

---

# Volterra Integral Equations with Vanishing Delay

Xiaoxuan Li, Weishan Zheng\*, Jiena Wu

Department of Mathematics and Statistics, Hanshan Normal University, Chaozhou, China

**Email address:**

weishan Zheng: [weishan Zheng@yeah.net](mailto:weishan Zheng@yeah.net) (Weishan Zheng)

**To cite this article:**

Xiaoxuan Li, Weishan Zheng, Jiena Wu. Volterra Integral Equations with Vanishing Delay. *Applied and Computational Mathematics*. Vol. 4, No. 3, 2015, pp. 152-161. doi: 10.11648/j.acm.20150403.18

---

**Abstract:** In this article, we use a Chebyshev spectral-collocation method to solve the Volterra integral equations with vanishing delay. Then a rigorous error analysis provided by the proposed method shows that the numerical error decay exponentially in the infinity norm and in the Chebyshev weighted Hilbert space norm. Numerical results are presented, which confirm the theoretical prediction of the exponential rate of convergence.

**Keywords:** Chebyshev Spectral-Collocation Method, Volterra Integral Equations, Vanishing Delay, Error Estimate, Convergence Analysis

---

## 1. Introduction

The Volterra integral equations (VIES) arise in many modeling problems in mathematical physics and chemical reaction, such as in the heat conduction, potential theory, fluid dynamic and radiative heat transfer problems. There are many methods to solve VIES, such as Legendre spectral-collocation method [1], Jacobi spectral-collocation method [2], spectral Galerkin method [3,4], Chebyshev spectral-collocation method [5] and so on. In this paper, according to [5], we use a Chebyshev spectral-collocation

method, where the collocation points are Chebyshev Gauss, Chebyshev Gauss-Radau, Chebyshev Gauss-Lobatto points to solve Volterra integral equations with vanishing delay.

## 2. Definition

The Volterra integral equations with the vanishing delay are defined as

$$y(\tau) = f(\tau) + \int_0^\tau R_1(\tau, \xi) y(\xi) d\xi + \int_0^{q\tau^2} R_2(\tau, \eta) y(\eta) d\eta, \quad \tau \in [0, T], \quad (1)$$

where  $0 < T < +\infty$ , and

$$f(\tau) \in C^m([0, T]), \quad R_1(\tau, \xi) \in C^m(\Omega_1), \quad R_2(\tau, \eta) \in C^m(\Omega_2), \quad m \geq 1, \quad (2)$$

where  $\Omega_1 := \{(\tau, \xi) : 0 \leq \xi \leq \tau \leq T\}$ ,  $\Omega_2 := \{(\tau, \eta) : 0 \leq \eta \leq q\tau^2 \leq \tau \leq T\}$ , and  $q$  is a constant,  $0 < q < 1$ .

## 3. Chebyshev Spectral-Collocation Method

For ease of analysis, we will transfer the integral interval  $[0, \tau]$  and  $[0, q\tau^2]$  to fixed interval  $[-1, 1]$ .

### 3.1. The Changes of Variables

Firstly, we use the changes of variables

$$\tau = \frac{T}{2}(1+x), \quad x := \frac{2}{T}\tau - 1,$$

$$\xi = \frac{T}{2}(1+s), \quad s := \frac{2}{T}\xi - 1,$$

$$\eta = \frac{T}{2}(1+t), \quad t := \frac{2}{T}\eta - 1.$$

$$\begin{aligned} \tilde{K}_1(x,s) &:= \frac{T}{2}R_1\left(\frac{T}{2}(1+x), \frac{T}{2}(1+s)\right), \\ \tilde{K}_2(x,t) &:= \frac{T}{2}R_2\left(\frac{T}{2}(1+x), \frac{T}{2}(1+t)\right). \end{aligned}$$

Then (1) becomes

$$u(x) = g(x) + \int_{-1}^x \tilde{K}_1(x,s)u(s)ds + \int_{-1}^{\frac{qT}{2}(1+x)^2-1} \tilde{K}_2(x,t)u(t)dt, \quad x \in [-1,1], \quad (3)$$

where  $u(x) := y\left(\frac{T}{2}(1+x)\right)$ ,  $g(x) := f\left(\frac{T}{2}(1+x)\right)$ ,

### 3.2. Set the Collocation Points

Now we assume that  $\{x_i\}_{i=0}^N$  (see, e.g., [6]) are the set of  $N+1$ -point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points, then (3) holds at  $x_i$ :

$$u(x_i) = g(x_i) + \int_{-1}^{x_i} \tilde{K}_1(x_i,s)u(s)ds + \int_{-1}^{\frac{qT}{2}(1+x_i)^2-1} \tilde{K}_2(x_i,t)u(t)dt, \quad 0 \leq i \leq N. \quad (4)$$

### 3.3. Linear Transformation

We make two simple linear transformations

$$s_x(z) = \frac{x+1}{2}z + \frac{x-1}{2}, \quad z \in [-1,1],$$

$$t_x(v) = \frac{qT}{4}(1+x)^2v + \frac{qT}{4}(1+x)^2 - 1, \quad v \in [-1,1].$$

Then (4) becomes

$$u(x_i) = g(x_i) + \int_{-1}^1 K_1(x_i,z)u(s_{x_i}(z))dz + \int_{-1}^1 K_2(x_i,v)u(t_{x_i}(v))dv, \quad i = 0,1,\dots,N, \quad (5)$$

where  $K_1(x,z) := \frac{x+1}{2}\tilde{K}_1(x,s_x(z))$ ,  $K_2(x,v) := \frac{qT(1+x)^2}{4}\tilde{K}_2(x,t_x(v))$ .

Applying appropriate  $N+1$ -point Gauss quadrature formula, we can obtain that

$$u(x_i) \approx g(x_i) + \sum_{k=0}^N K_1(x_i,z_k)u(s_{x_i}(z_k))\omega_k + \sum_{k=0}^N K_2(x_i,v_k)u(t_{x_i}(v_k))\omega_k, \quad i = 0,1,\dots,N,$$

where  $z_k = v_k$  are the  $N+1$ -point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points, corresponding weight  $\omega_k$ ,  $k = 0,1,\dots,N$  (see, e.g.[6]). We use  $u_i$  to approximate the function value  $u(x_i)$  and use

$u^N(x) := \sum_{j=0}^N u_j F_j(x)$  to approximate the function  $u(x)$ ,

where  $F_j(x)$  is the  $j$ -th Lagrange basic function.

First, deal with  $\sum_{k=0}^N K_1(x_i,z_k)u(s_{x_i}(z_k))\omega_k$ ,

$$\begin{aligned} & \sum_{k=0}^N K_1(x_i,z_k)u(s_{x_i}(z_k))\omega_k \\ &= \sum_{k=0}^N K_1(x_i,z_k) \sum_{j=0}^N u_j F_j(s_{x_i}(z_k))\omega_k \\ &= \sum_{k=0}^N K_1(x_i,z_k) (u_0 F_0(s_{x_i}(z_k))\omega_k + \dots + u_N F_N(s_{x_i}(z_k))\omega_k) \\ &= \sum_{k=0}^N K_1(x_i,z_k) u_0 F_0(s_{x_i}(z_k))\omega_k + \dots + \sum_{k=0}^N K_1(x_i,z_k) u_N F_N(s_{x_i}(z_k))\omega_k \\ &= K_1(x_i,z_0) u_0 F_0(s_{x_i}(z_0))\omega_0 + \dots + K_1(x_i,z_0) u_N F_N(s_{x_i}(z_0))\omega_0 \\ & \quad + K_1(x_i,z_1) u_0 F_0(s_{x_i}(z_1))\omega_1 + \dots + K_1(x_i,z_1) u_N F_N(s_{x_i}(z_1))\omega_1 + \dots \\ & \quad + K_1(x_i,z_N) u_0 F_0(s_{x_i}(z_N))\omega_N + \dots + K_1(x_i,z_N) u_N F_N(s_{x_i}(z_N))\omega_N \end{aligned}$$

$$\begin{aligned}
 &= u_0 \sum_{k=0}^N K_1(x_i, z_k) F_0(s_{x_i}(z_k)) \omega_k + \dots + u_N \sum_{k=0}^N K_1(x_i, z_k) F_N(s_{x_i}(z_k)) \omega_k \\
 &= \sum_{j=0}^N u_j \sum_{k=0}^N K_1(x_i, z_k) F_j(s_{x_i}(z_k)) \omega_k.
 \end{aligned}$$

Similarly,  $\sum_{k=0}^N K_2(x_i, v_k) u(t_{x_i}(v_k)) \omega_k = \sum_{j=0}^N u_j \sum_{k=0}^N K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \omega_k$ ,

then

$$u_i = g(x_i) + \sum_{j=0}^N u_j \sum_{k=0}^N \omega_k \left( K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right), \quad i = 0, 1, \dots, N. \tag{6}$$

The Chebyshev spectral-collocation method is to seek  $u_i = g(x_i) + \sum_{j=0}^N u_j \sum_{k=0}^N \omega_k \left( K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right)$ ,  $i = 0, 1, \dots, N$ .  $u^N(x)$  such that  $\{u_i\}_{i=0}^N$  satisfies the above equation.

### 3.4. Implementation of the Spectral Collocation Algorithm

(6) can be written in matrix form:  $U = G + AU$ ,

where  $U := [u_0, u_1, \dots, u_N]^T$ ,

$$G := [g(x_0), g(x_1), \dots, g(x_N)]^T,$$

$$A := (a_{ij})_{(N+1) \times (N+1)},$$

$$a_{ij} := \sum_{k=0}^N \omega_k \left( K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right).$$

We now discuss an efficient computation of  $F_j(s)$ . Considering Chebyshev function

$$F_j(s) = \sum_{p=0}^N a_{p,j} T_p(s), \tag{7}$$

where  $a_{p,j}$  is called the discrete polynomial coefficients of  $F_j$ . The inverse relation (see, e.g., [6]) is

$$a_{p,j} = \frac{1}{\gamma_p} \sum_{i=0}^N F_j(x_i) T_p(x_i) \omega_i^c = T_p(x_i) \omega_j^c / \gamma_p, \tag{8}$$

and  $\omega_j^c$  is the weight corresponding to  $x_i$ ,  $i = 0, 1, \dots, N$ ,

and  $\gamma_p = \sum_{i=0}^N T_p^2(x_i) \omega_j^c = \begin{cases} \pi, & p = 0, \\ \frac{\pi}{2}, & 1 \leq p \leq N. \end{cases}$  In addition,

$\gamma_N = \frac{\pi}{2}$  if  $\{x_i\}_{i=0}^N$  are the  $N+1$ -point Chebyshev Gauss, or Chebyshev Gauss-Radau points,  $\gamma_N = \pi$  if  $\{x_i\}_{i=0}^N$  are the  $N+1$ -point Chebyshev Gauss-Lobatto points.

## 4. Convergence Analysis

### 4.1. Some Spaces

For simplicity, we denote  $\left(\frac{\partial^k v}{\partial x^k}\right)(x)$  by  $\partial_x^k v(x)$ ,

$0 \leq k \leq m$ .

For non-negative integer  $m$ , we define  $H_{\omega^{\alpha,\beta}}^m(-1,1) := \{v := \partial_x^k v \in L^2_{\omega^{\alpha,\beta}}(-1,1), 0 \leq k \leq m\}$  with the

norm as  $\|v\|_{H_{\omega^{\alpha,\beta}}^m(-1,1)} := \left( \sum_{k=0}^m \|\partial_x^k v\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)}^2 \right)^{\frac{1}{2}}$ .

For a nonnegative integer  $N$ , we define the semi-norm

$$|v|_{H_{\omega^{\alpha,\beta}}^{m;N}(-1,1)} := \left( \sum_{k=\min(m,N+1)}^m \|\partial_x^k v\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)}^2 \right)^{\frac{1}{2}},$$

when  $\alpha = \beta = 0$ , we denote  $H_{\omega^{0,0}}^{m;N}(-1,1)$  by  $H^{m;N}(-1,1)$ .

When  $\alpha = \beta = -\frac{1}{2}$ , we denote  $\omega^c$  by  $\omega^{\frac{1}{2}}$ .

The space  $L^\infty(-1,1)$  is the Banach space of the measurable functions  $u : (-1,1) \rightarrow \mathbb{R}$  that is bounded outside a set of measure zero, equipped the norm  $\|u\|_{L^\infty(-1,1)} := \text{ess sup}_{x \in (-1,1)} |u(x)|$ .  $P_N$  is the space of all polynomials of degree not exceeding  $N$ .

### 4.2. Lemmas

Lemma 1. [6,7] If  $u \in H_{\omega^c}^m(-1,1)$ ,  $m \geq 1$ ,

$I_N v := \sum_{i=0}^N v(x_i) F_i(x)$ ,  $v \in C([-1,1])$ , then

$$\|u - I_N u\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m} |u|_{H_{\omega^c}^m(-1,1)}, \tag{9}$$

$$\|u - I_N u\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} |u|_{H_{\omega^c}^m(-1,1)}, \tag{10}$$

where  $I_N$  is the interpolation operator associated with the  $N+1$ -point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points  $\{x_i\}_{i=0}^N$ .

$$\left| \int_{-1}^1 u(x)\Psi(x)\omega^c(x)dx - \sum_{j=0}^N u(x_j)\Psi(x_j)\omega_j^c \right| \leq CN^{-m} |u|_{H_{\omega^c}^m(-1,1)} \|\Psi\|_{L_{\omega^c}^2(-1,1)},$$

$$\left| \int_{-1}^1 v(x)\Psi(x)dx - \sum_{j=0}^N v(x_j)\Psi(x_j)\omega_j \right| \leq CN^{-m} |v|_{H_{\omega^c}^m(-1,1)} \|\Psi\|_{L^2(-1,1)}.$$

where  $x_j$  is the  $N+1$ -point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto point, corresponding weight  $\omega_j^c$ ,  $j=0,1,\dots,N$  and  $z_j$  is  $N+1$ -point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto point, corresponding weight  $\omega^c$ ,  $j=0,1,\dots,N$ .

Lemma 3. If  $F_j(x)$ ,  $j=0,1,\dots,N$  are the  $j$ -th Lagrange interpolation polynomials associated with the  $N+1$ -point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points, then

$$\|I_N\|_{L^\infty(-1,1)} := \max_{x \in (-1,1)} \sum_{j=0}^N |F_j(x)| = O(\log N).$$

Lemma 4. [1,8] (Gronwall inequality) Suppose  $u(x)$  ( $-1 \leq x \leq 1$ ) is a non-negative, locally integrable function satisfying  $u(x) \leq v(x) + L \int_{-1}^x u(\tau) d\tau$ , where  $L \geq 0$  is a constant,  $v(x)$  is a integrable function, then there exists a constant  $C$  such that

$$u(x) \leq v(x) + C \int_{-1}^x v(\tau) d\tau, \quad \|u(x)\|_{L^\infty(-1,1)} \leq C \|v(x)\|_{L^\infty(-1,1)}.$$

Lemma 5. Assume that  $e(x)$  is a non-negative integrable function and satisfies

$$e(x) \leq v(x) + M_1 \int_{-1}^x e(t) dt + M_2 \int_{-1}^{\frac{qT}{2}(1+x)^2-1} e(\theta) d\theta, \quad (0 \leq M_1, M_2 < +\infty),$$

where  $v(x)$  is also a non-negative integrable function, then

$$e(x) \leq v(x) + C \int_{-1}^x v(\tau) d\tau, \quad \|e(x)\|_{L^\infty(-1,1)} \leq C \|v(x)\|_{L^\infty(-1,1)}.$$

$$\|u - u^N(x)\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} \left( |u|_{H_{\omega^c}^m(-1,1)} + (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \right), \tag{11}$$

where  $K_i^* := \max_{-1 \leq x \leq 1} |K_i(x, \cdot)|_{H_{\omega^c}^m(-1,1)}$ ,  $i=1,2$ .

Lemma 2. [6,7] If  $u \in H_{\omega^c}^m(-1,1)$ ,  $v \in H_{\omega^c}^m(-1,1)$ ,  $m \geq 1$  and  $\Psi \in P_N$ , then there exists a constant  $C$  independent of  $N$  such that

Proof. Since

$$q\tau^2 = \frac{qT}{2}(1+x)^2 - 1 < x,$$

Then

$$e(x) \leq v(x) + (M_1 + M_2) \int_{-1}^x e(\theta) d\theta.$$

Lemma 6. [9] For all measurable function  $f \geq 0$ , the following generalized Hardy's inequality  $\left( \int_a^b |Tf(x)|^q \omega_1(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b |f(x)|^p \omega_2(x) dx \right)^{\frac{1}{p}}$  holds if and only if

$$\sup_{a < x < b} \left( \int_x^b \omega_1(t) dt \right)^{\frac{1}{q}} \left( \int_a^x \omega_2(t)^{1-p} dt \right)^{\frac{1}{p}} < \infty, \quad p' = \frac{p}{p-1}, 1 < p \leq q < \infty$$

where  $T$  is an operator of the form  $(Tf)(x) = \int_a^x k(x,t)f(t) dt$ .

Lemma 7. [10] For every bounded function  $v(x)$ , there exists a constant  $C$  independent of  $v$  such that  $\sup_N \|I_N v\|_{L_{\omega^c}^2(-1,1)} \leq C \|v\|_{L^\infty(-1,1)}$ .

### 4.3. Theorems

#### 4.3.1. Convergence Analysis in $L^\infty(-1,1)$ Space

Theorem 1. Suppose  $u(x)$  is the exact solution to (3) and  $u^N(x)$  is the approximate solution obtained by using the spectral collocation schemes (6). Then for  $N$  sufficiently large, there is

Proof. Subtracting (6) from (5) gives

$$\begin{aligned}
 u(x_i) - u_i &= \int_{-1}^1 K_1(x_i, z) u(s_{x_i}(z_k)) dz + \int_{-1}^{-1} K_2(x_i, v) u(t_{x_i}(v_k)) dv \\
 &- \int_{-1}^1 K_1(x_i, z) u^N(s_{x_i}(z_k)) dz - \int_{-1}^{-1} K_2(x_i, v) u^N(t_{x_i}(v_k)) dv \\
 &+ \int_{-1}^1 K_1(x_i, z) u^N(s_{x_i}(z_k)) dz + \int_{-1}^{-1} K_2(x_i, v) u^N(t_{x_i}(v_k)) dv \\
 &- \sum_{j=0}^N u_j \sum_{k=0}^N \omega_k K_1(x_i, z_k) F_j(s_{x_i}(z_k)) - \sum_{j=0}^N u_j \sum_{k=0}^N \omega_k K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \\
 &= \int_{-1}^1 K_1(x_i, z) (u(s_{x_i}(z_k)) - u^N(s_{x_i}(z_k))) dz + \int_{-1}^{-1} K_2(x_i, v) (u(t_{x_i}(v_k)) - u^N(t_{x_i}(v_k))) dv \\
 &\quad + \int_{-1}^1 K_1(x_i, z) u^N(s_{x_i}(z_k)) dz - \sum_{k=0}^N K_1(x_i, z_k) u^N(s_{x_i}(z_k)) \omega_k \\
 &\quad + \int_{-1}^{-1} K_2(x_i, v) u^N(t_{x_i}(v_k)) dv - \sum_{k=0}^N K_2(x_i, v_k) u^N(t_{x_i}(v_k)) \omega_k \\
 &= \int_{-1}^1 K_1(x_i, z) e(s_{x_i}(z)) dz + \int_{-1}^{-1} K_2(x_i, v) e(t_{x_i}(v)) dv + J_1(x_i) + J_2(x_i),
 \end{aligned}$$

Then

$$e(x_i) = \int_{-1}^1 K_1(x_i, z) e(s_{x_i}(z)) dz + \int_{-1}^{-1} K_2(x_i, v) e(t_{x_i}(v)) dv + J_1(x_i) + J_2(x_i), \quad i = 0, 1, \dots, N, \tag{12}$$

where  $e(x) := u(x) - u^N(x)$ ,

$$\begin{aligned}
 J_1(x) &:= \int_{-1}^1 K_1(x, z) u^N(s_x(z)) dz - \sum_{k=0}^N K_1(x, z_k) u^N(s_x(z_k)) \omega_k, \\
 J_2(x) &:= \int_{-1}^{-1} K_2(x, v) u^N(t_x(v)) dz - \sum_{k=0}^N K_2(x, v_k) u^N(t_x(v_k)) \omega_k.
 \end{aligned}$$

By Lemma 2,

$$|J_1(x)| \leq CN^{-m} |K_1(x, \cdot)|_{H^{m;N}(-1,1)} \|u^N(s_x(\cdot))\|_{L^2(-1,1)}, \quad |J_2(x)| \leq CN^{-m} |K_2(x, \cdot)|_{H^{m;N}(-1,1)} \|u^N(t_x(\cdot))\|_{L^2(-1,1)}. \tag{13}$$

Multiplying  $F_j(x)$  on both sides of the error equation (12) and summing up from  $i = 0$  to  $i = N$  yield

$$\begin{aligned}
 \sum_{i=0}^N F_i(x) e(x_i) &= \sum_{i=0}^N F_i(x) \int_{-1}^1 K_1(x_i, z) e(s_{x_i}(z)) dz + \sum_{i=0}^N F_i(x) \int_{-1}^{-1} K_2(x_i, v) e(t_{x_i}(v)) dv + \sum_{i=0}^N F_i(x) J_1(x_i) + \sum_{i=0}^N F_i(x) J_2(x_i) \\
 &= I_N \int_{-1}^1 K_1(x_i, z) e(s_{x_i}(z)) dz + I_N \int_{-1}^{-1} K_2(x_i, v) e(t_{x_i}(v)) dv + I_N J_1(x_i) + I_N J_2(x_i),
 \end{aligned}$$

besides,

$$\sum_{i=0}^N F_i(x) e(x_i) = \sum_{i=0}^N (u(x_i) - u_i) F_i(x) = \sum_{i=0}^N u(x_i) F_i(x) - \sum_{i=0}^N u_i F_i(x) = I_N u - u^N,$$

So

$$I_N u - u^N = I_N \int_{-1}^1 K_1(x, z) e(s_x(z)) dz + I_N \int_{-1}^{-1} K_2(x, v) e(t_x(v)) dv + I_N J_1(x) + I_N J_2(x). \tag{14}$$

Consequently,

$$\begin{aligned}
 e(x) &= u(x) - u^N(x) = I_N u(x) - u^N(x) + u(x) - I_N u(x) \\
 &= I_N \int_{-1}^1 K_1(x, z) e(s_x(z)) dz + I_N \int_{-1}^1 K_2(x, v) e(t_x(v)) dv + I_N J_1(x) + I_N J_2(x) + u(x) - I_N u(x) \\
 &= \int_{-1}^1 K_1(x, z) e(s_x(z)) dz + \int_{-1}^1 K_2(x, v) e(t_x(v)) dv + u(x) - I_N u(x) + I_N J_1(x) + I_N J_2(x) + J_3(x) + J_4(x) \tag{15}
 \end{aligned}$$

where  $J_0 := u - I_N u$ ,

$$J_3 := I_N \int_{-1}^1 K_1(x, z) e(s_x(z)) dz - \int_{-1}^1 K_1(x, z) e(s_x(z)) dz, \quad J_4 := I_N \int_{-1}^1 K_2(x, v) e(t_x(v)) dv - \int_{-1}^1 K_2(x, v) e(t_x(v)) dv.$$

By Lemma 5, we get

$$\|e\|_{L^\infty(-1,1)} \leq C \left( \|J_0\|_{L^\infty(-1,1)} + \|I_N J_1\|_{L^\infty(-1,1)} + \|I_N J_2\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_4\|_{L^\infty(-1,1)} \right). \tag{16}$$

Using Lemma 1 for  $J_0(x)$  yields

$$\|J_0(x)\|_{L^\infty(-1,1)} = \|u - I_N u\|_{L^\infty(-1,1)} \leq CN^{\frac{1-m}{2}} |u|_{H_{\omega^c}^{m,N}(-1,1)}. \tag{17}$$

Using Lemma 7 and (13), we have

$$\max_{-1 \leq x \leq 1} |J_1(x)| \leq CN^{-m} K_1^* \max_{-1 \leq x \leq 1} \|u^N(s_x(\cdot))\|_{L^2(-1,1)} \leq CN^{-m} K_1^* \|u^N\|_{L^\infty(-1,1)} \leq CN^{-m} K_1^* \left( \|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right) \tag{18}$$

By Lemma 3,

$$\|I_N J_1(x)\|_{\infty(-1,1)} \leq \|I_N\|_{L^\infty(-1,1)} \|J_1(x)\|_{\infty(-1,1)} \leq CN^{-m} (\log N) K_1^* \left( \|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right). \tag{19}$$

Similarly,

$$\|I_N J_2(x)\|_{\infty(-1,1)} \leq CN^{-m} (\log N) K_2^* \left( \|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right). \tag{20}$$

According to Lemma 1 with  $m = 1$  to  $J_3(x)$  yields

$$\begin{aligned}
 \|J_3\|_{L^\infty(-1,1)} &\leq CN^{-\frac{1}{2}} \left| \int_{-1}^1 K_1(x, z) e(s_x(z)) dz \right|_{H_{\omega^c}^{1,N}(-1,1)} = CN^{-\frac{1}{2}} \left| \int_{-1}^x \tilde{K}_1(x, s) e(s) ds \right|_{H_{\omega^c}^{1,N}(-1,1)} \\
 &= CN^{-\frac{1}{2}} \left\| \tilde{K}_1(x, x) e(x) + \int_{-1}^x e(s) \frac{\partial}{\partial x} \tilde{K}_1(x, s) ds \right\|_{L_{\omega^c}^2(-1,1)} \\
 &\leq CN^{-\frac{1}{2}} \left\| \tilde{K}_1(x, x) e(x) \right\|_{L_{\omega^c}^2(-1,1)} + CN^{-\frac{1}{2}} \left\| \int_{-1}^x e(s) \frac{\partial}{\partial x} \tilde{K}_1(x, s) ds \right\|_{L_{\omega^c}^2(-1,1)} \\
 &= CN^{-\frac{1}{2}} \|e(x)\|_{L_{\omega^c}^2(-1,1)} + CN^{-\frac{1}{2}} \left\| \int_{-1}^{-1} e(s) ds \right\|_{L_{\omega^c}^2(-1,1)} \\
 &\leq CN^{-\frac{1}{2}} \|e(x)\|_{L^\infty(-1,1)}
 \end{aligned} \tag{21}$$

Similarly,

$$\|J_4\|_{L^\infty(-1,1)} = CN^{-\frac{1}{2}} \left| \int_{-1}^{\frac{qT}{2}(1+x)^2-1} \tilde{K}_2(x, t) e(t) dt \right|_{H_{\omega^c}^{1,N}(-1,1)}.$$

$$\begin{aligned}
 &= CN^{-\frac{1}{2}} \left\| \int_{-1}^x \tilde{K}_2 \left( x, \frac{qT}{2}(1+u)^2 - 1 \right) e \left( \frac{qT}{2}(1+u)^2 - 1 \right) (qt(1+u)) du \right\|_{H_{\omega^c}^{1;N}(-1,1)} \\
 &\leq CN^{-\frac{1}{2}} \left\| \tilde{K}_2 \left( x, \frac{qT}{2}(1+x)^2 - 1 \right) e \left( \frac{qT}{2}(1+x)^2 - 1 \right) (qt(1+x)) \right. \\
 &\quad \left. + \int_{-1}^x e \left( \frac{qT}{2}(1+x)^2 - 1 \right) (qt(1+x)) \frac{\partial}{\partial x} \tilde{K}_2 \left( x, \frac{qT}{2}(1+x)^2 - 1 \right) dx \right\|_{L_{\omega^c}^2(-1,1)} \leq CN^{-\frac{1}{2}} \|e\|_{L^\infty(-1,1)}. \tag{22}
 \end{aligned}$$

So

$$\|e\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m} \left( \|u\|_{H_{\omega^c}^{m;N}(-1,1)} + (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \right).$$

This completes the proof of this theorem.

$$\leq CN^{\frac{1}{2}-m} \|u\|_{H_{\omega^c}^{m;N}(-1,1)} + CN^{-m} (\log N) (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \tag{23}$$

Since  $\lim_{N \rightarrow \infty} \frac{N^{-m} \log N}{N^{\frac{1}{2}-m}} = \lim_{N \rightarrow \infty} \frac{\log N}{N^{\frac{1}{2}}} = 0$ , then, for  $N$

sufficiently large,  $N^{-m} \log N < N^{\frac{1}{2}-m}$ .

Hence

$$\|u(x) - u^N(x)\|_{L_{\omega^c}^2(-1,1)} \leq CN^{-m} (K_1^* + K_2^* + 1) \left( \|u'\|_{L_{\omega^c}^2(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u\|_{H_{\omega^c}^{m;N}} \right),$$

where  $K_i^* := \max_{-1 \leq x \leq 1} |K_i(x, \cdot)|_{H_{\omega^c}^m(-1,1)}$ ,  $i = 1, 2$ .

Proof. Applying Lemma 5, it follows from (15) that

$$e(x) \leq C \int_{-1}^x |J_0(t) + I_N J_1(t) + I_N J_2(t) + J_3(t) + J_4(t)| dt + |J_0(x) + I_N J_1(x) + I_N J_2(x) + J_3(x) + J_4(x)|.$$

By Lemma 6, we have

$$\|e\|_{L_{\omega^c}^2(-1,1)} \leq C \left( \|J_0\|_{L_{\omega^c}^2(-1,1)} + \|I_N J_1\|_{L_{\omega^c}^2(-1,1)} + \|I_N J_2\|_{L_{\omega^c}^2(-1,1)} + \|J_3\|_{L_{\omega^c}^2(-1,1)} + \|J_4\|_{L_{\omega^c}^2(-1,1)} \right).$$

Using Lemma 1 for  $J_0(x)$  yields

$$\|J_0(x)\|_{L_{\omega^c}^2(-1,1)} = \|u - I_N u\|_{L_{\omega^c}^2(-1,1)} \leq CN^{-m} \|u\|_{H_{\omega^c}^{m;N}(-1,1)}. \tag{24}$$

With Lemma 7, we get

$$\|I_N J_1(x)\|_{L_{\omega^c}^2(-1,1)} \leq C \max_{-1 \leq x \leq 1} |J_1(x)| \leq CN^{-m} K_1^* \left( \|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right). \tag{25}$$

From Theorem 1, let  $m = 1$ , then (11) becomes

$$\|e\|_{L^\infty(-1,1)} \leq C \left( \|u'\|_{L_{\omega^c}^2(-1,1)} + (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \right),$$

This makes (25) become

$$\|I_N J_1(x)\|_{L_{\omega^c}^2(-1,1)} \leq CN^{-m} K_1^* \left( \|u'\|_{L_{\omega^c}^2(-1,1)} + (K_1^* + K_2^* + 1) \|u\|_{L^\infty(-1,1)} \right). \tag{26}$$

Similarly,

### 4.3.2. Convergence Analysis in $L_{\omega^c}^2(-1,1)$ Space

Theorem 2. Suppose  $u(x)$  is the exact solution to (3), and  $u^N(x)$  is the approximate solution obtained by using the spectral collocation schemes (6). Then for  $N$  sufficiently large, we have

$$\|I_N J_2(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m} K_2^* \left( \|u'\|_{L^2_{\omega^c}(-1,1)} + (K_1^* + K_2^* + 1) \|u\|_{L^\infty(-1,1)} \right). \tag{27}$$

As the same analysis in (21), we obtain that  $\|J_3(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-1} \|e\|_{L^\infty(-1,1)}$ .

From Theorem 1, we get

$$\|J_3(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-\frac{1}{2}-m} K_2^* \left( |u|_{H^{m,N}_{\omega^c}(-1,1)} + (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \right). \tag{28}$$

Similarly,

$$\|J_4(x)\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-\frac{1}{2}-m} K_2^* \left( |u|_{H^{m,N}_{\omega^c}(-1,1)} + (K_1^* + K_2^*) \|u\|_{L^\infty(-1,1)} \right). \tag{29}$$

Hence, we have  $\|e\|_{L^2_{\omega^c}(-1,1)} \leq CN^{-m} (K_1^* + K_2^* + 1) \left( |u|_{H^{m,N}_{\omega^c}(-1,1)} + \|u'\|_{L^2_{\omega^c}(-1,1)} + \|u\|_{L^\infty(-1,1)} \right)$ .

This completes the proof of this Theorem.

### 5. Examples

#### 5.1. Example 1

From (1), let  $\tilde{K}_1(x,s) = e^{xs}, \tilde{K}_2(x,t) \equiv 0, g(x) = e^{2x} + \frac{1}{x+2} (e^{-(x+2)} - e^{x(x+2)})$ .

The corresponding exact solution is  $u(x) = e^{2x}, x \in [-1,1]$ .

We use the numerical scheme (6). Numerical errors versus several values of  $N$  are displayed in Table 1 and Figure ①. These results indicate that the desired spectral accuracy is obtained. Figure ② presents the approximate solution ( $N = 24$ ) and the exact solution, which are found in excellent agreement.

Table 1. The errors  $u - u^N$  versus the number of collocation points in  $L^\infty$  and  $L^2_{\omega^c}$  norms.

N	6	8	10	12	14
$L^\infty$ - error	2.4058e-004	2.0357e-006	1.3388e-008	6.6231e-011	2.5646e-013
$L^2_{\omega^c}$ - error	2.6350e-004	2.3946e-006	1.5653e-008	7.9650e-011	3.1722e-013
N	16	18	20	22	24
$L^\infty$ - error	2.1316e-014	7.9936e-015	7.9936e-015	7.1054e-015	1.1546e-014
$L^2_{\omega^c}$ - error	1.3511e-014	1.0471e-014	1.4066e-014	1.2207e-014	1.6227e-014

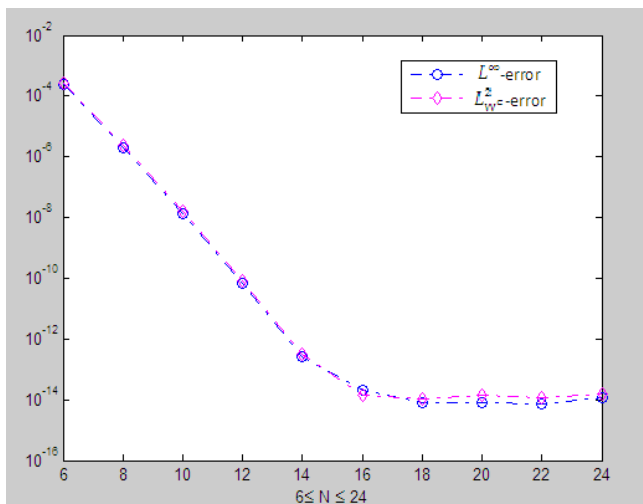


Figure 1. The errors  $u - u^N$  versus the number of collocation points in  $L^\infty$  and  $L^2_{\omega^c}$  norms.

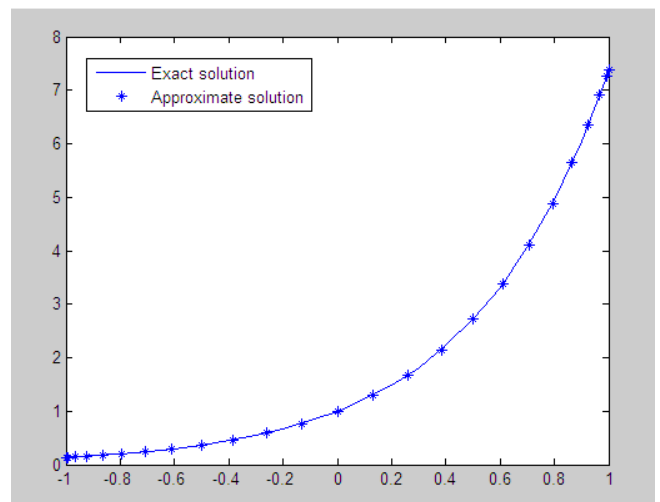


Figure 2. Comparison between approximate solution  $u^N$  and the exact solution  $u$ .



5.2. Example 2

From (1), let

$$R_1(\tau, \xi) = -(\tau + \xi), \quad R_2(\tau, \eta) = -(\tau - \eta), \quad T = 2, \quad q\tau = \frac{1}{2}\tau^2, \quad f(\tau) = -2\tau \cos \tau + 2\tau + 2 \sin \tau + \left(\frac{1}{2}\tau^2 - \tau\right) \cos\left(\frac{1}{2}\tau^2\right) - \sin\left(\frac{1}{2}\tau^2\right).$$

The corresponding exact solution is  $y(\tau) = \sin \tau, \quad \tau \in [0, 2]$ .

Figure ③ plots the errors for  $2 \leq N \leq 20$  in both  $L^\infty$  and  $L^2_{\omega^c}$  norms. The approximate solution ( $N = 20$ ) and the exact solution are displayed in Table 2. As expected, the errors decay exponentially which confirmed our theoretical predictions.

Table 2. The errors versus the number of collocation points in  $L^\infty$  and  $L^2_{\omega^c}$  norms.

N	2	4	6	8	10
$L^\infty$ - error	2.9023e-002	2.3295e-004	9.6846e-007	2.6571e-009	5.8210e-012
$L^2_{\omega^c}$ - error	3.5380e-002	2.6275e-004	1.0136e-006	2.8130e-009	5.8353e-012
N	12	14	16	18	20
$L^\infty$ - error	8.9928e-015	3.3307e-015	2.5535e-015	2.2204e-015	2.1094e-015
$L^2_{\omega^c}$ - error	8.8042e-015	2.1409e-015	1.7146e-015	2.4297e-015	2.4104e-015

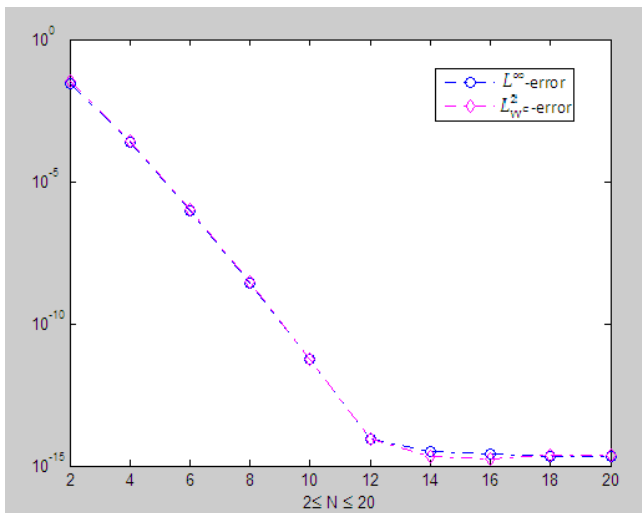


Figure 3. Plots the errors for  $2 \leq N \leq 20$  in both  $L^\infty$  and  $L^2_{\omega^c}$  norms.

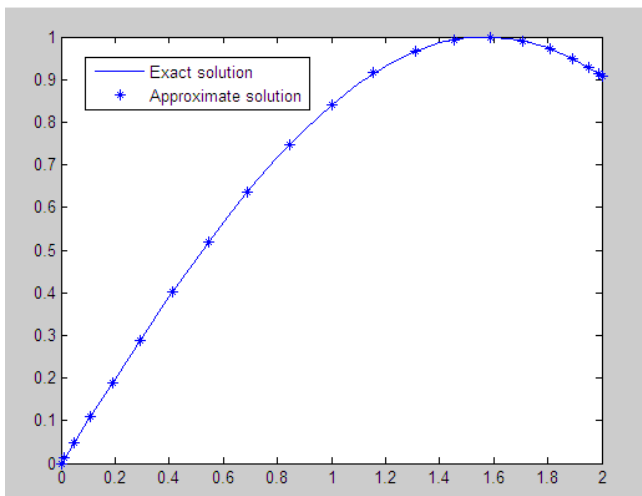


Figure 4. Comparison between approximate solution ( $N = 20$ ) and the exact solution.

6. Conclusion

We successfully solve the Volterra integral equation with vanishing delay by Chebyshev spectral-collocation method and provide a rigorous error analysis for this method. We get the conclusion that the error of the approximate solution decay exponentially in  $L^\infty$  norm and  $L^2_{\omega^c}$  norm. We also carry out the numerical experiment which confirm the theoretical prediction of the exponential rate of convergence.

References

- [1] T. Tang, X. Xu, J. Cheng, On spectral methods for Volterra type integral equations and the convergence analysis, *J. Comput. Math.* 26 (2008) 825-837.
- [2] Y. Chen, and T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equation with a weakly singular kernel, *Math. Comput.* 79(2010), pp. 147-167.
- [3] Z. Wan, Y. Chen, and Y. Huang, Legendre spectral Galerkin method for second-kind Volterra integral equations, *Front. Math. China*, 4(2009), pp. 181-193.
- [4] Z. Xie, X. Li and T. Tang, Convergence Analysis of Spectral Galerkin Methods for Volterra Type Integral Equations, *J. Sci. Comput.* 2012.
- [5] Z. Gu and Y. Chen, Chebyshev spectral collocation method for Volterra integral equations [D·Master's Thesis], *Contemporary Mathematics*, Volume 586, 2013, pp. 163-170.
- [6] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral method fundamentals in single domains*, Springer-Verlag, 2006.
- [7] J. Shen, and T. Tang, *Spectral and high-order methods with applications*, Science Press, Beijing, 2006.
- [8] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, 1989.

- [9] A. Kufner, and L. E. Persson, *Weighted inequality of Hardy's Type*, World scientific, NewYork, 2003.
- [10] P. Nevai, Mean convergence of Lagrange interpolation, III, *Trans. Amer. Math. Soc.*, 282(1984), pp. 669–698.