p may not have factors, ie either k or q is 1, and the other is equal to p^2 . Quite clearly q cannot be 1 since it is half the sum of the hypotenuse and the uneven side of a primitive right-angled triangle. Therefore k must be 1, which is in any case a perfect square.

It has thus been proven that whether p is a prime number or a composite number, k is always a perfect square, and thus g is always two times a perfect square, and if you please, **always half of a perfect square**.

3.3.

Since f is an uneven perfect square, and g half of an even perfect square, let us define the uneven number i and the even number j such that $i^2 = f$, and $j^2/2 = g$.

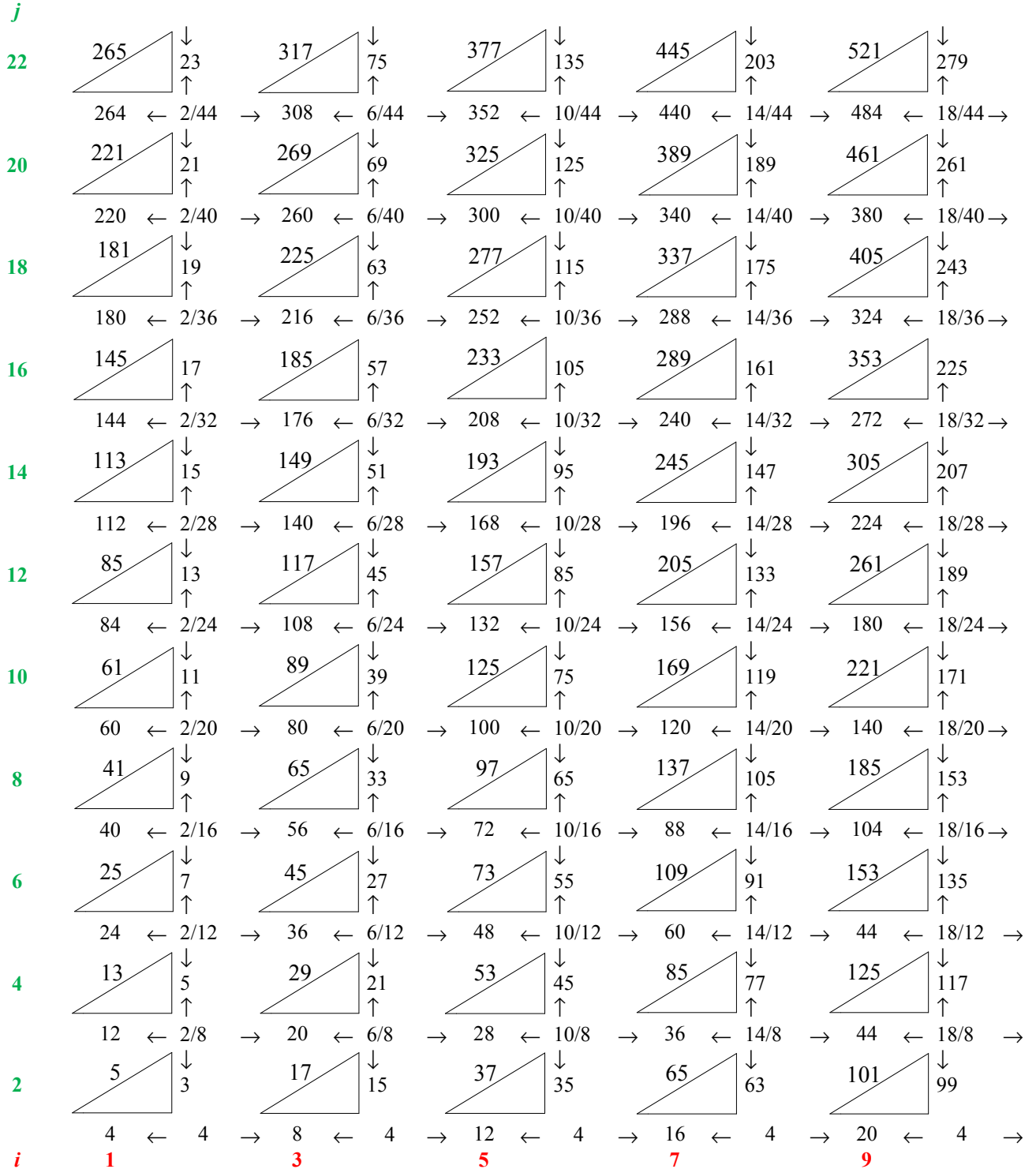
By Pythagorus:

$$h^2 = u^2 + e^2$$

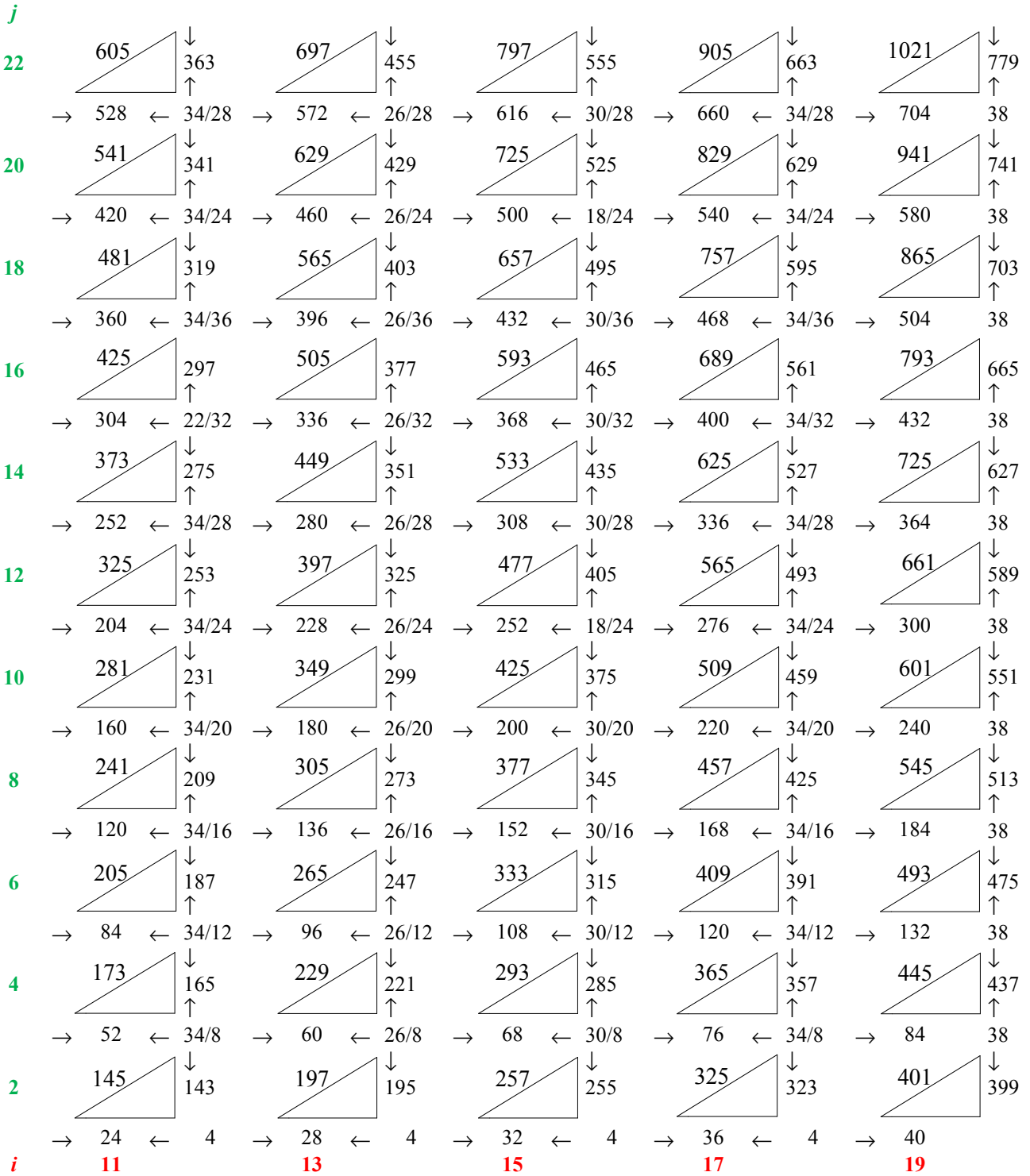
$$= (h - f)^2 + (h - g)^2 \quad (\text{since } u = h - f)$$

$$\begin{aligned}
 & \text{and } e = h - g & & = (h - i^2 - j^2/2)^2 - i^2j^2 & & (i^2 = f \text{ and } j^2/2 = g) \\
 & = h^2 - 2hf + f^2 + h^2 - 2hg + g^2 & & h - i^2 - j^2/2 = ij \\
 0 & = h^2 - 2h(f+g) + f^2 + g^2 & & \mathbf{h = i^2 + ij + j^2/2} \\
 & = h^2 - 2h(f+g) + (f+g)^2 - 2fg & & \text{(hypotenuse defined in terms of indices } i \text{ and } j) \\
 & = (h - f - g)^2 - 2fg
 \end{aligned}$$

Scheme: Matrix of Pythagorean triangles showing indices and the consistent differences in the legs



Scheme continued



$$\begin{aligned} \therefore u &= h - j^2/2 \\ &= i^2 + ij + j^2/2 - j^2/2 \\ u &= i^2 + ij \\ &\text{(uneven side defined in terms of indices } i \text{ and } j) \\ \therefore e &= h - i^2 \\ &\text{(since } i^2 = f) \end{aligned}$$

(since $j^2/2 = g$)

$$\begin{aligned} &= i^2 + ij + j^2/2 - i^2 \\ e &= j^2/2 + ij \\ &\text{(even side defined in terms of indices } i \text{ and } j) \end{aligned}$$

3.4.

Having proven that every primitive right-angled triangle is uniquely represented by a pair of indices (i, j), it remains

to be shown that if the indices are not relatively prime, the triangle is not primitive:

Let $i = vx$, and $j = vy$

$$\begin{aligned} e &= ij + j^2/2 \\ &= v^2xy + v^2y^2/2 \\ e &= v^2y(x + y/2) \end{aligned}$$

$$\begin{aligned} u &= ij + i^2 \\ &= v^2xy + v^2x^2 \\ u &= v^2x(y + x) \end{aligned}$$

$$\begin{aligned} h &= i^2 + ij + j^2/2 \\ &= v^2x^2 + v^2xy + v^2y^2/2 \\ h &= v^2(x^2 + xy + y^2/2) \end{aligned}$$

e , u and h have v^2 as a common factor which is the square of the factor v common to the indices. This thus proves that when the indices have a common factor, they represent a Pythagorean triple that is not primitive. When the indices and the Pythagorean triple have their common factors extracted, the simplified indices match the resulting primitive Pythagorean triple.

4. Conclusion

The theorem has been rigorously proven.

5. Applications and Corrolaries

Now with the three respective sides of right-angled triangles defined in terms of indices i and j , a matrix of right-angled triangles may be constructed for every uneven positive integer i and every even positive integer j (see above scheme). In this scheme, several observations can be made:

The triangles in a vertical column, having index i in common, have their uneven sides u increase by a factor of $2i$ in the triangle sequence going up the column. Likewise, the triangles in a horizontal row, having index j in common, have their even sides e increase by a factor of $2j$ in the triangle sequence from left to right.

As a corollary to this theorem, the sum of the even side e and the hypotenuse h of all primitive right-angled triangles is a perfect square. This may be deduced from the fact that $f(h + e) = u^2$ and f is proven to be a perfect square. Similarly the sum of the uneven side u and the hypotenuse h of all relatively prime right-angled triangles is half of a perfect square (and two times a perfect square), since $g(h + u) = e^2$ and g is proven to be half of (and twice) a perfect square.

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