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Series of primitive right-angled triangles

Martin W. Bredenkamp

Department of Science, Asia-Pacific International University; PO Box 4, MuakLek, Saraburi Province, 18180, Thailand

Email address:
martinb@apiu.edu (M. W. Bredenkamp)

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Abstract: From the infinite matrix of right-angled triangles, series of triangles are found that approach a right-angled triangle that has one irrational side such as the 45° triangle. This allows for the creation of a series of fractions that have as their limit an irrational number. Formulae for finding the next triangle in the triangle series, and thus the next fraction in the fraction series, are also developed. Such a series can be found for the square root of every uneven number that is not a perfect square, and for those of some of the even numbers as well.

Keywords: Pythagorean Triple Series, Non-Pythagorean Right-Angled Triangle Limit, Rational Series With Irrational Limits

1. Introduction

Having established defining indices that position all primitive right-angled triangles in an infinite two-dimensional matrix [1], which may be defined as the universal set, we have gone about finding subsets of right-angled triangles that are very similar in proportion. The obvious place to start is to find triangles that approximate the 45º triangle, and the 30/60º triangle that are the standard set squares. This quest has led to finding and defining series of primitive right-angled triangles that in proportion converge to the said triangles which have irrational proportions between all but one pair of sides. This also gives rise to series of fractions that converge to an irrational number.

2. Approximations to the 45º Triangle

In the previous paper [1] it has been established that all primitive right-angled triangles are comprised of an uneven-numbered leg (u), an even-numbered leg (e) and the hypotenuse (h) which in itself is uneven-numbered (all variables are positive integers). This eliminates the possibility of having the 45º triangle as part of the subset of approximating primitive right-angled triangles. This is well known in any case, since the hypotenuse is $\sqrt{2}$, which is an irrational number, disqualifying it from the set of primitive right-angled triangles.

Let us define the subset of triangles, and then set about finding them. Let us define them in terms of the indices (i,j) developed in the previous paper [1]. We define the subset of triangles (A) as follows:

$$A=\{(i,j)|\ u = i^2 + ij,\ \ e = j^2/2 + ij,\ \ h^2 = u^2+e^2,\ \text{ and } |e - u| = 1, \text{ where } (i + 1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

In the definition of the subset of triangles (A), an arbitrary value of 1 was chosen as the limit to the difference between the legs of the triangle. It does not take long to realise that such triangles exist, since the most well known primitive right-angled triangle, the 3/4/5-triangle [$(i,j) = (1,2)$], is one where 4 – 3 = 1. A closer look at the matrix of triangles shows that finding others is not such a daunting task, since for every i, $u_{j=2\text{-}\infty}$ is an arithmetic sequence with the interval being equal to $2i$. On the other hand, $e_{j=2\text{-}\infty}$ is a series with quadratic properties, since the interval in the series is defined by $[(j + 2)^2 - j^2]/2 + 2i$, which increases as j increases. Since for all cases except where $i = 1$, the series begins with e smaller than u, and since the intervals for e in a series are increasing quadratically, e will pass the value of u somewhere in the series. The task is therefore to find the place in each series where e passes u, enabling the easy identification of a case where $|e - u| = 1$.

The 3/4/5-triangle [$(i,j) = (1,2)$] belongs to the first series ($i = 1$). The second series, where $i = 3$, also has such a triangle: 21/20/29-triangle [$(i,j) = (3,4)$]. The third series ($i = 5$) does not have such a triangle: for $(i,j) = (5,6)$, e (48) is less than u (55), by 7; and for $(i,j) = (5,8)$, e (72) is greater than u (65) by 7. The fourth series ($i = 7$), in turn, has such a triangle again: 119/120/169-triangle [$(i,j) = (7,10)$]. The next series to have a qualifying triangle is the ninth ($i = 17$): 697/696/985-triangle [$(i,j) = (17,24)$]. We thus see that this subset of primitve right-angled triangles exists.

This subset of triangles may be ordered according to the numerical values of (i,j), forming a series of triangles. The size of the triangles increases in the series. Since the series is defined as having the legs u and e, differing by 1 only, the difference becomes less significant as the triangles increase in size. This means that the legs are more and more equal as the series progresses, and the triangles resemble the 45º triangle more and more. This also means that the ratio between h and u or e better and better approximates $\sqrt{2}$ as the series progresses.

As the series progresses, the interval from i to the next i, containing such a triangle also increases, making it more and more difficult to systematically and manually find the next member in the series. The question then arises: Is there a mathematical equation that defines this series of triangles, so that they may be determined algebraically? Having systematically found the first few triangles in the series, an investigation of (i,j) reveals an algebraic pattern:

For the series $(i,j)_n$, $(i,j)_{n+1} = (i_n + j_n,\ 2i_n + j_n)$ (1)

Table 1 contains the first fifteen triangles in this series, the latter part having been calculated by the above formula. Also included is $\sqrt{2}$, and the calculated approximation to $\sqrt{2}$, offered by each respective triangle. As the series progresses, the approximation improves, as indicated by the underlined digits that coincide with $\sqrt{2}$. The ratio of j/i is also included to show that it yields a series that also approximates a number, which happens to also be $\sqrt{2}$, but for another reason, which will come to light later in this paper.

***Table 1**. The series of triangles $(i,j)_n$ that have as their limit the 45º triangle.*

n	*i*	*j*	*u*	*e*	*h*	*h*/*e* ($\sqrt{2}$) **1.4142135623730950**	*j*/*i*
1	1	2	3	4	5	1.25	2
2	3	4	21	20	29	1.45	1.333333333333
3	7	10	119	120	169	1.4083333333333333	1.428571428571
4	17	24	697	696	985	1.4152298850574713	1.411764705882
5	41	58	4059	4060	5741	1.4140394088669951	1.414634146341
6	99	140	23661	23660	33461	1.4142434488588335	1.414141414141
7	239	338	137903	137904	195025	1.4142084348532312	1.414225941423
8	577	816	803761	803760	1136689	1.4142144421220265	1.414211438475
9	1393	1970	4684659	4684660	6625109	1.4142134114322064	1.414213926777
10	3363	4756	27304197	27304196	38613965	1.4142135882704622	1.414213499851
11	8119	11482	159140519	159140520	225058681	1.4142135579298095	1.414213573100
12	19601	27720	927538921	927538920	1311738121	1.4142135631354423	1.414213560533
13	47321	66922	5406093003	5406093004	7645370045	1.4142135622422969	1.414213562689
14	114243	161564	31509019101	31509019100	44560482149	1.4142135623955365	1.414213562319
15	275807	390050	183648021599	183648021600	259717522849	1.4142135623692447	1.414213562382

An interesting observation is that when n is uneven, u is less than e by 1, and when n is even, u is greater than e by 1. Another observation is that a similar series of rational numbers leading to $\sqrt{2}$ is h/u.

Having defined $(i,j)_{n+1}$ in terms of i_n and j_n, we can now define the sides of the triangle $(u,e,h)_{n+1}$ in terms of i_n and j_n, [(2), (6) and (8) respectively] and then in terms of u_n, e_n and h_n [(5), (7) and (9) respectively]. This calls for a good exercise in algebra, which sometimes includes factorisation:

$$u_{n+1} = i_{n+1}^2 + i_{n+1}j_{n+1} \qquad (u = i^2 + ij)$$

$$= (i_n + j_n)^2 + (i_n + j_n)(2i_n + j_n) \qquad [(i,j)_{n+1} = (i_n + j_n,\ 2i_n + j_n)] \qquad (1)$$

$$= i_n^2 + 2i_nj_n + j_n^2 + 2i_n^2 + 3i_nj_n + j_n^2$$

$$= 3i_n^2 + 5i_nj_n + 2j_n^2 \qquad (2)$$

However $h = i^2 + e$ by definition [1]

$$\therefore i = \sqrt{h - e} \qquad (3)$$

and $e = ij + j^2/2$ by definition (see also ref [1])

$\therefore 2e = 2j\sqrt{h-e} + j^2$ (3)

$\therefore 0 = j^2 + 2j\sqrt{h-e} - 2e$

$\therefore j = \frac{-2\sqrt{h-e}+\sqrt{4(h-e)+8e}}{2}$ roots of a quadratic equation

$= \frac{-2\sqrt{h-e}+2\sqrt{h-e+2e}}{2}$

$= \sqrt{h+e} - \sqrt{h-e}$ (4)

$\therefore u_{n+1} = 3\left(\sqrt{h_n - e_n}\right)^2 + 5\sqrt{h_n - e_n}\cdot\left(\sqrt{h_n+e_n} - \sqrt{h_n - e_n}\right) + 2\left(\sqrt{h_n+e_n} - \sqrt{h_n-e_n}\right)^2$

(2), (3) and (4)

$= 3h_n - 3e_n + 5\left(\sqrt{h_n^2 - e_n^2} - h_n + e_n\right) + 2\left(h_n + e_n - 2\sqrt{h_n^2 - e_n^2} + h_n - e_n\right)$

$= 3h_n - 3e_n + 5(u_n - h_n + e_n) + 2(2h_n - 2u_n)$ by Pythagorus

$= 2h_n + 2e_n + u_n$ (5)

$e_{n+1} = j_{n+1}^2/2 + i_{n+1}j_{n+1}$ ($e = j^2/2 + ij$)

$\therefore 2e_{n+1} = (2i_n + j_n)^2 + 2(i_n + j_n)(2i_n + j_n)$ [$(i,j)_{n+1} = (i_n + j_n, 2i_n + j_n)$] (1)

$= 4i_n^2 + 4i_nj_n + j_n^2 + 4i_n^2 + 6i_nj_n + 2j_n^2$

$= 8i_n^2 + 10i_nj_n + 3j_n^2$

$= (4i_n + 3j_n)(2i_n + j_n)$ (6)

$= \left[4\sqrt{h_n - e_n} + 3\left(\sqrt{h_n + e_n} - \sqrt{h_n - e_n}\right)\right]\left(2\sqrt{h_n - e_n} + \sqrt{h_n + e_n} - \sqrt{h_n - e_n}\right)$

(3) and (4)

$= \left(\sqrt{h_n - e_n} + 3\sqrt{h_n + e_n}\right)\left(\sqrt{h_n - e_n} + \sqrt{h_n + e_n}\right)$

$= h_n - e_n + 4u_n + 3h_n + 3e_n$ by Pythagorus and diff. of squares

$= 4h_n + 2e_n + 4u_n$

$\therefore e_{n+1} = 2h_n + e_n + 2u_n$ (7)

$h_{n+1} = i_{n+1}^2 + i_{n+1}j_{n+1} + j_{n+1}^2/2$ ($h = i^2 + ij + j^2/2$)

$= e_{n+1} + i_{n+1}^2$ ($e = j^2/2 + ij$)

$2h_{n+1} = 8i_n^2 + 10i_nj_n + 3j_n^2 + 2i_n^2 + 4i_nj_n + 2j_n^2$ [$(i,j)_{n+1} = (i_n + j_n, 2i_n + j_n)$] (1) and (6)

$= 10i_n^2 + 14i_nj_n + 5j_n^2$ (8)

$= 10\left(\sqrt{h_n - e_n}\right)^2 + 14\sqrt{h_n - e_n}\cdot\left(\sqrt{h_n + e_n} - \sqrt{h_n - e_n}\right) + 5\left(\sqrt{h_n + e_n} - \sqrt{h_n - e_n}\right)^2$

(3) and (4)

$= 10h_n - 10e_n + 14\left(\sqrt{h_n^2 - e_n^2} - h_n + e_n\right) + 5\left(h_n + e_n - 2\sqrt{h_n^2 - e_n^2} + h_n - e_n\right)$

$= 10h_n - 10e_n + 14(u_n - h_n + e_n) + 5(2h_n - 2u_n)$ by Pythagorus

$= 6h_n + 4e_n + 4u_n$

$\therefore h_{n+1} = 3h_n + 2e_n + 2u_n$ (9)

It takes more algebra to solve for h in terms of h only, etc. It is especially complicated with the even leg being greater than the uneven leg for the uneven positional-numbered members of the series, and the even leg being less than the uneven leg for the even positional-numbered members of the series.

$$h_{n+1} = 6h_n - h_{n-1} \quad (10)$$

For the next even positional-numbered member of the series ($n + 1$ as even):

$$e_{n+1} = 6e_n - e_{n-1} - 4 \quad (11)$$

$$\text{and } u_{n+1} = 6u_n - u_{n-1} + 4 \quad (12)$$

And for the next uneven positional-numbered members of the series ($n + 1$ as uneven):

$$e_{n+1} = 6e_n - e_{n-1} + 4 \quad (13)$$

$$\text{and } u_{n+1} = 6u_n - u_{n-1} - 4 \quad (14)$$

Since both the e and the u legs may be used for calculating $\sqrt{2}$, the legs may be dealt with as the greater number and the lesser number, irrespective of whether they are even or uneven. This simplifies the formulae to one each for each leg. Let the lesser leg be l, and the greater leg g, then:

$$g_{n+1} = 6g_n - g_{n-1} - 2 \quad (15)$$

$$\text{and } l_{n+1} = 6l_n - l_{n-1} + 2 \quad (16)$$

Now, the simple formulae for a series of rational numbers that have the square $\sqrt{2}$ as their limit are the following:

For the greater numbers: $\frac{h_{n+1}}{g_{n+1}}$ where $h_1 = 5$, $h_2 = 29$, $h_{n+1} = 6h_n - h_{n-1}$,

And $g_1 = 4$, $g_2 = 21$, $g_{n+1} = 6g_n - g_{n-1} - 2$. (17)

For the lesser numbers: $\frac{h_{n+1}}{l_{n+1}}$ where $h_1 = 5$, $h_2 = 29$, $h_{n+1} = 6h_n - h_{n-1}$,

and$l_1 = 3$, $l_2 = 20$, $l_{n+1} = 6l_n - l_{n-1} + 2$. (18)

3. Approximations to the 60º Triangle

The 60º triangle is one half of a bisected equilateral triangle with sides of 2 units. The hypotenuse, therefore, is 2 units long, and the short leg is 1 unit long. The leg of bisection is, by Pythagorus' theorem, the root of the difference of the squares of 2 and 1 *ie* $\sqrt{3}$. As with the 45º triangle, we will not only develop a series of triangles that approximate the 60º triangle more and more, but we will also develop a series of rational numbers that will have $\sqrt{3}$ as their limit.

We can define the subset of triangles (B), where the even leg (e) is the bisected leg, and the uneven leg (u) the short leg, as follows:

$$B=\{(i,j) \mid u = i^2 + ij,\ e = j^2/2 + ij,\ h^2 = u^2 + e^2,$$

$$\text{and } |h - 2u| = 1, \text{ where } (i + 1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

Let us call this series of triangles the ***e*-series of $\sqrt{3}$** since e represents the irrational leg in a system where the series are defined by their irrational legs. To qualify for this subset of triangles (B), a triangle must have its hypotenuse within 1 of two times the uneven leg. Again, the 3/4/5-triangle $[(i,j) = (1,2)]$ qualifies to be an element of B, where $|5 - 2(3)| = 1$. As before, for every series i, twice the uneven leg (u) starts off greater than h, but the latter increases quadratically, versus arithmetically, and a cross-over point is reached, where the presence of a qualifying triangle is sought for. The second series ($i = 3$) contains the second triangle: 33/56/65-triangle $[(i,j) = (3,8)]$; and the sixth series ($i = 11$) the third triangle: 451/780/901-triangle $[(i,j) = (11,30)]$, etc. The mathematical equation to define each member of this series is:

$$\text{For the series } (i,j)_n,\ (i,j)_{n+1} = (i_n+j_n,\ 2i_n+3j_n) \quad (19)$$

Table 2 contains the first eleven triangles in this series. Also included is $\sqrt{3}$, and the calculated approximation to $\sqrt{3}$ (e/u), offered by each respective triangle. Again the ratio of j/i is included to show that it yields a series that also approximates a number that will have meaning later in this section. Take note that the decimal portion of j/i is equal to the decimal portion of $\sqrt{3}$. The progressive underlined decimal part of each number shows to what extent it agrees with the decimal portion of $\sqrt{3}$.

Table 2. *The series of triangles $(i,j)_n$ that have as their limit the 60º triangle.*

	i	***j***	***u***	***e***	***h***	***e/u* ($\sqrt{3}$) 1.7320508075688773**	***j/i***
1	1	2	3	4	5	1.3333333333333333	2
2	3	8	33	56	65	1.6969696969696970	2.666666666667
3	11	30	451	780	901	1.7294900221729490	2.727272727273
4	41	112	6273	10864	12545	1.7318667304320102	2.731707317073
5	153	418	87363	151316	174725	1.7320375902842164	2.732026143791
6	571	1560	1216801	2107560	2433601	1.7320498586046527	2.732049036778
7	2131	5822	16947843	29354524	33895685	1.7320507394362811	2.732050680432
8	7953	21728	236052993	408855776	472105985	1.7320508026771768	2.732050798441
9	29681	81090	3287794051	5694626340	6575588101	1.7320508072176690	2.732050806914
10	110771	302632	45793063713	79315912984	91586127425	1.7320508075436616	2.732050807522
11	413403	1129438	637815097923	1104728155436	1275630195845	1.7320508075670668	2.732050807565

In this series of triangles, $2u$ is always 1 greater than h.

As before the sides of the triangles $(u,e,h)_{n+1}$ may be defined in terms of i_n and j_n, [(20), (22) and (24) respectively] and in terms of u_n, e_n and h_n [(21), (23) and (25) respectively]. The algebra leading up to the equations is not included:

$$u_{n+1} = 3i_n^2 + 7i_nj_n + 4j_n^2 \quad (20)$$

$$= 4h_n + 4e_n - u_n \quad (21)$$

$$2e_{n+1} = (2i_n + 3j_n)(4i_n + 5j_n) \quad (22)$$

$$\text{and } e_{n+1} = 8h_n + 7e_n - 4u_n \quad (23)$$

$$2h_{n+1} = 10i_n^2 + 26i_nj_n + 19j_n^2 \quad (24)$$

$$\text{and } h_{n+1} = 9h_n + 8e_n - 4u_n \quad (25)$$

Once again h may be solved in terms of h only, etc. In this series there is no alternation, so the formulae are simple:

$$h_{n+1} = 15h_n - 15h_{n-1} + h_{n-2} \quad (26)$$

$$e_{n+1} = 15e_n - 15e_{n-1} + e_{n-2} \quad (27)$$

$$\text{and } u_{n+1} = 15u_n - 15u_{n-1} + u_{n-2} \quad (28)$$

Therefore the simple formula for a series of rational numbers that has $\sqrt{3}$ as its limit is the following:

$$\frac{e_{n+1}}{u_{n+1}} \text{ where } e_1 = 4,\ e_2 = 56,\ e_3 = 780,\ e_{n+1} = 15e_n - 15e_{n-1} + e_{n-2},$$

$$\text{and } u_1 = 3,\ u_2 = 33,\ u_3 = 451,\ u_{n+1} = 15u_n - 15u_{n-1} + u_{n-2} \quad (29)$$

In the beginning of this section we arbitrarily chose the even leg (e) to represent the $\sqrt{3}$-leg of the 60º triangle forming the e-series of $\sqrt{3}$. Now, let the even leg (e) represent the short leg, and the uneven leg (u) represent the bisected leg, defining the subset of triangles (C) as follows:

$$C = \{(i,j) \mid u = i^2 + ij,\ e = j^2/2 + ij,\ h^2 = u^2 + e^2,$$

$$\text{and } |h - 2e| = 1, \text{ where } (i + 1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

This then is the ***u*-series of $\sqrt{3}$** where u represents the irrational leg. Here, the first qualifying triangle belongs to the second series (i = 3): the 15/8/17-triangle [(i,j) = (3,2)], where |17 – 2(8)| = 1. The second triangle belongs to the sixth series (i = 11): 209/120/241-triangle [(i,j) = (11,8)], etc. The mathematical equation to define each member of this series is:

$$\text{For the series } (i,j)_n,\ (i,j)_{n+1} = (3i_n+j_n,\ 2i_n+j_n) \quad (30)$$

Table 3 contains the first ten triangles in this series. As before, $\sqrt{3}$ is included, and the calculated approximation to $\sqrt{3}$ (u/e), offered by each respective triangle. The ratio of j/i is also included again.

Table 3. *The alternate series of triangles $(i,j)_n$ that have as their limit the 60º triangle.*

n	***i***	***j***	***u***	***e***	***h***	***u/e* ($\sqrt{3}$) 1.7320508075688773**	***j/i***
1	3	2	15	8	17	1.875	0.666666666667
2	11	8	209	120	241	1.7416666666666667	0.727272727273
3	41	30	2911	1680	3361	1.7327380952380952	0.731707317073
4	153	112	40545	23408	46817	1.7321001367053999	0.732026143791
5	571	418	564719	326040	652081	1.7320543491596123	0.732049036778
6	2131	1560	7865521	4541160	9082321	1.7320510618432295	0.732050680432
7	7953	5822	109552575	63250208	126500417	1.7320508258249522	0.732050798441
8	29681	21728	1525870529	880961760	1761923521	1.7320508088796045	0.732050806914
9	110771	81090	21252634831	12270214440	24540428881	1.7320508076629833	0.732050807522
10	413403	302632	296011017105	170902040408	341804080817	1.7320508075756338	0.732050807565

The product of the j/i ratios of the e-series and the u-series is 2, *viz*. 2.732050807565 × 0.732050807565 = 2. An uncanny property of these two ratios is that the decimal portions of the two ratios are identical, and the integer portion is 2 for the e-series, and 0 for the u-series. Furthermore, these decimal portions are identical with that of $\sqrt{3}$, which has the intermediate, 1, as the integer portion of the number. This phenomenon can be justified algebraically. If we break up the ratio of the e-series into its integer and decimal components, and set the decimal as an unknown, x, to solve where the product of the ratios must be equal to 2, we have the following:

$$(2 + x)x = 2$$

$$\therefore 2x + x^2 = 2$$

$$\therefore x^2 + 2x - 2 = 0$$

$$x = \frac{-2 \mp \sqrt{4+8}}{2} \text{ quadratic equation}$$

$$= -1 \mp \sqrt{3}$$

$$= 0.7320508075688773$$

(rejecting the negative root)

Interestingly the product of the two ratios of the 45° is also 2, where the legs are the same length and the two series are intertwined *ie* $\sqrt{2}$ squared is 2.

Another interesting phenomenon shows up: The developing decimal portions of the two series are also the same, out of phase by one position in the respective series (see Table 4). This indicates that the respective quotients in the ratios must be the same, and that the numerator of the *e*-series must be greater than that of the *u*-series by an interval of two times the denominator [eg. *e*-series (3,8), *u*-series (3,2)]. A further surprising phenomenon is that the *j*-indices of both series are identical, and the *i*-indices also except that the *u*-series begins at 3, causing it to be out of phase with the *i*-indices of the *e*-series. This all goes to show that the indices, even though they may be perceived to have been chosen arbitrarily [1], are meaningful and correct.

***Table 4**. A comparison of the indices and their ratios of the e- and u-series of $\sqrt{3}$*

***e*-series of $\sqrt{3}$**			***u*-series of $\sqrt{3}$**		
i	***j***	***j/i***	***i***	***j***	***j/i***
1	2	2	3	2	0.666666666667
3	8	2.666666666667	11	8	0.727272727273
11	30	2.727272727273	41	30	0.731707317073
41	112	2.731707317073	153	112	0.732026143791
153	418	2.732026143791	571	418	0.732049036778
571	1560	2.732049036778	2131	1560	0.732050680432
2131	5822	2.732050680432	7953	5822	0.732050798441
7953	21728	2.732050798441	29681	21728	0.732050806914
29681	81090	2.732050806914	110771	81090	0.732050807522
110771	302632	2.732050807522	413403	302632	0.732050807565
413403	1129438	2.732050807565			

In the triangles of the *u*-series of $\sqrt{3}$, $2e$ is also always 1 less than h.

The sides of the triangles $(u,e,h)_{n+1}$ may be defined in terms of i_n and j_n, [(31), (33) and (35) respectively] and in terms of u_n, e_n and h_n [(32), (34) and (36) respectively]:

$$u_{n+1} = (3i_n + j_n)(5i_n + j_n) \quad (31)$$

$$= 8h_n - 4e_n + 7u_n \quad (32)$$

$$2e_{n+1} = (8i_n + 3j_n)(2i_n + j_n) \quad (33)$$

$$\text{and } e_{n+1} = 4h_n - e_n + 4u_n \quad (34)$$

$$2h_{n+1} = 34i_n^2 + 26i_nj_n + 5j_n^2 \quad (35)$$

$$\text{and } h_{n+1} = 9h_n - 4e_n + 8u_n \quad (36)$$

h may be solved in terms of h only, etc:

$$h_{n+1} = 15h_n - 15h_{n-1} + h_{n-2} \quad (26)$$

$$e_{n+1} = 15e_n - 15e_{n-1} + e_{n-2} \quad (27)$$

$$\text{and } u_{n+1} = 15u_n - 15u_{n-1} + u_{n-2} \quad (28)$$

The general formulae for the three respective sides of the triangles are the same as for the *e*-series. The simple formulae are different, however, in that the members in the respective series differ. The second series of rational numbers that has $\sqrt{3}$ as its limit is therefore the following:

$\frac{u_{n+1}}{e_{n+1}}$ where $u_1 = 15$, $u_2 = 209$, $u_3 = 2911$,

$$u_{n+1} = 15u_n - 15u_{n-1} + u_{n-2},$$

and $e_1 = 8$, $e_2 = 120$, $e_3 = 1680$,

$$e_{n+1} = 15e_n - 15e_{n-1} + e_{n-2} \quad (37)$$

4. Series that Have as Their Respective Limits the Roots of All the Uneven Numbers that are Not Perfect Squares

We have shown that there are series of triangles that have as their limit the 45° triangle, and the 60° triangle. These triangles also produce series of rational numbers that have as their limit $\sqrt{2}$ and $\sqrt{3}$, respectively. Which other triangles are there that have two rational sides and one irrational leg? It turns out that the 60° triangle is the first in a series of irrational right-angled triangles where roots of all the uneven numbers form one leg, and the hypotenuses are consecutive numbers and the other leg, consecutive numbers that are one less than the hypotenuse in each triangle. Table 5 illustrates this with the first triangles of this infinite series.

***Table 5**. A series of triangles that represent the roots of all the uneven numbers.*

hypotenuse	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
rational leg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
irrational leg	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	3	$\sqrt{11}$	$\sqrt{13}$	$\sqrt{15}$	$\sqrt{17}$	$\sqrt{19}$	$\sqrt{21}$	$\sqrt{23}$	5	$\sqrt{27}$	$\sqrt{29}$	$\sqrt{31}$	$\sqrt{33}$

$\sqrt{9}$ and $\sqrt{25}$ are not irrational, and therefore are not a part of the argument, but they do form part of the picture in which the roots of all the uneven numbers are involved. The second triangle in this series forms the limiting triangle of a series of triangles, that will allow us to develop a series of rational numbers that have $\sqrt{5}$ as its limit. Here too, there is an *e*-series and a *u*-series.

The ***e*-series of $\sqrt{5}$** is defined by a subset of triangles (D) as follows:

$$D = \{(i,j) \mid u = i^2 + ij,\ e = j^2/2 + ij,\ h^2 = u^2 + e^2,$$

$$\text{and } |h - 3u/2| \le 1, \text{ where } (i+1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

The mathematical equation to define each member of this series is:

$$\text{For the series } (i,j)_n,\ (i,j)_{n+1} = (i_n+2j_n,\ 2i_n+3j_n) \qquad (38)$$

Table 6 contains the first ten triangles in this series. $\sqrt{5}$ is included, and the calculated approximation to $\sqrt{5}$ ($2e/u$), offered by each respective triangle. The difference between h and $3u/2$ is provided, to reveal the eligibility of the triangles for this series. As with $\sqrt{2}$, the difference is alternating, this time in halves.

Table 6. *The e-series of triangles $(i,j)_n$ that are used to define a rational series of numbers that have $\sqrt{5}$ as limit.*

n	*i*	*j*	*u*	*e*	*h*	3*u*/2	Diff	2*e*/*u* ($\sqrt{5}$) 2.2360679774997897
1	1	2	3	4	5	4.5	-0.5	2.6666666666666667
2	5	8	65	72	97	97.5	0.5	2.2153846153846154
3	21	34	1155	1292	1733	1732.5	-0.5	2.2372294372294372
4	89	144	20737	23184	31105	31105.5	0.5	2.2360032791628490
5	377	610	372099	416020	558149	558148.5	-0.5	2.2360715831001965
6	1597	2584	6677057	7465176	10015585	10015585.5	0.5	2.2360677765668318
7	6765	10946	119814915	133957148	179722373	179722372.5	-0.5	2.2360679886974005
8	28657	46368	2149991425	2403763488	3224987137	3224987137.5	0.5	2.2360679768757682
9	121393	196418	38580030723	43133785636	57870046085	57870046084.5	-0.5	2.2360679775345652
10	514229	832040	692290561601	774004377960	1038435842401	1038435842401.5	0.5	2.2360679774978517

For the last triangle in the table, the ratio of j/i is determined to be 1.618033988748.

The sides of the triangles $(u,e,h)_{n+1}$ may be defined in terms of i_n and j_n, [(39), (41) and (43) respectively] and in terms of u_n, e_n and h_n [(40), (42) and (44) respectively]:

$$u_{n+1} = (3i_n + 5j_n)(i_n + 2j_n) \qquad (39)$$

$$= 12h_n + 8e_n - 9u_n \qquad (40)$$

$$2e_{n+1} = 8i_n^2 + 26i_nj_n + 21j_n^2 \qquad (41)$$

$$\text{and } e_{n+1} = 12h_n + 9e_n - 8u_n \qquad (42)$$

$$2h_{n+1} = 10i_n^2 + 32i_nj_n + 35j_n^2 \qquad (43)$$

$$\text{and } h_{n+1} = 17h_n + 12e_n - 12u_n \qquad (44)$$

h may be solved in terms of h only, etc:

$$h_{n+1} = 17h_n + 17h_{n-1} - h_{n-2} \qquad (45)$$

$$e_{n+1} = 17e_n + 17e_{n-1} - e_{n-2} \qquad (46)$$

$$\text{and } u_{n+1} = 17u_n + 17u_{n-1} - u_{n-2} \qquad (47)$$

The first series of rational numbers that has $\sqrt{5}$ as its limit is therefore the following:

$$\frac{2e_{n+1}}{u_{n+1}} \text{ where } u_1 = 3,\ u_2 = 65,\ u_3 = 1155,$$

$$u_{n+1} = 17u_n + 17u_{n-1} - u_{n-2},$$

$$\text{and } e_1 = 4,\ e_2 = 72,\ e_3 = 1292,$$

$$e_{n+1} = 17e_n + 17e_{n-1} - e_{n-2} \qquad (48)$$

The ***u*-series of $\sqrt{5}$** is defined by a subset of triangles (E) as follows:

$$E = \{(i,j) \mid u = i^2 + ij,\ e = j^2/2 + ij,\ h^2 = u^2 + e^2,$$

$$\text{and } |h - 3e/2| \le 1, \text{ where } (i+1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

The mathematical equation to define each member of this series is:

$$\text{For the series } (i,j)_n,\ (i,j)_{n+1} = (3i_n + j_n,\ 4i_n + j_n) \qquad (49)$$

Table 7 contains the first nineteen triangles in this series. The $\sqrt{5}$ is included, and the calculated approximation to $\sqrt{5}$ ($2u/e$), offered by each respective triangle. The difference between h and $3e/2$ is provided, to reveal the eligibility of the triangles for this series. The numbers (n) of the triangles are designated *n*a and *n*b, since the algebra that relates these triangles relates every second triangle. The interval between the indices of *n*a and *n*b, is equal to the interval between *n*b and (n+1)a, whereas the interval between the indices in a series is always geometric. Viewed differently: we thus have two interspersed series of triangles in one. This phenomenon becomes fairly general beyond $\sqrt{5}$. The u-series of both $\sqrt{93}$ and $\sqrt{97}$ have thirty interspersed series, respectively. As with $\sqrt{2}$ and the e-series of $\sqrt{5}$, this interspersed pair has the difference alternating in sign.

For the last triangle in the table, the ratio of j/i is 1.236067977496. The product of the j/i ratios of the e-series and the u-series is 2 again, *viz.* 1.618033988748 × 1.236067977496 = 2. This phenomenon is universal, and can be proven as follows:

For an e-series and a u-series in root n, select triangles in the respective series that approximate the limiting triangle well, and are of similar size. Let subscript e represent the e-series, and likewise for the u-series.

Then $h_e = h_u$, $e_e = u_u$, and $u_e = e_u$

$h = i^2 + e$, and $h = j^2/2 + u$

By definition of i and j, ref [1]

$\therefore h_e = h_u = i_e^2 + e_e = j_u^2/2 + u_u$

$\therefore i_e^2 = j_u^2/2$ Since $e_e = u_u$

and $i_e\sqrt{2} = j_u$

Likewise $i_u\sqrt{2} = j_e$

Table 7. *The u-series of triangles $(i,j)_n$ that are used to define a rational series of numbers that have $\sqrt{5}$ as limit.*

n	*i*	*j*	*u*	*e*	*h*	3*e*/2	Diff	2*u*/*e* ($\sqrt{5}$) 2.2360679774997897
1a	1	2	3	4	5	6	-1	1.5
1b	3	4	21	20	29	30	-1	2.1
2a	5	6	55	48	73	72	1	2.2916666666666667
2b	13	16	377	336	505	504	1	2.2440476190476190
3a	21	26	987	884	1325	1326	-1	2.2330316742081448
3b	55	68	6765	6052	9077	9078	-1	2.2356245869134171
4a	89	110	17711	15840	23761	23760	1	2.2362373737373737
4b	233	288	121393	108576	162865	162864	1	2.2360926908340701
5a	377	466	317811	284260	426389	426390	-1	2.2360585379582073
5b	987	1220	2178309	1948340	2922509	2922510	-1	2.2360666002853711
6a	1597	1974	5702887	5100816	7651225	7651224	1	2.2360685035492360
6b	4181	5168	39088169	34961520	52442281	52442280	1	2.2360680542493576
7a	6765	8362	102334155	91530452	137295677	137295678	-1	2.2360679481840645
7b	17711	21892	701408733	627359044	941038565	941038566	-1	2.2360679732226830
8a	28657	35422	1836311903	1642447296	2463670945	2463670944	1	2.2360679791334991
8b	75025	92736	12586269025	11257501248	16886251873	16886251872	1	2.2360679777381447
9a	121393	150050	32951280099	29472520900	44208781349	44208781350	-1	2.2360679774087462
9b	317811	392836	225851433717	202007663444	303011495165	303011495166	-1	2.2360679774865066
10a	514229	635622	591286729879	528862928880	793294393321	793294393320	1	2.2360679775048634

Therefore the product of the j/i-ratios of the respective series yields the following:

$$j_e/i_e \times j_u/i_u = \frac{j_e j_u}{i_e i_u}$$

$$= \frac{i_u\sqrt{2}\, i_e\sqrt{2}}{i_e i_u}$$

$$= 2$$

The sides of the triangles $(u,e,h)_{n+1}$ may be defined in terms of i_n and j_n, [(50), (52) and (54) respectively] and in terms of u_n, e_n and h_n [(51), (53) and (55) respectively]:

$$u_{n+1} = (3i_n + j_n)(7i_n + 2j_n) \quad (50)$$

$$= 12h_n - 8e_n + 9u_n \quad (51)$$

$$2e_{n+1} = (10i_n + 3j_n)(4i_n + j_n) \quad (52)$$

$$\text{and } e_{n+1} = 12h_n - 9e_n + 8u_n \quad (53)$$

$$2h_{n+1} = 58i_n^2 + 34i_nj_n + 5j_n^2 \quad (54)$$

$$\text{and } h_{n+1} = 17h_n - 12e_n + 12u_n \quad (55)$$

h may be solved in terms of h only, etc, and the general equation is identical to that of the e-series:

$$h_{n+1} = 17h_n + 17h_{n-1} - h_{n-2} \quad (45)$$

$$e_{n+1} = 17e_n + 17e_{n-1} - e_{n-2} \quad (46)$$

$$\text{and } u_{n+1} = 17u_n + 17u_{n-1} - u_{n-2} \quad (47)$$

The second and third series (interspersed series) of rational numbers that have $\sqrt{5}$ as their respective limits are therefore the following:

$\frac{2u_{n+1}}{e_{n+1}}$ where $u_1 = 3, u_2 = 55, u_3 = 987,$
$u_{n+1} = 17u_n + 17u_{n-1} - u_{n-2},$

and $e_1 = 4,\ e_2 = 48,\ e_3 = 884,$

$$e_{n+1} = 17e_n + 17e_{n-1} - e_{n-2} \quad (56)$$

and

$\frac{2u_{n+1}}{e_{n+1}}$ where $u_1 = 21, u_2 = 377, u_3 = 6765,$
$u_{n+1} = 17u_n + 17u_{n-1} - u_{n-2},$

and $e_1 = 20, e_2 = 336, e_3 = 6052,$
$e_{n+1} = 17e_n + 17e_{n-1} - e_{n-2}$ (57)

An interesting phenomenon, but not unexpected, is the symmetry that there is between the e- and u-series with respect to $(u,e,h)_{n+1}$ in terms of $(u,e,h)_n$. Table 8 illustrates this phenomenon.

Table 8. *A comparison of $(u,e,h)_{n+1}$ in terms of $(u,e,h)_n$ of the e- and u-series of $\sqrt{3}$ and $\sqrt{5}$.*

	e-series	*u-series*
$\sqrt{3}$	$u_{n+1} = 4h_n + 4e_n - u_n$ (21)	$u_{n+1} = 8h_n - 4e_n + 7u_n$ (32)
	$e_{n+1} = 8h_n + 7e_n - 4u_n$ (23)	$e_{n+1} = 4h_n - e_n + 4u_n$ (34)
	$h_{n+1} = 9h_n + 8e_n - 4u_n$ (25)	$h_{n+1} = 9h_n - 4e_n + 8u_n$ (36)
$\sqrt{5}$	$u_{n+1} = 12h_n + 8e_n - 9u_n$ (40)	$u_{n+1} = 12h_n - 8e_n + 9u_n$ (51)
	$e_{n+1} = 12h_n + 9e_n - 8u_n$ (42)	$e_{n+1} = 12h_n - 9e_n + 8u_n$ (53)
	$h_{n+1} = 17h_n + 12e_n - 12u_n$ (44)	$h_{n+1} = 17h_n - 12e_n + 12u_n$ (55)

In a similar way series of triangles and therefore rational numbers may be found which have the square roots of all the uneven numbers, that are not perfect squares, as limits, respectively. In table 9 the indices of the first triangle in a series, and the formula for finding the indices for the following triangles in that series is provided for the next few series of triangles. Since there are several interspersed subseries per series, the first triangle of each subseries is provided. All subseries in one series have the same formula for finding the next triangle in that subseries.

Table 9. *The means to finding the series of several subsets of triangles that follow those discussed.*

	***e*-series**	***u*-series**
$\sqrt{7}$	$(i,j)_{n+1} = (5i_n+9j_n,\ 6i_n+11j_n)$	$(i,j)_{n+1} = (11i_n+3j_n,\ 18i_n+5j_n)$
	(1,2); (3,4); (5,6)	(1,2); (5,8); (11,18)
$\sqrt{11}$	$(i,j)_{n+1} = (7i_n+15j_n,\ 6i_n+13j_n)$	$(i,j)_{n+1} = (13i_n+3j_n,\ 30i_n+7j_n)$
	(3,2); (5,4); (7,6)	(1,2); (7,16); (13,30)
$\sqrt{13}$	$(i,j)_{n+1} = (13i_n+30j_n,\ 10i_n+23j_n)$	$(i,j)_{n+1} = (23i_n+5j_n,\ 60i_n+13j_n)$
	(3,2); (5,4); (13,10)	(1,2); (43,112); (3,8); (7,18); (13,34); (23,60)
$\sqrt{15}$	$(i,j)_{n+1} = (3i_n+7j_n,\ 2i_n+5j_n)$	$(i,j)_{n+1} = (5i_n+j_n,\ 14i_n+3j_n)$
	(3,2)	(1,2); (9,26); (3,8); (5,14)
$\sqrt{17}$	$(i,j)_{n+1} = (3i_n+8j_n,\ 2i_n+5j_n)$	$(i,j)_{n+1} = (5i_n+j_n,\ 16i_n+3j_n)$
	(3,2)	(1,2); (1,4); (3,10); (5,16)
$\sqrt{19}$	$(i,j)_{n+1} = (131i_n+351j_n,\ 78i_n+209j_n)$	$(i,j)_{n+1} = (209i_n+39j_n,\ 702i_n+131j_n)$
	(3,2); (7,4); (17,10); (27,16); (37,22); (47,28); (131,78); (393,234)	(1,2); (1,4); (3,10); (9,30); (25,84); (53,178); (131,440); (209,702)
$\sqrt{21}$	$(i,j)_{n+1} = (43i_n+120j_n,\ 24i_n+67j_n)$	$(i,j)_{n+1} = (67i_n+12j_n,\ 240i_n+43j_n)$
	(3,2); (7,4); (11,6); (25,14); (43,24); (129,72)	(1,2); (1,4); (3,10); (5,18); (9,32); (19,68); (43,154); (67,240)
$\sqrt{23}$	$(i,j)_{n+1} = (19i_n+55j_n,\ 10i_n+29j_n)$	$(i,j)_{n+1} = (29i_n+5j_n,\ 110i_n+19j_n)$
	(3,2); (7,4); (11,6); (15,8); (19,10); (57,30)	(1,2); (1,4); (3,12); (9,34); (19,72); (29,110)

Several observations may be made when studying the table above:

At the u-series of $\sqrt{13}$ and $\sqrt{15}$, the second subseries seems to be out of place since it has greater indices than all the rest. This is because the respective second subseries do not have a triangle in the first "cycle" of triangles, but starts in the second cycle. If the formula is applied to all the other subseries, replacing the given indices, the second set of indices will fall into place. Stated in another way: the second subseries begins at what should have been the second triangle in the series.

Another observation is that the *i*-index of one of the *e*-subseries is identical to the *i*-index of one of the *u*-subseries throughout the respective series. For example: the fifth subseries of the *e*-series of $\sqrt{21}$ [(43,24), (4729,2640), (520147,290376), ….] has the same *i*-index as the seventh subseries of the *u*-series of $\sqrt{21}$ [(43,154), (4729,16942), (520147,1863466), ….].

Then, as the root increases, going down the table, there are triangles that may be matched, or correlated. In the *e*-series, starting from $\sqrt{11}$, all in the table have triangle (3,2) as first in the first subseries. Triangle (5,4) occurs in both $\sqrt{11}$ and $\sqrt{13}$, etc.

Studying the formulae for propagating the subseries, show interesting systematic correlations. The i_n-coefficient of both *i* and *j* of every such formula in both the series of all the roots, correlate to a starting triangle of one of the subseries being described by the respective formulae. For example: in the *e*-series of $\sqrt{23}$, the respective i_n-coefficients of the formula is 19 and 10. There is a starting triangle (19,10).

Comparing the formulae of the *e*- and *u*-series of any root has the *ii*-coefficient of the *e*-series equal to the *jj*-coefficient of the *u*-series, and the *ji*-coefficient of the *e*-series equal to twice the *ij*-coefficient of the *u*-series. A similar pattern exists for the remaining coefficients.

Testing this technology on a primitive right-angled triangle as limit instead of one with one irrational leg, such as the 3/4/5-triangle, reveals that the *e*-series does not have triangles that fall within 1 when applying the formula: difference between 4*e*/3 and *h*. The difference diverges. The *u*-series produces a series of triangles where the difference between 4*u*/3 and *h* is 0, when the indices are multiples of (1,2), the indices of the 3/4/5-triangle. Therefore there is no convergence, since the series of triangles is the repetition of the 3/4/5-triangle.

A last observation with respect to this family of series of triangles is that when there are several subseries, and a table is created with all the interspersed subseries in ascending order, from a little way down the list of triangles, each triangle that follows may be described as an exact combination of previous triangles. Table 10 illustrates this using the *u*-series of $\sqrt{93}$ as an example. As mentioned earlier in the paper, this series has 30 interspersed subseries, designated 1a, …..1z, 1A, 1B, 1C, 1D. Thereafter the second member of each subseries follows: 2a, 2b etc. In the last column, the composition of the indices of that particular triangle is indicated by only the letter of triangles that occur closely above that triangle in the table. Numbers have algebraic meaning such as 2f + a, for g. This is true for 1g and 2g, etc. It is interesting to note that the composition is always comprised of an uneven number of components. This is because the *i*-indices are uneven, and to remain uneven, the number of components must be uneven.

Table 10. *The u-series of* $\sqrt{93}$ *illustrates the composition of indices from previous triangles.*

n	*i*	*j*	*u*	*e*	*h*	**47*e*/46**	**Diff**	**Comp**
1a	1	2	3	4	5	4.1	-0.91	
1b	1	4	5	12	13	12.3	-0.74	
1c	1	6	7	24	25	24.5	-0.48	
1d	1	8	9	40	41	40.9	-0.13	
1e	1	10	11	60	61	61.3	0.30	
1f	1	12	13	84	85	85.8	0.83	
1g	3	26	87	416	425	425.0	0.04	2f+a
1h	5	44	245	1188	1213	1213.8	0.83	g+e+d
1i	7	60	469	2220	2269	2268.3	-0.74	2g+d
1j	9	78	783	3744	3825	3825.4	0.39	3g
1k	13	112	1625	7728	7897	7896.0	-1	i+h+d
1l	19	164	3477	16564	16925	16924.1	-0.91	2i+h
1m	25	216	6025	28728	29353	29352.5	-0.48	2j+i
1n	31	268	9269	44220	45181	45181.3	0.30	k+2j
1o	59	510	33571	160140	163621	163621.3	0.30	n+l+j
1p	87	752	72993	348176	355745	355745.0	0.04	2n+m
1q	115	994	127535	608328	621553	621552.5	-0.48	o+n+m
1r	261	2256	656937	3133584	3201705	3201705.4	0.39	3p
1s	289	2498	805443	3841924	3925445	3925444.1	-0.91	q+2p

1t	463	4002	2067295	9860928	10075297	10075296.0	-1	r+q+p
1u	637	5506	3913091	18665340	19071109	19071108.3	-0.74	2r+q
1v	811	7010	6342831	30255160	30912881	30912880.9	-0.13	s+2r
1w	985	8514	9356515	44630388	45600613	45600613.8	0.83	t+2r
1x	1709	14772	28166029	134351340	137272021	137272021.3	0.30	v+u+r
1y	3331	28792	107001713	510395784	521491345	521491344.5	-0.48	x+2v
1z	4229	36554	172471307	822684324	840568765	840568765.8	0.83	2x+v
1A	5851	50574	330142675	1574773212	1609007413	1609007412.3	-0.74	y+x+v
1B	8371	72356	675765717	3223387444	3293461085	3293461084.1	-0.91	2y+x
1C	10891	94138	1143870839	5456238480	5574852361	5574852360.0	-1	z+2y
1D	13411	115920	1734458041	8273326320	8453181241	8453181240.0	-1	A+z+y
2a	15931	137702	2447527323	11674650964	11928447725	11928447724.1	-0.91	2A+z
2b	18451	159484	3283078685	15660212412	16000651813	16000651812.3	-0.74	B+A+z
2c	20971	181266	4241112127	20230010664	20669793505	20669793504.5	-0.48	B+2z
2d	23491	203048	5321627649	25384045720	25935872801	25935872800.9	-0.13	C+B+z
2e	26011	224830	6524625251	31122317580	31798889701	31798889701.3	0.30	2C+z
2f	28531	246612	7850104933	37444826244	38258844205	38258844205.8	0.83	D+C+z
2g	72993	630926	51381159567	245086990256	250414968305	250414968305.0	0.04	2f+a
2h	122495	1058804	144703221005	690231151188	705236176213	705236176213.8	0.83	g+e+d
2i	169477	1464900	276989310829	1321232862300	1349955315829	1349955315828.3	-0.74	2g+d
2j	218979	1892778	462430436103	2205782912304	2253734714745	2253734714745.4	0.39	3g
2k	315463	2726752	959706270545	4577777600928	4677294505297	4677294505296.0	-1	i+h+d
2l	461449	3988604	2053472506797	9795018261604	10007953441205	10007953441204.1	-0.91	2i+h
2m	607435	5250456	3558288019585	16972954844328	17341932123553	17341932123552.5	-0.48	2j+i
2n	753421	6512308	5474152808909	26111587349100	26679230552341	26679230552341.3	0.30	k+2j
2o	1433849	12393690	19826602967611	94572455920860	96628378875661	96628378875661.3	0.30	n+l+j
2p	2114277	18275072	43108731635673	205627692705536	210097859938265	210097859938265.0	0.04	2n+m
2q	2794705	24156454	75320538813095	359277297703128	367087673740153	367087673740152.0	-0.48	o+n+m

5. Other Families of Series of Triangles That Have Even and Uneven Roots

The family of series of triangles are not just limited to all the uneven roots created by a family where the hypotenuse and the rational leg differ by 1. Another family exists where the hypotenuse and the rational leg differ by 2, and then one with a difference of 3 etc. Table 11 shows the family where the hypotenuse and the rational leg differ by two. Every second member of the family, though, has all three the sides as multiples of two, and are thus identical to a triangle in the previous family, once having extracted 2 from all the sides. The triangles that offer new limits to a series of triangles, are those that have an uneven hypotenuse and rational leg. The irrational leg is a root of an even number, which is new, since the previous family had only roots of uneven numbers as limits. Not all even numbers are covered in this way, but every eighth number, beginning with $\sqrt{8}$, is new. When we get to $\sqrt{16}$, we have a perfect square, and this triangle (3/4/5) is of no use as a limit to a series of triangles.

Table 11. *A series of triangles that have the hypotenuse 2 greater than the rational leg.*

hypotenuse	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
rational leg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
irrational leg	$\sqrt{8}$	$\sqrt{12}$	4	$\sqrt{20}$	$\sqrt{24}$	$\sqrt{28}$	$\sqrt{32}$	6	$\sqrt{40}$	$\sqrt{44}$	$\sqrt{48}$	$\sqrt{52}$	$\sqrt{56}$	$\sqrt{60}$	8	$\sqrt{68}$

Something else that needs to be mentioned, is that at times the arbitrary limit of 1, set for qualifying a triangle into a series, is too small, excluding all possible triangles. If the limit is then raised to 5 or 10, series of triangles may be found. An example is the *e*-series of triangle $1/\sqrt{8}/3$. Three interspersed subseries are generated with a difference between $3u$ and h, of 4, 2 and -4.

There is another family of triangles that will complete the picture, and they are those where the hypotenuse is the irrational number. The $1/1/\sqrt{2}$, where we started this paper, is such a triangle. Table 12 shows the family of triangles, which act as limits to series, where the two rational legs differ by 1. These families differ from the previous, in that the difference between the numbers being square-rooted is not arithmetic, but geometric.

Table 12. *A series of triangles where the hypotenuse is irrational, and the legs differ by 1.*

hypotenuse	$\sqrt{5}$	$\sqrt{13}$	5	$\sqrt{41}$	$\sqrt{61}$	$\sqrt{85}$	$\sqrt{113}$	$\sqrt{145}$	$\sqrt{181}$	$\sqrt{221}$	$\sqrt{265}$	$\sqrt{313}$	$\sqrt{365}$	$\sqrt{421}$
rational leg 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
rational leg 2	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Other series exist here too where the two rational legs differ by 2, 3 etc. These offer alternate series to a root, compared to those we have dealt with already, where all the uneven numbers are accounted for. Some roots have several solutions. For example $\sqrt{65}$ has eight different series, four pairs (e- and u-series of): $\sqrt{65}$/32/33; $\sqrt{65}$/4/9; 7/4/$\sqrt{65}$ and 1/8/$\sqrt{65}$.

Table 13 shows all the roots under $\sqrt{100}$, that serve as a limit to a series of triangles and thus rational numbers in this way. Each number has a pair of series excepting $\sqrt{2}$. Where more than one pair is present, the number of pairs are indicated in parentheses. In total there are therefore 107½ pairs of series. The number of subseries may also be counted, but they are dependent on what the limit of the difference is set to. The limit is 1 most of the time, but at times 5 or 10.

Table 13. *The number of pairs of series that have limits of less than* $\sqrt{100}$.

2 (½)	13 (2)	26	35 (2)	47	57 (2)	69 (2)	79	89 (2)
3	15 (2)	27	37 (2)	48 (2)	58 (2)	71	80 (2)	91 (2)
5 (2)	17 (2)	29 (2)	39 (2)	50	59 (2)	72 (2)	82	93 (2)
7	19	31	40 (2)	51 (2)	61 (2)	73 (2)	83	95 (2)
8	21 (2)	32	41 (2)	53 (2)	63 (2)	74	85 (4)	96 (2)
10	23	33 (2)	43	55 (2)	65 (4)	75 (2)	87 (2)	97 (2)
11	24 (2)	34	45 (2)	56 (2)	67	77 (2)	88 (2)	99 (2)

6. Conclusion

The infinite two dimensional array of Pythagorean triples, developed in the previous paper [1], offers a system of subsets of triangles that approximate right-angled triangles with one side that is irrational. These subsets may be ordered into series, that better and better approximate the limiting triangle with one irrational side. Ratios of the sides of these series, produce series of rational numbers that approach an irrational limit. The way to find these series has been shown, and many other interesting asides with respect to these series of triangles, and thus rational numbers.

References

[1] MW Bredenkamp, Pure and Applied Mathematics Journal, 2013, **2**, 36-41.