
Exact, polynomial, determination solution method of the subset sum problem

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Abstract: In this paper we give original geometrical interpretation to the domain of definition of integer and combinatorial problems. The solution of the problems concerning NP class has been carried out on the hyperarches. The existence criterion of the solution on the hyperarches has been defined. The method for establishing the sequence of approximation to the solution on the hyperarches was constructed. Calculation experiments were conducted, and the obtained polynomial algorithm, practically and theoretically solved exactly the (SSP) problem.

Keywords: Knapsack Problem (KP), Subset Sum Problem (SSP), NP Class, Integer Programming, N-Dimensional Cube, Hyper Plane, Hyper Circle, Hyper Arch

1. Introduction

The problems in computing mathematics are divided into two classes according to their difficulty: 1) the first class contains the problems in which the number of simple operations for its solution (addition, subtraction, division, comparison, and square root) is polynomial expressed by the parameters (by the number of n -variables, the number of m -restriction) of the problem.

This class is said to be a polynomial class and in short is called a P class. 2) The problems in which the number of simple operations for its solution is proportional to 2^n enter to NP class. This class is called NP hard class. The knapsack problem is one of the classic problems in the linear integer programming problems. Up to present time, it was not possible to solve this problem exactly by a polynomial algorithm.

The knapsack problem can be formulated as a solution of the following linear integer programming formulation:

$$(KP) \text{ maximize } \sum_{j=1}^n p_j x_j \quad (1.1)$$

$$\text{subject to } \sum_{j=1}^n w_j x_j \leq c \quad (1.2)$$

$$x_j \in \{0,1\} \quad j = 1,2,\dots,n \quad (1.3)$$

We will denote the optimal solution vector by $X^* = (x_1^*, x_2^*, \dots, x_n^*)$ and the optimal solution value by Z^* . The set X^* denotes the optimal solution set, corresponding to the optimal solution vector.

Problem (KP) is the simplest non-linear integer programming model with binary variables, only one single constraint and only positive coefficients. Nevertheless, adding the integrality condition (1.3) to the simple linear program (1.1)-(1.2) already puts (KP) into the class of "difficult" problems [1].

The knapsack problem has been studied for centuries as it is the simplest prototype of a maximization problem. Already in 1897 Mathews showed how several constraints may be aggregated into one single knapsack constrain. This is somehow a prototype of a reduction of a general integer program to (KP), thus proving that (KP) is at least as hard to solve as an integer program. It is however unclear how the name "Knapsack Problem" was invented. Dantzig is using the expression in his early work and thus the name could be a kind of folklore [1].

When $p_j = w_j$ in (KP), the resulting optimization problem is known as the subset sum problem (SSP) because we are looking for a subset of the values w_i with the sum being as close as possible to, but not exceeding the given target value C .

$$\begin{aligned}
 (SSP) \text{ maximize } & \sum_{j=1}^n w_j x_j ; \\
 \text{subject to } & \sum_{j=1}^n w_j x_j \leq c \\
 & x_j \in \{0,1\}, \quad j=1,\dots,n
 \end{aligned}$$

Although (SSP) is a special case of (KP) it is still NP-hard and is NP-complete [1].

The class of NP-complete problems NPC is the set of decision problems Q satisfying the following two properties:

$$\begin{aligned}
 Q \in NP \\
 \forall R \in NP : R \leq_p Q
 \end{aligned}$$

To see the latter, one may simply assume that an NP-complete problem Q could be solved in polynomial time, then by transforming R to Q we would also get a polynomial algorithm for R contradicting the assumption.

The theory of NP-completeness gives us a framework for showing that it is very doubtful that a polynomial algorithm for solving e.g. the subset sum problem, then we would also be able to solve numerous famous optimization problems like the travelling salesman problem, general integer programming, and we would even be able to efficiently find mathematical proofs of theorems, as stated in Cook [1].

In this paper, a new solution method of knapsack problem is given by an original approach; some problems are solved exactly by a polynomial algorithm. One is subset sum problem.

2. Problem Statement

At first the engineers and economists has been engaged integer problems. These problems appeared as a result of economic and engineering demands. Therefore, these problems have got engineering and economic interpretation. But these interpretations are not enough for the solution of the problem.

In order to solve any mathematical problem, it is necessary to give its geometrical interpretation. It is the work of professional mathematicians. To this end, this matter requires special attentive in this paper.

We consider that, reader no need to use any references by reading the paper.

When constructing the new method we'll take three principles:

1. Definition of existence criterion of the solution integer point in the considered domain;
2. Construction of the sequence approximating to the solution using the found point;
3. Definition of necessary and sufficient conditions for the found solution to be optimal.

3. Investigation of n-Dimensional Cube

S is the set of vertex points of n-dimensional cube whose rib equals a unit, one of the vertexes is on the origin of coordinate, the ribs originate from the origin of coordinates and are arranged on the coordinate axes. The set is equal to the set $x_j \in \{0; 1\}, j = 1, 2, \dots, n$ [2].

Let's investigate the structure of n-dimensional cube and mathematical relations between its elements. When a rib of n-dimensional cube equals a unit, the coordinates of its vertex points consists of a unit and zero. The number of these vertices is $S = \sum_{k=0}^n C_n^k = 2^n$. For the indicated k the C_n^k number points are such that their k number coordinates consist of a unit, n-k number coordinates consist of zero. The equation of planes holding these points is as follows:

$$x_1 + x_2 + \dots + x_n = k; \quad k = \overline{1, n-1} \quad (3.1)$$

It is seen from equation (3.1) that the planes holding the C_n^k number integer points are parallel to each other and these plane equations change only depending on the free term k in the right hand side. Denote these hyper planes by M_{nk} . $R_k = \sqrt{k}$ is the distance of these points from the origin.

$d_k = \frac{k}{\sqrt{n}}$ is the distance of hyper planes M_{nk} from the origin, $O_{nkj} = \frac{k}{n}; j = \overline{1, n}$ is the center of a hyper circle located on hyper planes M_{nk} and holding the vertices of

n-dimensional cube, contained in M_{nk} . $r_{nk} = \sqrt{\frac{nk-k^2}{n}}$ Is a radius of this hyper circle? Denote by K_{nk} a hyper circle of a radius r_{nk} and center at the point O_{nk} located on the hyper plane M_{nk} . On each n-dimensional space we can group the set S being the vertices of n-dimensional cube with a rib equal a unit on parallel hyper planes M_{nk} , we solve problem (KP) on each K_{nk} and find general optimal solution. It is a general geometric principle. By solving the optimization problem considered in any domain we solve the problem in the parts whose unions equal to the domain and intersections are empty sets, find optimal solution at each part, compare them and then find general optimal solution.

4. Correlation of Constraint Hyper Plane and Hyper Circles

Let's study correlations of the constraint plane $M : \left\{ \sum_{j=1}^n w_j x_j = c \right\}$ and hyper circle K_{nk} . Let's normalize the vector $W(w_1, w_2, \dots, w_n)$ and denote it by W_n . $|W_n| = 1$ Find the projection of the vector W on the hyper plane M_{nk} and denote it by W_p

$$W_p = W_n - (W_n, N_n) \cdot N_n.$$

$N_n \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)$ are the normal of the hyper plane M_{nk} . One of the three relations between the hyper plane M and hyper circles K_{nk} is possible:

1) $c > (W, D_1)$ (Figure 1: Illustration of the correlation of constraint hyper plane and hyper circles). Here D_1 is the point obtained from the intersection of the straight line passing through the point O_{nk} in the direction W_p and the hyper circle K_{nk} and giving the maximum value to the inner product (W, X) on K_{nk} . In this case, any point lying on K_{nk}

satisfies the constraint: $\sum_{j=1}^n w_j x_j \leq c$.

2) $c < (W, D_2)$. Here the points D_2 is the point obtained from the intersection of the straight line passing through the point O_{nk} in the direction W_p and giving the minimal value to the inner product (W, X) on the hyper circle K_{nk} . In this case none of the points located on the K_{nk} satisfies the

constraint: $\sum_{j=1}^n w_j x_j \leq c$.

3) $(W, D_2) \leq c \leq (W, D_1)$. In this case either the hyper plane M intersects the hyper circle K_{nk} or touches it at the points D_1, D_2 . In order to solve problem (KP) we solve it on each hyper circle $K_{nk} \subset M_{nk}$ and denote these solutions by X_{nk}^* .

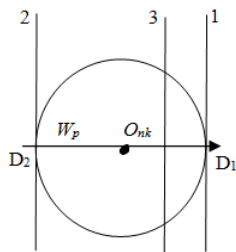


Figure 1. Illustration of the correlation of constraint hyper plane and hyper circles

$$\max_S \left(\sum_{j=1}^n p_j x_j \right) = \max_k \left(\sum_{j=1}^n p_j x_{nkj}^* \right)$$

$$\left. \begin{aligned} \sum_{j=1}^n p_j x_j &\leq c \\ \sum_{j=1}^n x_j &= k \\ \sum_{j=1}^n (x_j - o_{nkj})^2 &= r_{nk}^2 \end{aligned} \right\}$$

In the first case the solution of the problem (KP) on the K_{nk} is found as follows. We give unit values to x_j

corresponding to k number p_j taking the greatest value. To other $n-k$ number x_j we give zero value.

In the second case the problem has no solution on K_{nk} . In this case we don't look for the solution on K_{nk} and M_{nk} , respectively.

In the third case the solution of problem (KP) on K_{nk} is of special importance. The general solution of the problem reduces to its solution in the third case [3]. In the considered case, the domain for the solutions of the problem is a hyper arch being a part of a hyper circle. We express the hyper arch analytically in the following way

5. Criterion on the Existence of Solution Points on a Hyper Arch

In order to solve the problem we have to define the criterion on the existence of integer points on a hyper arch. By $X(n, k)$ we denote a set of integer points contained in the hyper circle K_{nk} whose k number coordinates equal a unit, $n-k$ number coordinates equal a zero. The hyper plane

$M : \left\{ \sum_{j=1}^n w_j x_j = c \right\}$ intersects the hyper circle lying on

M_{nk} and divides it into two parts (figure 2: Illustration of the criterion on the existence of solution points on a hyper arch).

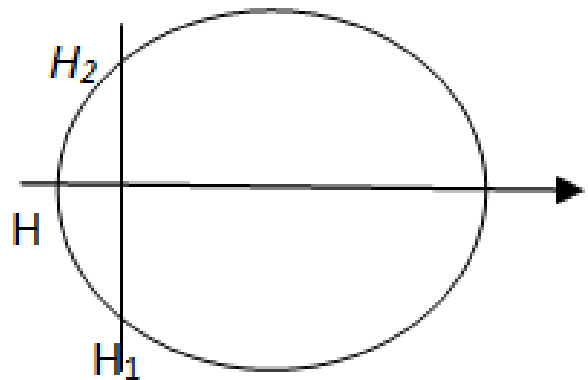


Figure 2. Illustration of the criterion on the existence of solution points on a hyper arch

The hyper arch H_1HH_2 satisfies the constraint $\sum_{j=1}^n w_j x_j \leq c$.

Let's give the criterion on the existence of even if one or none

of these points on this hyper arch.

For this we find the k -th minimum element among the elements of the vector W and denote it by w_{km} . We give unit value to the x_{1j} corresponding to the k -number w_j satisfying the condition $w_{km} - w_j \geq 0$, the zero value to other $n-k$ number x_{1j} . Let's verify if the point $X_1(n, k)$

obtained in such a way satisfies the constraint $\sum_{j=1}^n w_j x_j \leq c$. If

the point $X_1(n, k)$ doesn't satisfy the constraint the problem (KP) has no solution in K_{nk} and M_{nk} respectively.

If $X_1(n, k)$ satisfies the constraint of problem (KP) has a solution in K_{nk} and M_{nk} , respectively. $X_1(n, k)$ is a point or one of the points contained in $X(n, k)$ and giving the minimum value to the inner product (W, X) .

So, we defined the criterion on the existence of solution point on a hyper arch [3].

6. The Local Property Any Point Lying on the Hyper Circle

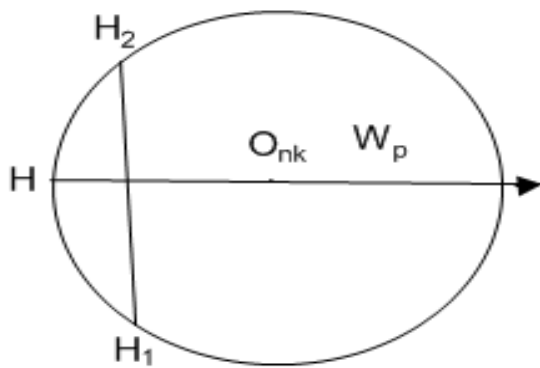


Figure 3. Illustration of the local property any point lying on the hyper circle

Now let's consider the vector HO_{nk} (fig. 3: Illustration of the local property any point lying on the hyper circle). Let's define relation between coordinates of the vector HO_{nk} and initial point of this vector H. $H(h_1, h_2, \dots, h_n)$; $Onk(\frac{k}{n}, \frac{k}{n}, \dots, \frac{k}{n})$, $j = \overline{1, n}$. By ho_j we denote the coordinates of the vector HO_{nk} . $ho_j = \frac{k}{n} - h_j, j = \overline{1, n}$.

Here the maximum values of ho_j correspond to the minimal of ho_j . As ho_j consist of the difference of the constant quantity $\frac{k}{n}$ and h_j values

If we consider the vector HO_{nk} as a normal of a plane and take the minimal k number ho_j and give to corresponding x_{1j} a unit and the others $n-k$ number x_{1j} zero values we can determine the point $X_1(n, k) \in X(n, k)$ or one of these points

close to H in the vicinity of the point H. This is a local property of any point lying on the hyper circle K_{nk} .

In future we'll use this property in constructing the sequence of approximating to solution of integer points on a hyper arch.

7. The Construction of a Sequence Converging to the Solution on a Hyper Arch

Let's give analytic expression of the problem on a hyper

arch:

$$\left. \begin{aligned} \sum_{j=1}^n p_j x_j &\rightarrow \max \\ \sum_{j=1}^n w_j x_j &\leq c \\ \sum_{j=1}^n x_j &= k \\ \sum_{j=1}^n (x_j - o_{nj})^2 &= r_{nk}^2 \\ x_j &\in \{0; 1\} \end{aligned} \right\} \quad (7.1)$$

Assume that the point $X_1(n, k)$ is determined by the existence criterion and is located on the hyper circle K_{nk} . We use this point and construct a sequence converging to the solution of problem (7.1). The terms of the sequence should satisfy the constraint and give strong monotonically increasing values to the functional.

1) Monotony condition: take the first of the coordinates equal zero of the point $X_1(n, k)$ and compare it with the corresponding coordinates of the point $X_1(n, k)$ taking unit values p_i and p_j . $p_i - p_j > 0$.

2) Constraint satisfaction condition. Let's give constraint satisfaction condition for the indices i and j satisfying the monotony condition $p_i - p_j > 0$. We calculate the quantity $d = c - \sum_{j=1}^n w_j x_{1j}$; $d + w_j - w_i \geq 0$ are constraint satisfaction condition.

$$\left\{ \begin{aligned} p_i - p_j &> 0 \\ d + w_j - w_i &\geq 0 \end{aligned} \right. \quad (7.2)$$

If condition (7.2) is satisfied, we accept $x_{1i} = 0; x_{1i} = 1$. The point $X_2(n, k)$ found in such a way increases the functional and satisfies the constraint and is located on K_{nk} .

We continue this process until the relation (7.2) is not satisfied between the indices i and j .

We conduct the process $k = \overline{1, n-1}$ for all K_{nk} to $k = \overline{1, n-1}$ find the points $X_q(n, k)$ compare the values that they give to the functional and find the solution of problem (KP)

$$Z^* = \max_k (P, X_q^*(n, k)); k = \overline{1, n}; P(p_1, p_2, \dots, p_n) \text{ is object vector.}$$

8. Necessary and Sufficient Condition of the Integer Point Lying on the Hyper Arch to be Local Optimal Solution on a Hyper Arch

Denote by A the negation of condition (7.2).

Theorem: The necessary and sufficient condition for the integer point on the edge hyper arch to be local optimal

solution, is condition A between coefficients p_j and w_j according to $x_j = 1$ coordinates and coefficients p_i and w_i according to $x_i = 0$ coordinates.

Necessities: Suppose that the point $X^*(n, k)$ is a local solution point on the edge hyper arch, on the hyper circle K_n .

$$\max(P, X(n, k)) = (P, X^*(n, k))$$

Assume contrary. If the point $X^*(n, k)$ does not satisfy condition A, when there exist the coordinates $x_j=1$ and $x_i=0$ such that $p_i - p_j > 0$. If we accept $x_j=0$ and $x_i=1$ then we can find the point $\bar{X}(n, k)$ such that

$$(P, X^*(n, k)) < (P, \bar{X}(n, k))$$

$$Z(X^*(n, k)) < Z(\bar{X}(n, k))$$

This contradicts the local optimal solution of the point $X^*(n, k)$. The obtained contradiction prove the necessities.

Suffices: Assume that the point $X^*(n, k)$ satisfies the condition A. Prove that this point is a local optimal solution on a hyper arch. Prove that is a maximum value of the functional on the edge of the hyper arch $Z^* = (P, X^*(n, k))$. Assume contrary. Z^* is not a maximum value of the functional on the edge of the hyper arch. Here is a point $\bar{X}(n, k)$ such that $\bar{Z} = (P, \bar{X}(n, k))$ is maximum value of the functional. Show that in this case the point $X^*(n, k)$ do not satisfy the condition A.

k number non zero coordinates of the point $X^*(n, k)$ denote by $x_{i1}, x_{i2}, \dots, x_{ik}$. By $x_{q1}, x_{q2}, \dots, x_{qk}$ denote non zero coordinates at the point $\bar{X}(n, k)$. Then

$$Z^* = p_{i1}^* + p_{i2}^* + \dots + p_{ik}^* \quad p_{ij} > 0; j = \overline{1, k}$$

$$\bar{Z} = \bar{p}_{q1} + \bar{p}_{q2} + \dots + \bar{p}_{qk} > 0; j = \overline{1, k}$$

If $\bar{Z} > Z^*$ hens

$$\bar{p}_{q1} + \bar{p}_{q2} + \dots + \bar{p}_{qk} > p_{i1}^* + p_{i2}^* + \dots + p_{ik}^*$$

There is $\bar{p}_{qj} > p_{ij}^*; l_j \neq qj$ such that $\bar{p}_{qj} - p_{ij}^* > 0$. This fact contradicts fulfillment of the condition A at the point $X^*(n, k)$. As the obtained contradiction, contradicts the theorems condition. Suffices is proved.

9. Estimation of the Number of Iterations and its Decrease

Now estimate the number of iterations. If any point $X_1(n, k)$ is an initial point, then the number of iterations is

$$N_{it} \leq c - (W, X_1(n, k))$$

$p_i - p_j \geq 1$ at each iteration, the value of the functional increases at last by a unit. Find a closer initial point to hyper plane $(W, X) = c$, we can provide the condition $N_{it} \leq n$. In order to solve these problem, we must chose the integer c_1 that satisfies condition $(W, X_1(n, k)) < c_1 < c$ close to c . $c - c_1 < n$. Find the intersection point of the hyper plane $(W, X) = c_1$ and the straight line $W_{min}W_{max}$ and denote it by T (Fig.4: Illustration of the decrease number of iterations). Then we must find the vector L parallel to the intersection of hyper planes $(W, X) = c$

and $x_1 + x_2 + \dots + x_n = k$. Transfer in parallel way the point T to on the hyper circle K_{nk} , in the direction of L and $-L$ and find the points H_1 and H_2 on the hyper circle K_{nk} . Show these operations concretely.

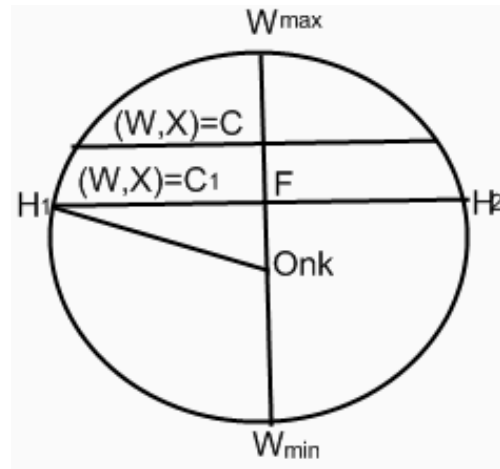


Figure 4. Illustration of the decrease number of iterations

Normalize the vector W and denote it by W_n . Find the projections of this vector on the hyper plane M_{nk} .

$$W_p = W_n - (W_n, E_n)E_n; E_n = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$$

Normalize the vector W_p and denote it by W_{pn} . Transfer in a parallel way the point O_{nk} by r_{nk} in the direction of W_{pn} and $-W_{pn}$. Denote the obtained points by W_{max} and W_{min} . $W_{max} = O_{nk} + r_{nk}W_{pn}; W_{min} = O_{nk} - r_{nk}W_{pn}$.

Find the intersection points of the straight line passing through the point W_{min} in the direction W_{pn} and the hyper plane $(W, X) = c_1$ and denote it by T .

$$T = W_{min} + [(c_1 - (W, W_{min})) / (W, W_{pn})] W_{pn}$$

Find the vector L parallel to the intersection of the hyper plane $w_1x_1 + w_2x_2 + \dots + w_nx_n = c_1$ and hyper plane $x_1 + x_2 + \dots + x_n = k$.

$$\begin{cases} (W, L) = 0 \\ (E, L) = 0 \end{cases}$$

Here $E = (1, 1, \dots, 1)$. When the dimension of the space is $n > 3$ the problem has infinitely number solutions.

$$\begin{cases} w_1l_1 + w_2l_2 + \dots + w_nl_n = 0 \\ l_1 + l_2 + \dots + l_n = 0 \end{cases} \quad (9.1)$$

In order to solve the system (9.1) we use the following principle. Chose $w_k \neq w_l$ and give the values $l_j = p_j$ to the elements of the vector in the direction L different from the elements $j \neq 1; j \neq k$. The determinant of the obtained system of equation is

$$D = \begin{vmatrix} w_1 & w_k \\ 1 & 1 \end{vmatrix} = w_1 - w_k$$

This system always has solution.

$$\begin{cases} w_1 l_1 + w_k l_k = s_1 \\ l_1 + l_k = s_2 \end{cases}$$

$$s_1 = - \sum_{j=1}^n w_j p_j + w_1 p_1 + w_k p_k$$

$$s_2 = - \sum_{j=1}^n p_j + p_1 + p_k$$

$$l_1 = \frac{s_1 - w_k s_2}{w_1 - w_k}, \quad l_2 = \frac{w_1 s_2 - s_1}{w_1 - w_k}$$

Thus we determined the L II ($M \cap M_{nk}$) direction vector. Normalize the vector L and denote it by L_n .

$$d = W_{min} F(\text{figure})$$

$$d = \frac{c_1 - (W, W_{min})}{(W, W_{pn})}$$

$$d_1 = O_{nk} F = I r_{nk} - dI$$

$$d_2 = FH_1 = FH_2$$

$$d_2 = \sqrt{r_{nk}^2 - r_{nk}^2 + 2r_{nk}d - d^2} = \sqrt{2r_{nk}d - d^2}$$

$$H_1 = W_{min} + dW_{pn} + d_2 L_n$$

$$H_2 = W_{min} + dW_{pn} - d_2 L_n$$

Find the points $X_1(n, k)$ and $X_2(n, k)$ close to the points H_1 and H_2 and from these points construct a sequence approximating to the solution of the both of ends of the hyper arch. Compare the points according to the value given to the functional and find the local solution on the hyper circle $X^*(n, k)$.

Apply these processes for all the hyper circle for the $k = \overline{1, n}$. $\bar{X}(n, k)$ satisfying condition

$$Z = \max(W, X^*(n, k)); k = \overline{1, n}.$$

will be an optimal solution of the problem.

It showed be noted that while pasting to other end, the following principle may be as well.

If any point $X_1(n, k)$ on the hyper arch is known, we find intersection hyper plane $(W, X) = (W, X_1(n, k))$ and the line $W_{min} W_{max}$ and denote it by T . Then transfer in a parallel way the point T on the hyper circle in the direction of the vector $L = \overrightarrow{X_1(n, k)T}$. $X_2(n, k) = T + L$.

Calculation experiments given in [4].

10. Investigation of Difficulty Order of the Problem

To understand difficulty order we must consider the cycles contained in each other in the algorithm:

$$k = \overline{1, n-1}; j = \overline{1, n}; i = \overline{1, n}; kk = \overline{1, n};$$

The maximal number of embedded cycles is four.

Hence we proved polynomial property of the method and that its difficulty order is $O(n^4)$.

11. Conclusion

In this work we obtained the following new results.

- 1) Geometric interpretation of domain of definition of the solution of combinatorial and discrete problems is given.
- 2) A method for reducing the solution of any linear integer programming problem to its solution on a hyper arch is given.
- 3) Existence criterion of this problem on a hyper arch is defined.
- 4) Using the initial point found on a hyper arch a method for constructing the sequence approximating to the solution is given.
- 5) The difficulty order of the algorithm is a polynomial.
- 6) Necessary and sufficient conditions for the found point to be an optimal solution on the edge of hyper arch is given. This condition is analytic.
- 7) Obtained method is exact, polynomial and determination of the subset sum problem (SSP).

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