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# Existence and uniqueness of mild solutions for fractional integrodifferential equations

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**Abstract:** The aim of this paper is to prove the existence and uniqueness of mild solution of a class of l nonlinear fractional integrodifferential equations $\begin{cases}\frac{d^q u(t)}{dt^q} + Au(t) = \int_0^t f(t,s,x(s))ds + \int_0^t a(t-s)g(s,y(s))ds, \quad t \in [0,T], \\ u(0) = u_{o.}\end{cases}$ in a Banach space $X$, where $0 < q < 1.$ Results are obtained by fixed point theorem. The results are established by using Krasnoselskii's fixed point theorem and the contraction mapping principle.

**Keywords:** Integrodifferential Equation, Fractional Equation, Mild Solution, Compact Semigroup, Krasnoselskii Theorem, Semi Group of Linear Operators

## 1. Introduction

An integrodifferential equation is one which involves both integrals and derivatives of an unknown function. It arises in many fields like electronic, fluid dynamics, biological models and chemical kinetics. A well known example is the equations of basic electric circuit analysis. In recent years, the theory of various integrodifferential equations in Banach spaces has been studied deeply due to their important values in sciences and technologies and many significant results have been established [ 3, 4, 5, 6,7, 12, 13, 22, 23, 24, 25, 26, 27] and references therein. On the other hand, many phenomena in Engineering, Physics, Economics, Chemistry, Aerodynamics and Electrodynamics of complex medium can be modeled by fractional differential equations. During the past decades, such problem attracted many researchers (see[3, 7, 8, 9, 10, 16, 17, 18, 19, 20, 28] and references therein).

However, among the previous researches on the fractional integrodifferential equations, few are concerned with mild solutions of the fractional integrodifferential equations as follows:

$$\begin{cases}\frac{d^q u(t)}{dt^q} + Au(t) = \int_0^t f(t,s,x(s))ds + \int_0^t a(t-s)g(s,y(s))ds, \\ \qquad\qquad t \in [0,T], \\ u(0) = u_{o.}\end{cases} \tag{1.1}$$

where $0 < q < 1$, and the fractional derivative is understood in the Caputo sense.

In this paper, motivated by [1-28] (especially the estimating approaches given in [11, 12, 15, 24, 26]), we investigate the existence and uniqueness of mild solution of the equation (1.1) in a Banach space $X$, $-A$ generates a compact semigroup $S(.)$ of uniformly bounded linear operators on a Banach space $X$.The function $a(.)$ is real valued and locally integrable on $[0,\infty)$, and the nonlinear maps $f: [0,T] \times [0,T] \times X \to X$ and $g: [0,T] \times X \to X$ are continuous functions. New existence and uniqueness results are given.

## 2. Preliminaries

In this paper, we set $I = [0,T]$, a compact interval in$\mathbb{R}$. We denote by $X$ a Banach space with norm $\|.\|$. Let $-A: D(A) \to X$ be the infinitesimal generator of compact semigroup $S(.)$of uniformly bounded linear operators. Then there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for $t \geq 0$. According to [23, 24], a mild solution of (1.1) can be defined as follows:

### 2.1. Definition

A continuous function $u: I \to X$ satisfying the equation

$$u(t) = Q(t)u_0 + \int_0^t R(t-s)[\int_0^s f(s,\tau,u(\tau))d\tau + K(u)(s)]ds \quad (2.1)$$

for $t \in I$ is called a mild solution of (1.1), where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma,$$

$$R(t) = q\int_0^\infty \sigma t^{q-1}\,\xi_q(\sigma)S(t^q\sigma)d\sigma,$$

$$K(u)(t) = \int_0^t a(t-s)g(s,u(s))ds,$$

and $\xi_q$ is a probability density function on $(0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\sigma x}\,\xi_q(\sigma)d\sigma = \sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+qj)}, 0 < q \le 1,\ \ x > 0.$$

### 2.2. Remark

Noting that $\int_0^\infty \sigma\,\xi_q(\sigma)d\sigma = 1$ in reference [24], we can see that $\|R(t)\| \le qMt^{q-1}, t > 0$. In this paper, we use $\|f\|_p$ to denote the $L^p$ norm of $f$ and $f \in L^p(0,T)$ for some $p$ with $1 \le p < \infty$. $C([0,T],X)$ denotes the Banach space of all continuous functions $[0,T] \to X$ endowed with the sup-norm given by

$$\|u\|_\infty := \sup_{t\in I}\|u\|$$

for $u \in C([0,T],X)$. Set $a_T := \int_0^T |a(s)|\,ds$.

The following well known theorem will be used later.

### 2.3. Theorem (Krasnoselskii)

Let $\Omega$ be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that

(i)$Ax + By \in \Omega$ whenever $x, y \in \Omega$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in \Omega$ such that $z = Az + Bz$.

## 3. Main Results

We will require the following assumptions:

$(H1)$ The function $f: [0,T]\times[0,T]\times X \to X$ is continuous and there exists $L > 0$ such that

$$\|f(t,s,u) - f(t,s,v)\| \le L\|u - v\|, u, v \in C([0,T],X).$$

$(H2)$ The function $g: [0,T]\times X \to X$ is continuous and there exists $L_1 > 0$ such that

$$\|g(t,u) - g(t,v)\| \le L_1\|u - v\|, u, v \in C([0,T],X).$$

$(H3)$ The function $L_q: I \to R^+, 0 < q < 1,$ satisfies

$$L_q(t) = MT^q(LT + L_1 a_T) \le \omega \le 1, t \in [0,T].$$

### 3.1. Theorem

Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ with $\|S(t)\| \le M,\ t \ge 0$. If the maps $f$ and $g$ satisfy(H1)and (H2), $L_q(t)$ satisfies (H2) and $L \le \gamma[MT^q(1 + a_T)]^{-1}, 0 < \gamma < 1,$ then (1.1) has a unique mild solution for every $u_0 \in X$.

*Proof:*Define the mapping $\mathcal{F}: C([0,T],X) \to C([0,T],X)$ by

$$(\mathcal{F}u)(t) = Q(t)u_0 + \int_0^t R(t-s)\left[\int_0^s f(s,\tau,u(\tau))d\tau + K(u)(s)\right]ds \quad (3.1)$$

Set

$$\sup_{t\in[0,T]}\|f(t,s,0)\| = M_1,$$

and

$$\sup_{t\in[0,T]}\|g(t,0)\| = M_2.$$

Choose $r$ such that

$$M\{\|u_0\| + T^q((Lr + M_1) + (L_1 r + M_2)a_T)\} \le r.$$

Let $B_r$ be the nonempty closed and convex set given by

$$B_r = \{u \in C([0,T],X)\,|\,\|u\|_\infty \le r\}.$$

Then for $u \in B_r$, we have

$$\|(\mathcal{F}u)(t)\|$$

$$\le \|Q(t)u_0\| + \int_0^t \|R(t-s)\|\,.\left\|\int_0^s f(s,\tau,u(\tau))d\tau + K(u)(s)\right\|ds$$

$$\le M\|u_0\| + qM\int_0^t (t-s)^{q-1}\left\|\int_0^s f(s,\tau,u(\tau))d\tau + K(u)(s)\right\|ds$$

$$\le M\|u_0\| + qM\int_0^t (t-s)^{q-1} \times \left\|\int_0^s [f(s,\tau,u(\tau)) - f(s,\tau,0) + f(s,\tau,0)]d\tau + K(u)(s)\right\|ds$$

$$\le M\|u_0\| + qM\int_0^t[(t-s)^{q-1} \times \int_0^s (\|f(s,\tau,u(\tau)) - f(s,\tau,0)\| + \|f(s,\tau,0)\|)d\tau]ds$$

$$+qM\int_0^t (t-s)^{q-1}\|K(u)(s)\|ds.$$

Noting that

$$\|K(u)(s)\| = \left\|\int_0^s a(s-\tau)g(\tau,u(\tau))d\tau\right\|$$

$$\leq \int_0^s |a(s-\tau)|[\|g(\tau,u(\tau))-g(\tau,0)\|+\|g(\tau,0)\|]\,d\tau \leq (L_1r+M_2)a_T, \qquad (3.2)$$

we obtain

$$\|(\mathcal{F}u)(t)\| \leq M\|u_0\| + MT^q[((Lr+M_1)T+(L_1r+M_2)a_T] \leq r,$$

for $t\in[0,T]$. Hence $\mathcal{F}: B_r \to B_r$.
Let $u$ and $v$ be two elements in $C([0,T],X)$. Then

$$\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|$$

$$\leq qM\int_0^t (t-s)^{q-1}\left[\int_0^s \|f(s,\tau,u(\tau))-f(s,\tau,v(\tau))\|d\tau + \|K(u)(s)-K(v)(s)\|\right]ds$$

$$\leq qM\int_0^t (t-s)^{q-1}\left[\int_0^s \|f(s,\tau,u(\tau))-f(s,\tau,v(\tau))\|d\tau + \int_0^s |a(s-\tau)|\,\|g(\tau,u(\tau)) - g(\tau,v(\tau))\|d\tau\right]ds$$

$$\leq MT^q(LT+L_1a_T)\|u-v\|$$
$$= L_q(t)\|u-v\|.$$

So

$$\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\infty \leq L_q(T)\|u-v\|_\infty.$$

The conclusion follows by contraction mapping principle. □

We assume the following,

$(H4)$ The function $f: I\times I\times X\to X$ is continuous, and there exists a positive function

$$\mu(.)\in L^p_{loc}(I,\mathbb{R}^+)(p>1/q>1)$$

such that $\|f(t,s,u(s))\|\leq\mu(t)$, the function $s\to\frac{\mu(s)}{(t-s)^{1-q}}$ belongs to $L^1([0,T],\mathbb{R}^+)$, and set $T_{p,q}:=max\{T^{q-1/p},T^q\}$.

Let $-A$ be the infinitesimal generator of a compact semigroup $S(.)$ of uniformly bounded linear operators. Then there exists a constant $M\geq 1$ such that $\|S(t)\|\leq M$, for $t\geq 0$.

### *3.2. Theorem*

If the maps $f$ and $g$ satisfy (H1),(H2),(H4) respectively, and $L_1\leq\lambda(M.T_{p,q}.a_T)^{-1}$, $0<\lambda<1$, Then(1.1) has a mild solution for every $u_0\in X$.

Proof: Define

$$(\phi u)(t) = \int_0^t R(t-s)\int_0^s f(s,\tau,u(\tau))d\tau ds$$

$$(\psi u)(t) = Q(t)u_0 + \int_0^t R(t-s)K(u)(s)ds$$

Choose $r$ such that

$$\left\{M\|u_0\| + MT_{p,q}[qM_{p,q}T\|\mu\|_{L^p_{loc}(I,\mathbb{R}^+)} + (L_1r+M_2)a_T]\right\} \leq r,$$

where $M_{p,q}:=(\frac{p-1}{pq-1})^{(p-1)/p}$.

Let $B_r=\{u\in C([0,T],X), |\|u\|_\infty\leq r\}$ be the closed convex and nonempty subset of the subspace $C([0,T],X)$.

Letting $u,v\in B_r$, we have

$$\|(\phi v)(t)+(\psi u)(t)\|$$

$$\leq \int_0^t \left\|R(t-s)\int_0^s f(s,\tau,u(\tau))d\tau\right\|ds + \|Q(t)u_0\| + \int_0^t \|R(t-s)K(u)(s)\|ds$$

$$\leq M\|u_0\| + qM\int_0^t (t-s)^{q-1}\int_0^s \|f(s,\tau,u(\tau))\|d\tau ds + qM\int_0^t (t-s)^{q-1}\int_0^s \|K(u)(s))\|ds.$$

According to Holder inequality, (H1), (H2) and (3.2), for $t\in I$, we have

$$\|(\phi v)(t)+(\psi u)(t)\|$$

$$\leq M\|u_0\| + qM\int_0^t (t-s)^{q-1}\int_0^s \|f(s,\tau,u(\tau))\|d\tau ds + qM\int_0^t (t-s)^{q-1}\int_0^s \|K(u)(s)\|\,ds$$

$$\leq M\|u_0\| + MT_{p,q}[qM_{p,q}T\|\mu\|_{L^p_{loc}(I,\mathbb{R}^+)} + (L_1r+M_2)a_T]$$

$$\leq r.$$

Thus, $(\phi v)+(\psi u)\in B_r$.

For $u,v\in C([0,T],X)$ and $t\in[0,T]$, using (H2), we obtain

$$\|(\psi u)(t)-(\psi v)(t)\|$$

$$\leq qM\int_0^t (t-s)^{q-1}\,\|K(u)(s)-K(v)(s)\|ds$$

$$\leq qM\int_0^t (t-s)^{q-1}\left\|\int_0^s a(s-\tau)[g(\tau,u(\tau)) - g(\tau,v(\tau))]d\tau\right\|ds$$

$$\leq MT^q.a_T.L_1\|u-v\|_\infty$$

$\leq \lambda \|u-v\|_\infty$.

So, we know that $\psi$ is a contraction mapping.
Set $u(t) = \{(\phi u)(t) | u \in B_r\}$.
Fix $t \in [0,T]$.For $0 < \varepsilon < t$, set

$$(\phi_\varepsilon u)(t) = \int_0^{t-\varepsilon} R(t-s) \int_0^s f(s,\tau,u(\tau)) d\tau ds$$

$$= qS(\varepsilon^q \sigma) \int_0^{t-\varepsilon} (t-s)^{q-1} \int_0^s f(s,\tau,u(\tau)) d\tau \int_0^\infty \sigma \xi_q(\sigma) \times S((t-s)^q)\,\sigma - \varepsilon^q \sigma) d\sigma ds.$$

Since $S(t)$ is compact for each $t \in [0,T]$, the sets $u_\varepsilon(t) = \{(\phi_\varepsilon u)(t) | u \in B_r\}$ are relatively compact in $X$ for each $\varepsilon$, $0 < \varepsilon < t$. Furthermore,

$$\|(\phi u)(t) - (\phi_\varepsilon u)(t)\|$$

$$\leq qM \int_{t-\varepsilon}^t (t-s)^{q-1} \left\| \int_0^s f(s,\tau,u(\tau)) d\tau \right\| ds$$

$$\leq qM.M_{p,q}.\|\mu\|_{L^p_{loc}(I,\mathbb{R}^+)}.T.\varepsilon^{q-1/p},$$

which implies that $u(t)$ is relatively compact in $X$.
Next, we prove that $(\phi u)(t)$ is equicontinuous.
For $0 < t_2 < t_1 < T$, we have

$$\|(\phi u)(t_1) - (\phi u)(t_2)\|$$

$$= \left\| \int_0^{t_1} R(t_1-s) \int_0^s f(s,\tau,u(\tau)) d\tau ds - \int_0^{t_2} R(t_2-s) \int_0^s f(s,\tau,u(\tau)) d\tau ds \right\|$$

$$= \left\| \int_0^{t_2} [R(t_1-s) - R(t_2-s)] \times \int_0^s f(s,\tau,u(\tau)) d\tau ds + \int_{t_2}^{t_1} R(t_1-s) \int_0^s f(s,\tau,u(\tau)) d\tau ds \right\|$$

$$\leq q \left\| \int_0^{t_2} \int_0^\infty [(t_1-s)^{q-1} - (t_2-s)^{q-1}] \xi_q(\sigma) \times S(\,(t_1-s)^q \sigma \times \int_0^s f(s,\tau,u(\tau)) d\tau d\sigma ds) \right\|$$

$$+ \int_{t_2}^{t_1} \|R(t_1-s)\| \int_o^s \left\| f(s,\tau,u(\tau)) d\tau \right\| ds$$

$$+q \left\| \int_0^{t_2} \int_0^\infty \sigma [(t_2-s)^{q-1}] \xi_q(\sigma) [S(\,(t_1-s)^q \sigma) - S((t_2-s)^q \sigma)] \times \int_0^s f(s,\tau,u(\tau)) d\tau d\sigma ds) \right\|$$

$$= I_1 + I_2 + I_3.$$

By (H4), we get

$$I_1 \leq qM \int_0^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}| \left\| \int_0^s f(s,\tau,u(\tau)) d\tau \right\| ds$$

$$\leq qM \int_0^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}|\, \mu(s).T.ds$$

In view of the assumption of $\mu(s)$, we see that $I_1 \to 0$ as $t_2 \to t_1$, and one

$$I_2 \leq qM \int_{t_2}^{t_1} (t_1-s)^{q-1} \left\| \int_0^s f(s,\tau,u(\tau)) d\tau \right\| ds$$

$$\leq qM \int_{t_2}^{t_1} (t_1-s)^{q-1} \mu(s).T.ds$$

Clearly, the last term tends to 0 as $t_2 \to t_1$. Hence, $I_2 \to 0$ as $t_2 \to t_1$, and

$$I_3 \leq q \left\| \int_0^{t_2} \int_0^\infty \sigma [(t_2-s)^{q-1}] \xi_q(\sigma) [S(\,(t_1-s)^q \sigma) - S((t_2-s)^q \sigma)] \times \int_0^s f(s,\tau,u(\tau)) d\tau d\sigma ds) \right\|$$

$$\leq q \int_0^{t_2} (t_2-s)^{q-1} \mu(s).T.ds \int_0^\infty \sigma \xi_q(\sigma)\ \|S(\,(t_1-s)^q \sigma) - S((t_2-s)^q \sigma)\|\ d\sigma ds \quad (3.2)$$

The right-hand side of (3.2) tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$ by the compactness of $S(t)$. So $I_3 \to 0$ as $t_2 \to t_1$.

Thus, $\|(\phi u)(t_1) - (\phi u)(t_2)\| \to 0$ as $t_2 \to t_1$,which is independent of $u$ .Therefore $\phi$ is compact by the Arzela-Ascoli theorem.

Next we show that $\phi$ is continuous.

Let $\{u_n\}$ be a sequence of $B_r$ such that $u_n \to u$ in $B_r$. By the continuity of $f$ on $I \times I \times X$, we have

$$f(t,s,u_n(s)) \to f(t,s,u), \quad n \to \infty.$$

Noting to continuity of $f$, we get

$$\|(\phi u_n)(t) - (\phi u)(t)\|$$

$$= \left\| \int_0^t R(t-s) \int_0^s [f(s,\tau,u_n(\tau)) - f(s,\tau,u(\tau))] d\tau ds \right\|$$

$$\leq q\,M \int_0^t (t-s)^{q-1} \int_0^s \|f(s,\tau,u_n(\tau)) - f(s,\tau,u(\tau))\|\, d\tau ds$$

$$\leq MT^q \|f(\cdot,\cdot,u_n(\cdot)) - f(\cdot,\cdot,u(\cdot))\|_\infty \to 0 \ as\ n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty} \|\phi u_n - \phi u\| = 0.$$

So $\phi$ is continuous.

By Krasnoskelskii's theorem, we have the conclusion of the theorem.

### *3.3. Remark*

In theorem 3.2, if we furthermore suppose that the hypothesis (H5)

$$\|f(t,s,u(t)) - f(t,s,v(t))\| \le L^1\|u - v\|, \quad L^1 > 0,$$

holds, then we can obtain the uniqueness of the mild solution in theorem 3.2.

Actually, from what we have just proved,(1.1) has a mild solution $u(t)$ and

$$u(t) = Q(t)u_0 + \int_0^t R(t-s)[\int_0^s f(s,\tau,u(\tau))d\tau + K(u)(s)]ds$$

Let $v(t)$ be another mild solution of(1.1). Then

$$\|u(t) - v(t)\| \le \int_0^t \|R(t-s)\|[\int_0^s \|f(s,\tau,u(\tau)) - f(s,\tau,v(\tau))\|d\tau$$

$$+\|K(u)(s) - K(v)(s)\|]ds$$

$$\le qM\int_0^t (t-s)^{q-1}(L_1 a_T + L^1 T)\|u(s) - v(s)\|ds,$$

which implies by Gronwall's inequality that (1.1) has a mild solution $u(t)$.

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