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Small gain theorem for distributed feedback control of Sturm-Liouville dynamics

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Abstract: This paper constructs the small-gain theorem upon a general class of Sturm-Liouville systems. It appears that the feedback connection of two Sturm-Liouville sub-systems is guaranteed of well-posedness, Hurwitz, dissipativity and passivity in L_2-spaces provided the loop gain is less than 1. To construct the theorem, spatiotemporal transfer-function and geometrical isomorphism between the space-time domain and the mode-frequency domain are developed, whereof the H_∞-norm is extended to be 2D-H_∞ norm in mode-frequency domain. On grounds of this small-gain theorem, robust performance of any Sturm-Liouville plant can be formulated as robust stability of a feedback connection, whereupon feedback syntheses can be performed via modal-spectral μ-loopshaping.

Keywords: Small Gain Theorem, Distributed Control, Robust Control, nD Transfer Function

1. Introduction

The small-gain theorem says that the feedback connection of two systems is guaranteed of dissipativity in L_2-spaces provided the product of the system gains is less than 1. Through Small Gain theorem (SGT), robust performance of any generalized plant that includes performance requirements and modelling uncertainties can be transformed into robust stability of a feedback-connection of nominal plant and uncertainty. Thereupon feedback syntheses are performed via frequency-domain loopshaping or state-space bounded-real inequalities, which is beyond the reach of Circle or Popov criterion. Therefore, SGT is usually regarded as the first law as to Robust Control.

Consider a feedback loop G interconnected by two systems: M and Δ that are minimum dynamics to realize their input-output operations. Dependent on what types of systems M and Δ are considered, there are four classes of SGT organized as follows:

(C1) In L_2-spaces, if the product of system gains $\|M\|\|\Delta\|<1$, then the feedback loop G is guaranteed of dissipativity for generic dynamics M and Δ [1]. With the assumption that the feedback-loop is well-posed, G is further guaranteed of Lyapunov passivity.

(C2) In Banach spaces, the loop gain $\|M\Delta\|<1$ guarantees the feedback loop G to be well-posed and dissipative, as both M and Δ are of linearity and infinite states. This can be derived from elementary functional analysis, say [2].

(C3) As both M and Δ are of linearity and finite states, the loop gain $\|M\Delta\|<1$ in L_2-spaces guarantees the feedback loop G to be well-posed, dissipative, passive and Hurwitz. This has been largely documented in control literature, say in [3-4].

(C4) As M and Δ are Sturm-Liouville dynamics in properly chosen spatiotemporal L_2-spaces, the loop gain $\|M\Delta\|<1$ guarantees the feedback loop G to be well-posed, dissipative, passive and Hurwitz in some sense to be defined. This paper is aimed to explore this class of SGT.

Civilization has been encountered a great quantity of Sturm-Liouville plants- heat transfers, acoustic waves, structure vibrations, quantum mechanics, electromagnetic waves, non-Fourier heat transfers [5-6], thermoacoustic vibrations [7], and so on. In a bounded region, Sturm-Liouville dynamics mostly behave as standing waves, wherein modes of larger (smaller) variations in space are with faster (slower) motions in time. Therefore, to track spatially subtle distribution under feedback control with pointed actuation will always be accompanied by temporally abrupt transience. Inasmuch as distributed sensors and actuators are getting well-developed nowadays [8-12],

distributed control is capable of remedying this situation. The small-gain theorem of Class C4 thus comes to play for robust syntheses of distributed control, on condition that the feedback controller is also belonging to the defined Sturm-Liouville dynamics.

In this paper, a general class of Sturm-Liouville dynamics is defined with Spatiotemporal Transfer Function (STF) obtained though the composite of modal decomposition and Laplace transform [5-7, 13-15]. The values of STF along imaginary axis thus define mode-frequency responses, whereon a 2D-H_∞ norm is created in mode-frequency domain that is employed to prove the small-gain theorem in Class C4. This newly developed mode-frequency domain makes possible to loop-shape feasible Sturm-Liouville controllers to guarantee dissipativity, well-posedness, passivity and exponential stability of the resulting closed-loop. Without the small-gain theorem in Class C4, robust control of Sturm-Liouville plants can merely be based on the small-gain theorem in Class C2 or C3, which will results in less vigorous robustness or performance, which is discussed in more detail in the next paragraph.

There were candidate approaches to Sturm-Liouville dynamics control. Some are based on order-truncated plants, for examples [16-20], which can employ the small-gain theorem in Class C3. Thereof, the spatiotemporally robust performance is not guaranteed at the stage of finite-dimensional syntheses. Another two types are involved in modelling with infinite-order transfer-functions [21-24] and identification with fraction-order transfer-functions [25-29], respectively, which can apply the small-gain theorem in Class C2 toward robust performance with merely well-posedness and dissipativity. Besides, they were not developed originally for distributed sensing and actuation. The other is nD state-space robust control [30-35] extended from 1D robust state-space synthesis to nD version in space-time domain. It can merely employ the small-gain theorem in Class C2 to guarantee well-posedness and dissipativity. For these reasons, this work show how the small-gain theorem in Class C4 leads to modal-spectral loopshaping of Sturm-Liouville controllers to guarantee Hurwitz, dissipativity, passivity and well-posedness, which is beyond the capacities of the above approaches.

Including this introduction section, this paper is organized into seven sections. Section 2 defines through spatiotemporal transfer-functions the fraction-order class of Sturm-Liouville dynamics. Section 3 develops mode-frequency responses. Therein the H_∞-norm is extended to be 2D-H_∞ norm in mode-frequency domain and geometrical isomorphism is created between space-time domain and mode-frequency domain. With such a 2D-H_∞ norm, Section 4 proves that Hurwitz, passivity and dissipativity are equivalent to one another for the defined class of Sturm-Liouville dynamics. On grounds of the above sections, Section 5 proves the small-gain theorem of Class C4. Section 6 applies this theorem into μ-loopshaping in mode-frequency domain. Section 7 concludes the present work.

2. Fraction-Order Sturm-Liouville Dynamics

Consider the following two types of Sturm-Liouville operator A operated in bounded regions Ω:

Type I: $\mathsf{A}\phi = -(1/\rho)\nabla\cdot(k\nabla\phi)$ in Ω,

$$\alpha\phi + \beta\nabla\phi\cdot\hat{n} = 0 \text{ on } \partial\Omega; \tag{1}$$

Type II: $\mathsf{A}\phi = (1/\rho)\nabla^2(k\nabla^2\phi)$ in Ω,

$$\alpha\phi + \beta\nabla\phi\cdot\hat{n} = 0 \wedge \alpha k\nabla^2\phi + \beta\nabla(k\nabla^2\phi)\cdot\hat{n} = 0 \text{ on } \partial\Omega, \tag{2}$$

where $\rho(x) > 0$, $k(x) > 0$, $\forall x \in \Omega$, and $\alpha(x)\cdot\beta(x) \geq 0$ for all $x \in \partial\Omega$. Here we are concerned with von-Neumann, Dirichlet and Robin types of boundary conditions. The spatial operator A above is positive-definite and self-adjoint under the inner-product:

$$\langle\psi,\phi\rangle_\Omega = \int_\Omega \rho(x)\phi^*(x)\psi(x)dV, \tag{3}$$

and its inverse is a compact operator in $L_2(\Omega)$. Thus, its eigenvalues are all positive, and its eigenfunctions can constitute an real, feasible, orthonormal, and complete basis of $L_2(\Omega)$. Here we denote by Λ the countable set of A's eigenvalues and by $\{\phi_\lambda\}_{\lambda\in\Lambda}$ the set of corresponding eigenfunctions, i.e.

$$\mathsf{A}\phi_\lambda = \lambda\phi_\lambda \text{ in } \Omega, \ \alpha\phi_\lambda + \beta\nabla\phi_\lambda\cdot\hat{n} = 0 \text{ on } \partial\Omega.$$

With respect to the eigenfunctions set $\{\phi_\lambda\}_{\lambda\in\Lambda}$, Galerkin transform G from spatial functions to modal functions, $F(\lambda) = \mathsf{G}[f(x)]$, is defined by

$$F(\lambda) \equiv \int_\Omega \rho(x)\phi_\lambda(x)f(x)dx.$$

Completeness and orthonormality of $\{\phi_\lambda\}_{\lambda\in\Lambda}$ jointly imply that the Galerkin transform G has a unique inverse G^{-1},

$$f(x) = \mathsf{G}^{-1}[F(\lambda)]:$$

$$f(x) \equiv \sum_{\lambda\in\Lambda} F(\lambda)\phi_\lambda(x).$$

Then, Laplace-Galerkin transform H from spatial-temporal functions to modal-complex functions is defined by the composite of Galerkin transform and Laplace transform:

$$\mathsf{H} = \mathsf{LG} = \mathsf{GL};$$

explicitly,

$$F(\lambda,s) \equiv \mathsf{H}[f(x,t)] = \int_{0^-}^\infty\int_\Omega e^{-st}\rho(x)\phi_\lambda(x)f(x,t)dxdt. \tag{4}$$

Here the domain of one-side Laplace transform, denoted by Γ, is an infinite line parallel to the imaginary axis, whereon the integral in (4) is converged. Accordingly, the inverse of Laplace-Galerkin transform H^{-1} is the composite of the inverse of Laplace transform and that of Galerkin transform, that is,

$$\mathsf{H}^{-1} = \mathsf{G}^{-1}\mathsf{L}^{-1} = \mathsf{L}^{-1}\mathsf{G}^{-1};$$

explicitly,

$$f(x,t) \equiv \mathsf{H}^{-1}[F(\lambda,s)] = \frac{1}{2\pi j}\sum_{\lambda\in\Lambda}\int_{\Gamma} F(\lambda,s)\phi_\lambda(x)e^{ts}ds\,. \quad (5)$$

There are two basic properties about Galerkin and Laplace transforms:

$$\mathsf{G}[\mathsf{A}f(x,t)] = \lambda\mathsf{G}[f(x,t)]$$

based on Green's identity, and

$$\mathsf{L}[\mathsf{D}_t f(x,t)] = s\mathsf{L}[f(x,t)],$$

where D_t denotes the temporal differentiation. Following these two properties, the Laplace-Galerkin transform H is of

$$\mathsf{H}[h(\mathsf{A},\mathsf{D}_t)f(x,t)] = h(\lambda,s)\cdot\mathsf{H}[f(x,t)], \quad (6)$$

where h is a ratio of two expressions of finite or some infinite length constructed from two independent variables, one standing for space and the other for time, allowing for the operations of addition, subtraction, multiplication, integer exponents in time, and fraction-order exponents in space. For example,

$$\mathsf{H}[\frac{\mathsf{D}_t - \mathsf{A}^{1/2}}{\mathsf{D}_t + \mathsf{A}^{1/2}}f(x,t)] = \frac{s-\sqrt{\lambda}}{s+\sqrt{\lambda}}F(\lambda,s)\,. \quad (7)$$

This work considers a general Sturm-Liouville dynamics $\hat{P}$ including the feedback controller that is to be synthesized:

$$M(\mathsf{A},\mathsf{D}_t)\psi = N(\mathsf{A},\mathsf{D}_t)q \quad \text{in} \quad \Omega\times[0,\infty), \quad (8)$$

$$\mathsf{B}(\mathsf{A}^{j-1}\psi) = 0,\ j = 1,2,\cdots \quad \text{on} \quad \Omega\times[0,\infty), \quad (9)$$

where the operator B indicates the Robin boundary condition defined in Type I and Type II Sturm-Liouville operators. Therein M and N are of temporal integral-order and allow for spatial fraction-order of A, but the spatiotemporal orders of N is not larger than that of M. Performing Laplace-Galerkin transform H on the Sturm-Liouville dynamics in (8)-(9) yields its spatiotemporal transfer function:

$$P(\lambda,s) = \frac{N(\lambda,s)}{M(\lambda,s)}\,. \quad (10)$$

For any proper dynamics $\hat{P}$ as described in (10), it is defined that the set $\{s : M(\lambda,s) = 0, \forall\lambda\in\Lambda\}$ contains all poles of P, and similarly to zeros. Note that unstable poles-zeros cancellation is not considered in this work. Besides, the impulse response of $\hat{P}$ is defined by $p(x,t) = \mathsf{H}^{-1}[P(\lambda,s)]$.

3. Geometrical Isomorphism

Define the Fourier-Galerkin transform R by the composite of Galerkin transform and Fourier transform, that is, $\mathsf{R} = \mathsf{FG} = \mathsf{GF}$; explicitly,

$$\begin{aligned}\hat{F}(\lambda,\omega) &\equiv \mathsf{R}[f(x,t)] \\ &= \int_{-\infty}^{\infty}\int_{\Omega}\rho(x)\varphi^*(\lambda,\omega,x,t)f(x,t)\,dxdt\end{aligned}. \quad (11)$$

where the kernel $\phi : \Lambda\times\Re\times\Omega\times\Re \to \mathrm{C}$ is

$$\varphi(\lambda,\omega,x,t) \equiv \frac{e^{j\omega t}}{\sqrt{2\pi}}\phi_\lambda(x)\,. \quad (12)$$

Accordingly, the inverse of Fourier-Galerkin transform $\mathsf{R}^{-1} = \mathsf{G}^{-1}\mathsf{F}^{-1} = \mathsf{F}^{-1}\mathsf{G}^{-1}$ is explicitly

$$f(x,t) \equiv \mathsf{R}^{-1}[\hat{F}(\lambda,\omega)] = \sum_{\lambda\in\Lambda}\int_{-\infty}^{\infty}\varphi(\lambda,\omega,x,t)\hat{F}(\lambda,\omega)\,d\omega\,. \quad (13)$$

Fourier-Galerkin transform for one-side temporally spatial functions can be calculated by Laplace-Galerkin transform in (4)-(5) with the infinite line Γ being the imaginary axis, provided that the underlying integral in (11) is converged. Consider a proper dynamics $\hat{P}$ with the spatiotemporal transfer function $P(\lambda,s)$, and denote its input by f and output by y, i.e. $P(\lambda,s) = Y(\lambda,s)/F(\lambda,s)$. Then,

$$\hat{Y}(\lambda,\omega) = P(\lambda,j\omega)\cdot\hat{F}(\lambda,\omega), \quad (14)$$

where

$$\hat{Y}(\lambda,\omega) = \mathsf{R}[y(x,t)] = 1/\sqrt{2\pi}\,Y(\lambda,j\omega),$$

$$\hat{F}(\lambda,\omega) = \mathsf{R}[f(x,t)] = 1/\sqrt{2\pi}\,F(\lambda,j\omega),$$

and $P(\lambda,j\omega)$ defines the mode-frequency response of the dynamics $\hat{P}$.

Since both Fourier transform and Galerkin transform are geometric isomorphic, i.e. one to one, linear and inner-product conserved, the pair of Fourier-Galerkin transforms (R, R^{-1}) is of geometric isomorphism between space-time domain and mode-frequency domain. Specifically it implies 2-norm conservation, that is,

$$\int_0^{\infty}\int_{\Omega}\rho(x)\left|f(x,t)\right|^2 dx\,dt = \sum_{\lambda\in\Lambda}\int_{-\infty}^{\infty}\left|\hat{F}(\lambda,\omega)\right|^2 d\omega\,, \quad (15)$$

or succinctly written by $\|f\| = \|\hat{F}\|$. Further to define the 2D-H_∞ norm of P by

$$\|P\|_\infty \equiv \max_{\lambda\in\Lambda,\,\omega\in\Re^+} |P(\lambda, j\omega)|.$$

Then, based on the 2-norm conservation in (15), the L_2-gain of the dynamics $\hat{P}$ equals the 2D-H_∞ norm of P, since

$$\|\hat{P}\| \equiv \sup_f \frac{\|y\|}{\|f\|} = \sup_{\hat{F}} \frac{\|\hat{Y}\|}{\|\hat{F}\|} \le \max_{\lambda\in\Lambda,\,\omega\in\Re^+} |P(\lambda, j\omega)| = \|P\|_\infty, \quad (16)$$

where the equality holds when $|\hat{F}|$ is a Dirac distribution in the mode-frequency domain at the argument of the maximum. With (16), $\|P\|_\infty < \gamma$ corresponds to the following L_2-gain performance in space-time domain:

$$\int_0^T \int_\Omega \rho(x)|y(x,t)|^2 dx\,dt < \gamma^2 \int_0^T \int_\Omega \rho(x)|f(x,t)|^2 dx\,dt, \forall T \in [0,\infty). \quad (17)$$

4. Spatiotemporal Stability

The legitimacy of 2D-H_∞ norm as a metric of robustness is justified by the proof that bounded 2D-H_∞ norm implies Hurwitz, passivity, and dissipativity.

Definition 1: *Hurwitz* of a proper dynamics $\hat{P}$ is defined by that all of its poles are in the left-half plane, which implies exponential decay. *Passivity* is defined by that its impulse response $p:\Omega\times\Re^+ \to \Re$ has $\lim_{t\to\infty} p(x,t) = 0$ for almost everywhere $x \in \Omega$ in the sense of $L_2(\Omega)$. *Dissipativity* is defined by bounded L_2-gain, i.e.

$$\sup_{\|f\|_{\Omega\times\Re^+}=1} \|\hat{P}f\|_{\Omega\times\Re^+} < \infty. \#$$

It follows to prove that Hurwitz, passivity, and dissipativity are equivalent to one another.

Lemma 1: Hurwitz and passivity are equivalent to each other.

Proof. For a proper dynamics $\hat{P}$, its impulse response p is known as

$$p(x,t) = \mathsf{G}^{-1}\mathsf{L}^{-1}[P(\lambda,s)] \equiv \mathsf{G}^{-1}[p_\lambda(t)] = \sum_{\lambda\in\Lambda} \phi_\lambda(x) p_\lambda(t) \quad (18)$$

By Definition 1, if $\hat{P}$ is not Hurwitz, then there exists at least one pole on the imaginary axis or in the right-half plane. This implies that some λ-mode response $p_\lambda(t)$ will not converge to zero as the time $t \to \infty$. Since $\{\phi_\lambda\}_{\lambda\in\Lambda}$ is a linearly independent set because of being orthogonal to one another, the impulse response $p(x,t)$ will not converge to zero for all $x \in \Omega$ as $t \to \infty$. That is, passivity implies Hurwitz.

Since $\{\phi_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal, complete basis of $L_2(\Omega)$, (18) implies

$$\|p\|_\Omega^2(t) = \sum_{\lambda\in\Lambda} |p_\lambda(t)|^2,$$

for any time t. If the dynamics $\hat{P}$ is Hurwitz, then there exist bounded values $c_\lambda > 0$, $a_\lambda < 0$, $t_\lambda \ge 0$ such that $\|p_\lambda(t)\|^2 \le c_\lambda e^{a_\lambda t}$ as $t \ge t_\lambda$ for all $\lambda \in \Lambda$, which is known from 1D stability. This furthers to imply

$$\|p\|_\Omega^2(t) \le c_0 \sum_{\lambda\in\Lambda} e^{a_\lambda t} \quad \text{for} \quad \forall t \ge t_0,$$

where $t_0 \equiv \max_{\lambda\in\Lambda} t_\lambda$ and $c_0 \equiv \max_{\lambda\in\Lambda} c_\lambda$. The set $\{a_\lambda : \lambda \in \Lambda\}$ can be chosen to have distinct members, therefore there exists $\varepsilon > 0$ small enough such that

$$\|p\|_\Omega^2(t) \le c_0 (e^{a_0 t} + e^{(a_0-\varepsilon)t} + e^{(a_0-2\varepsilon)t} + \cdots) \quad \text{for} \quad \forall t \ge t_0,$$

where $a_0 \equiv \max_{\lambda\in\Lambda} a_\lambda$. The above equation implies

$$\|p\|_\Omega^2(t) \le c_0 \frac{e^{a_0 t}}{1 - e^{-\varepsilon t}} \quad \text{for} \quad \forall t \ge t_0,$$

thereby $\|p\|_\Omega^2(t) \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} p(x,t) = 0$ for almost everywhere $x \in \Omega$ in the sense of $L_2(\Omega)$, so Hurwitz implies passivity. #

Lemma 2: Hurwitz and dissipativity are equivalent to each other.

Proof. As the signal $f(x,t) = \phi_\lambda(x) v(t)$ with a specific $\lambda \in \Lambda$ is input to the dynamics $\hat{P}$, it will output

$$y(x,t) = \phi_\lambda(x) \int_{0^-}^t p_\lambda(t-\tau) v(\tau) d\tau \equiv \phi_\lambda(x) y_\lambda(t).$$

Suppose some roots s of $M(\lambda,s) = 0$ are on imaginary axis or in right-half plane, then p_λ is not exponentially bounded. Known from 1D convolution, y_λ will be unbounded in the sense of $L_2(\Re^+)$ as v is bounded by $\|v\|_{\Re^+} = 1$. Since $\|y\|_{\Omega\times\Re^+} = \|y_\lambda\|_{\Re^+}$ in such a case, this furthers to imply that y is $L_2(\Omega\times\Re^+)$-unbounded as $\|f\|_{\Omega\times\Re^+} = \|v\|_{\Re^+} = 1$. Therefore, the dynamics $\hat{P}$ is not dissipative if it is not Hurwitz, that is, dissipativity implies Hurwitz. As of such, for any non-Hurwitz Sturm-Liouville dynamics, its 2D-H_∞ can be defined to be unbounded.

Suppose the dynamics $\hat{P}$ to be Hurwitz, that is, all poles of P are in the left-half plane. This implies that its 2D-H_∞ norm $\|P\|_\infty \equiv \max_{\lambda\in\Lambda,\,\omega\in\Re^+} |P(\lambda, j\omega)|$ is bounded, since there is no singularities $(\lambda,\omega) \in \Omega\times\Re^+$ in the denominator $M(\lambda, j\omega)$.

Since the L_2-gain of $\hat{P}$ equals the 2D-H_∞ norm of P, as shown in (16), the dynamics P is dissipative. That is, Hurwitz implies dissipativity. #

Based on Lemma 1 and Lemma 2, we conclude the above discussion with the following theorem:

Theorem 1: For a proper Sturm-Liouville dynamics, Hurwitz, passivity, dissipativity, and bounded 2D-H_∞-norm are equivalent to one another. #

For a finite-dimensional, matrix-valued dynamics $\hat{P}$ with all entries being proper Sturm-Liouville dynamics, the 2D-H_∞ norm of P is defined by

$$\|P\|_\infty \equiv \max_{\lambda\in\Lambda,\omega\in\Re^+} \bar{\sigma}(P(\lambda, j\omega)), \tag{19}$$

where $\bar{\sigma}$ denotes the singular value of the underlying matrix. As of such, the L_2-gain of $\hat{P}$ is equal to the 2D-H_∞ norm of P, since

$$\|P\| \equiv \sup_f \frac{\|y\|_{\Omega\times\Re^+}}{\|f\|_{\Omega\times\Re^+}} = \sup_{\hat{F}} \frac{\|\hat{Y}\|_{\Lambda\times\Re}}{\|\hat{F}\|_{\Lambda\times\Re}} = \max_{\lambda\in\Lambda,\omega\in\Re^+} \bar{\sigma}(P(\lambda, j\omega)) \equiv \|P\|_\infty . \tag{20}$$

Definition 2: Any finite-dimensionally matrix-valued, proper Sturm-Liouville dynamics is defined to be spatiotemporally stable if all of its entries are Hurwitz, passive or dissipative. #

Due to Pythagorean Proposition, a Sturm-Liouville dynamics matrix has bounded L_2-gain if and only if all of its entries are dissipative. Therefore, Theorem 1 still holds as Sturm-Liouville dynamics expands on being finite-dimensionally matrix-valued.

5. Small Gain Theorem

Suppose a nominal dynamics M is feedback-connected by an unstructured uncertainty Δ bounded by 2D-H_∞ norm, $\|\Delta\|_\infty \le \gamma^{-1}$, as shown in Figure 1, then the closed-loop dynamics is guaranteed to be well-posed and spatiotemporally stable if and only if $\|M\|_\infty < \gamma$. This is an equivalent statement of the small-gain theorem of Class C4, and proved here.

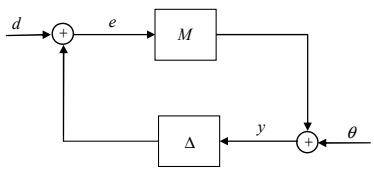

Figure 1. *For Spatiotemporal Small Gain theorem*

Theorem 2: Consider the closed-loop dynamics feedback-connected by any two proper Sturm-Liouville dynamics $\hat{M}$ and $\hat{\Delta}$ with compatible matrix dimensions. If $\|M\|_\infty < \gamma$ and $\|\Delta\|_\infty \le \gamma^{-1}$, the closed loop must be of well-posedness and spatiotemporal stability.

Proof. Define the loop gain A by $A = M\Delta$ or $A = \Delta M$. Firstly, we prove that if $\|A\|_\infty < 1$, then firstly the inverse of $I - A$ is uniquely existent and has bounded 2D-H_∞ norm.

For any function $f \in L_2(\Lambda\times\Re)$, the sequence $((I + A + \cdots + A^n)f)_{n=1}^\infty$ is Cauchy in $L_2(\Lambda\times\Re)$, for if $m > n$,

$$\begin{aligned}
&\left\|(I + A + \cdots + A^m)f - (I + A + \cdots + A^n)f\right\|_2 \\
&= \left\|A^{n+1}f + \cdots + A^m f\right\|_2 \le \left\|A^{n+1}\right\|_\infty \|f\|_2 + \cdots + \left\|A^m\right\|_\infty \|f\|_2 \\
&\le \|f\|_2 \{\|A\|_\infty^{n+1} + \cdots + \|A\|_\infty^m\} \le \|f\|_2 \sum_{r=n+1}^{\infty} \|A\|_\infty^r \\
&= \|f\|_2 \frac{\|A\|_\infty^{n+1}}{1 - \|A\|_\infty}.
\end{aligned}$$

Since $\|A\|_\infty^{n+1} \to 0$ as $n \to \infty$, the left-hand side tends to zero as $m, n \to \infty$ with $m > n$. Since $L_2(\Lambda\times\Re)$ is a Banach space, the Cauchy sequence $((I + A + \cdots + A^n)f)_n$ converges to a limit point $Tf \in L_2(\Lambda\times\Re)$. On letting $m \to \infty$ in the above inequality we find

$$\left\|Tf - (I + A + \cdots + A^n)f\right\|_2 \le \|f\|_2 \frac{\|A\|_\infty^{n+1}}{1 - \|A\|_\infty}.$$

This shows that $\left\|T - (I + A + \cdots + A^n)\right\|_\infty$ is bounded, and hence that $\|T\|_\infty$ is. Furthermore,

$$\left\|T - (I + A + \cdots + A^n)\right\|_\infty \le \frac{\|A\|_\infty^{n+1}}{1 - \|A\|_\infty},$$

which implies that

$$I + A + \cdots + A^n \to T \quad \text{as} \quad n \to \infty.$$

It remains to show that $T = (I - A)^{-1}$. For any $f \in L_2(\Lambda\times\Re)$, bounded 2D-$H_\infty$ norm of $I - A$ gives

$$\begin{aligned}
(I - A)Tf &= (I - A)\lim_{n\to\infty}(I + A + \cdots + A^n)f \\
&= \lim_{n\to\infty}(I - A)(I + A + \cdots + A^n)f \\
&= \lim_{n\to\infty} f - A^{n+1}f.
\end{aligned}$$

We have

$$\left\|A^{n+1}f\right\|_2 \le \left\|A^{n+1}\right\|_\infty \|f\|_2 \le \|A\|_\infty^{n+1} \|f\|_2,$$

and thus $(I - A)Tf = f$, since $A^{n+1}f \to 0$ as $n \to \infty$. Similarly, $T(I - A)f = f$, and then $T = (I - A)^{-1}$,

measured by the 2D- H_∞ norm.

Secondly, the internal behavior of the closed-loop of Figure 1 is governed by

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} (I-\Delta M)^{-1} & \Delta(I-M\Delta)^{-1} \\ M(I-\Delta M)^{-1} & (I-M\Delta)^{-1} \end{bmatrix} \begin{bmatrix} d \\ \theta \end{bmatrix}, \tag{21}$$

On grounds of the above deduction, $\|\Delta M\|_\infty \le \|\Delta\|_\infty \cdot \|M\|_\infty < 1$ implies that $(I-\Delta M)^{-1}$ is uniquely existent and 2D- H_∞ norm -bounded, and similarly to $(I-M\Delta)^{-1}$. Therefore, all entries of the 2×2 block matrix in (21) are uniquely existent and 2D- H_∞ norm -bounded. Further to note that the feedback connection of two Sturm-Liouville dynamics is also belonging to Sturm-Liouville dynamics. Based on Theorem 1, the closed-loop dynamics is of well-posedness and spatiotemporally stable. Moreover, there is no any structure constraint on the uncertainty Δ, so $\|\Delta M\|_\infty < 1$ or $\|M\Delta\|_\infty < 1$ for all $\|\Delta\|_\infty \le \gamma^{-1}$ implies $\|M\|_\infty < \gamma$. #

6. Application to Loop Shaping

Accommodating some extent of model uncertainties, the mode-frequency responses can be curve-fitted in the mode-Bode plot as a dense set:

$$\{G_0(1+W_1\Delta_1) : \|\Delta_1\|_\infty \le 1\},$$

where G_0 stands for a nominal plant. Therein the set $\{W_1\Delta_1 : \|\Delta_1\|_\infty \le 1\}$ contains model uncertainties, wherein the robustness weighting W_1 envelops all multiplicative perturbations $W_1\Delta_1$ in the modal-Bode magnitude plot. As shown in Figure 2a, the Sturm-Liouville controller K, pre- or post- composite to the plant, together with a negative unit feedback -1 executes the feedback compensation. The controller K is to guarantee specified performance for all uncertain plants under consideration, i.e. robust performance. Here the performance is specified by L_2 -gain as

$$\int_0^T \int_\Omega \rho(x)|\hat{z}_2(x,t)|^2 dx\,dt < \gamma^2 \int_0^T \int_\Omega \rho(x)|\hat{w}_2(x,t)|^2 dx\,dt \quad \forall T \in [0,\infty) \tag{22}$$

where the exogenous disturbances w_2 comprises the slow-time reference command and sensor noises of higher frequencies. Therein, the tracking error e has been weighted by the performance weighting W_2, i.e. $z_2 = W_2 e$.

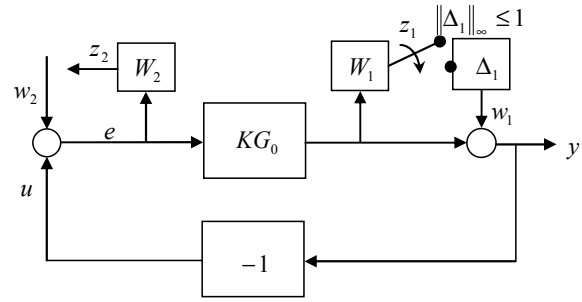

Figure 2a. *Construction of generalized plant*

Tracing the path of signal flow in Figure 2a, one can transform the interconnection of blocks in Figure 2a into the feedback-interconnection of three blocks:

Generalized plant:

$$P: \begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & W_1KG_0 & W_1KG_0 \\ 0 & W_2 & W_2 \\ 1 & KG_0 & KG_0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}; \tag{23}$$

External feedback:

$$u = -y; \tag{24}$$

Internal feedback 1:

$$w_1 = \Delta_1 z_1,\ \|\Delta_1\|_\infty \le 1. \tag{25}$$

Therein the internal disturbance w_1 and stability variable z_1 are induced from modelling uncertainty Δ_1. Moreover, if we replace the L_2 -gain performance in (22) by such an internal feedback as

Internal feedback 2:

$$w_2 = \Delta_2 z_2,\ \|\Delta_2\|_\infty \le 1, \tag{26}$$

then the robust performance of the original setting in Figure 2a is equivalent to the robust stability of the closed-loop system in Figure 2b. This equivalency is inferred from the small-gain theorem of Theorem 2. Substituting (24) into (23) yields the lower fractional transformation of Figure 2b, which is shown in Figure 2c. It becomes the feedback-interconnection of two blocks:

$$\mathrm{M} = \begin{bmatrix} -\mathrm{W_1T} & \mathrm{W_1T} \\ -\mathrm{W_2S} & \mathrm{W_2S} \end{bmatrix} \text{and}\ \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}; \quad \mathrm{z} = \begin{bmatrix} \mathrm{z_1} \\ \mathrm{z_2} \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} \mathrm{w_1} \\ \mathrm{w_2} \end{bmatrix} \tag{27}$$

where S is the sensitivity function $1/(1+KG_0)$, and T is its complementary function, i.e. $S+T=1$. The first requirement of the controller K is to make the sensitivity function S spatiotemporally stable.

If the Δ -structured singular value μ_Δ is defined by

$$\mu_\Delta(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0\}}, \tag{28}$$

then based on Theorem 2 assisted with Theorem 1 and Nyquist criterion, the spatiotemporally robust stability of Figure 2c is guaranteed, so is spatiotemporally robust performance of Figure 2a, if and only if

$$\mu_\Delta(M(\lambda, j\omega)) \le 1, \text{ for all } (\lambda, \omega) \in \Lambda \times \Re^+.$$

Substituting (27) into (28) yields

$$\mu_\Delta(M) = \frac{1}{\min\{\max(|\Delta_1|, |\Delta_2|) : 1 + W_1 T \Delta_1 - W_2 S \Delta_2 = 0\}},$$

which, after careful calculation, is explicitly to be

$$\mu_\Delta(M) = |W_1 T| + |W_2 S| \tag{29}$$

Therefore, the set of sensitivity functions with spatiotemporally robust performance in the original setting of Figure 2a is

$$\left\{ \begin{array}{l} S : |W_1(\lambda, j\omega)(1 - S(\lambda, j\omega))| \\ + |W_2(\lambda, j\omega) S(\lambda, j\omega)| < 1, \forall (\lambda, \omega) \in \Lambda \times \Re^+ \end{array} \right\} \tag{30}$$

Finally, we know that any feasible loop S is a spatiotemporally stable transfer function and a member of (30).

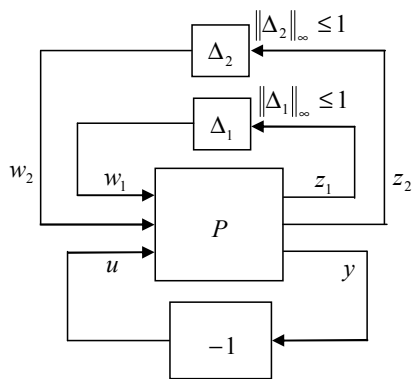

Figure 2b. *Robust stability of Figure 2a*

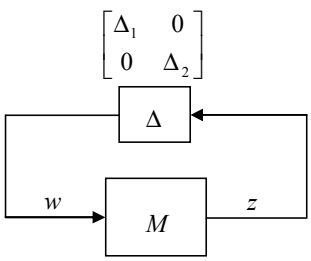

Figure 2c. *Fractional transformation of Figure 2b*

A graphical interpretation of robust performance is shown in Figure 3, where the line segment of $S + T = 1$ inside the diamond of $\mu_\Delta(M) = 1$ represents the set of all feasible loops S. In figure 3, we also demonstrate the conservatism of singular value $\bar{\sigma}(M)$ and the risk of spectral radius $\rho(M)$ as metrics of robust performance.

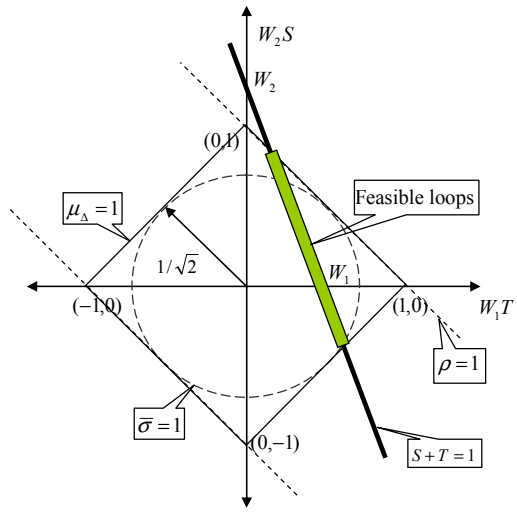

Figure 3. *Graphical interpretation of robust performance*

7. Conclusion

Small Gain theorem is always regarded to the most fundamental as to Robust Control. This paper implants an elegant small-gain theorem to Sturm-Liouville systems control, the closed-loop of which is guaranteed of well-posedness, exponential decay, passivity and dissipativity. To approach this, it compromises:

1. The definition of fraction-order Sturm-Liouville dynamics through spatiotemporal transfer-functions;
2. The creation of geometrical isomorphism and 2D-H_∞ norm in the mode-frequency domain;
3. The proofs of the equivalency of Hurwitz, passivity, dissipativity to bounded 2D-H_∞ norm, and the Sturm-Liouville Small Gain theorem; and
4. The derivation of μ -loopshaping in the mode-frequency domain as an application example of Sturm-Liouville Small Gain theorem.

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