



# Modelling Infectiology of Dengue Epidemic

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**Abstract:** In this paper a mathematical model for the transmission dynamics of dengue fever disease is presented. We present a SITR (susceptible, infected, treated, recovery) and ASI (aquatic, susceptible, infected) epidemic model to describe the interaction between human and dengue fever mosquito populations. In order to assess the transmission of Dengue fever disease, the susceptible population is divided into two, namely, careful and careless human susceptible population. The model presents four possible equilibria: two disease-free and two endemic equilibrium. The results show that the disease-free equilibrium point is locally and globally asymptotically stable if the reproduction number is less than unity. Endemic equilibrium point is locally and globally asymptotically stable under certain conditions using additive compound matrix and Lyapunov method respectively. Sensitivity analysis of the model is implemented in order to investigate the sensitivity of certain key parameters of dengue fever disease with treatment, Careful and Careless Susceptibles on the transmission of Dengue fever Disease.

**Keywords:** Dengue Fever Disease, Careful, Careless, Susceptibles, Equilibrium, Stability, Reproduction Number

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## 1. Introduction

Dengue is a major health problem found in tropical and sub-tropical climates worldwide, mostly in urban and semi-urban areas [1]. Dengue fever disease can cause a severe flu-like illness, and sometimes Dengue fever can vary from mild to severe. The more severe forms of dengue fever include dengue hemorrhagic fever and dengue shock syndrome. Dengue fever (DF) is a vector-borne disease transmitted by female *Aedes aegypti* and *Aedes albopictus* mosquitoes because they require blood meal for the development of their eggs. Four different serotypes can cause dengue fever. A human infected by one serotype, on recovery, gains total immunity to that serotype and only partial and transient immunity with respect to the other three. Preventing or reducing dengue virus transmission depends entirely on the control of mosquito. The spread of dengue is attributed to expanding geographic distribution of the four dengue viruses and their mosquito vectors, the most important of which is the predominantly urban species *Aedes aegypti* [2].

Mathematical models have played a major role in increasing our understanding of the dynamics of infectious diseases. Several models have been proposed to study the effects of some factors on the transmission dynamics of these

infectious diseases including Dengue fever and to provide guidelines as to how the spread can be controlled [3].

Mathematical modelling also became considerable important tool in the study of epidemiology because it helped us to understand the observed epidemiological patterns, disease control and provide understanding of the underlying mechanisms which influence the spread of disease and may suggest control strategies [4,5,6,7,8,2,9,10,11]. Moreover in [12] the authors presented a dynamical model that studied the temporal model for dengue disease with treatment. So far no one considered a dynamical system that incorporates the effects of treated individual, Careful and Careless Susceptibles on the transmission of Dengue fever in the society. In this paper, an extension of the model of [12] is presented to include temporary immunity and Susceptibles with different behaviour i.e. the dynamical system that incorporates the effects of Careful and Careless Susceptibles on the transmission of Dengue fever in the society.

Thus, we study and analyse a non-linear mathematical model showing the effect of Treatment, Careful and Careless Susceptibles on the transmission of dengue fever disease in the population.

## 2. Formulation of the Model

In this section, a deterministic model is developed that describes the dynamics of Dengue fever of population size  $N$  [13]. Two types of population are considered: humans and mosquito. The humans are divided into five mutually-exclusive compartments indexed by  $h$  are given by: careful human susceptible,  $S_{h_1}(t)$ , careless human susceptible,  $S_{h_2}(t)$ , in which the possibility of careless human Susceptibles contracting the disease is higher than that for careful human Susceptibles,  $I_h(t)$ , individuals capable of transmitting dengue fever disease to others;  $T_h(t)$ , individual who are treated and  $R_h(t)$ , individuals who have acquired immunity

at time  $t$ . The total number of human is constant, which means that  $N(t) = S_{h_1}(t) + S_{h_2}(t) + I_h(t) + T_h(t) + R_h(t)$ . Similarly, the model has also three compartments for the mosquito (mosquitoes) indexed by  $m$  are given by:  $A_m(t)$ , which represents the aquatic phase of the mosquito (including egg, pupae and larvae) and the adult phase of the mosquito, with  $S_m(t)$  and  $I_m(t)$ , susceptible and infected, respectively. It is also assumed that  $N_m = S_m(t) + I_m(t)$ .

Considering the above considerations and assumptions, we then have the following schematic model flow diagram for dengue fever disease with treatment, Careful and Careless Susceptibles:

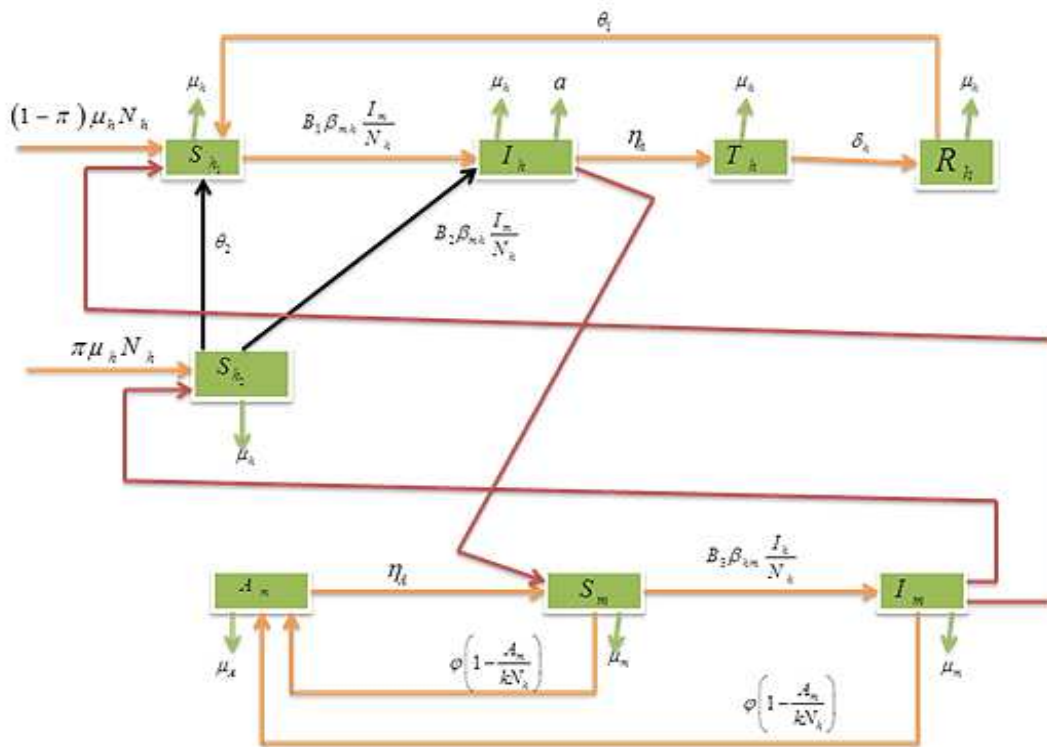


Figure 1. Model Flow diagram for dengue fever disease with treatment, Careful and Careless human susceptible.

From the above flow diagram, the model is described by an initial value problem with a system of eight differential equations given as follows:

$$\begin{aligned} \frac{dS_{h_1}}{dt} &= (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2} \\ \frac{dS_{h_2}}{dt} &= \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} \end{aligned} \quad (1)$$

$$\frac{dI_h}{dt} = (B_1S_{h_1} + B_2S_{h_2})\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h$$

$$\frac{dT_h}{dt} = \eta_h I_h - (\mu_h + \delta_h)T_h$$

$$\frac{dR_h}{dt} = \delta_h T_h - (\mu_h + \theta_1)R_h$$

$$\frac{dA_m}{dt} = \phi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - (\mu_A + \eta_A)A_m$$

$$\frac{dS_m}{dt} = \eta_A A_m - \left(B_3\beta_{hm}\frac{I_h}{N_h} + \mu_m\right)S_m$$

$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m$$

where  $S_{h_1}(0) > 0$ ,  $S_{h_2}(0) > 0$ ,  $I_h(0) \geq 0$ ,  $T_h(0) \geq 0$ ,  $R_h(0) \geq 0$ ,  $A_m(0) \geq 0$ ,  $S_m(0) > 0$ ,  $I_m(0) \geq 0$ , for all  $t \geq 0$ .

## 3. Mathematical Analysis of the Model

The dynamics of dengue fever disease is determined by the

basic reproduction number  $R_0$  which is a key concept and is defined as the average number of secondary infection arising from a single infected individual introduced into the susceptible class during its entire infectious period in a totally susceptible population [14,15], for if  $R_0 < 1$  the result is disease free-equilibrium and if  $R_0 > 1$  means that there exists endemic equilibrium point. The model system of equations (1) will be analysed qualitatively to get a better understanding of the effects of treated individual, careful and Careless human Susceptibles of Dengue fever disease.

**3.1. Disease Free Equilibrium (DFE)**

For the disease free equilibrium, it is assumed that there is no infection for both populations of human and mosquitoes i.e.  $I_h(t) = 0$  and  $I_m(t) = 0$ , denoted by ' $E_0$ '. Thus  $E_0$  of the model system (1) is obtained as  $E_0 = (S_h(t), S_{h_2}(t), 0, 0, 0, A_m(t), S_m(t), 0) =$

$$\left( \frac{(1-\pi)N_h(\mu_h + \theta_2) + \theta_2\pi N_h}{\mu_h + \theta_2}, \frac{\pi\mu_h N_h}{\mu_h + \theta_2}, 0, 0, 0, \frac{kN_h q}{\eta_A \varphi}, \frac{kN_h q}{\varphi\mu_m}, 0 \right)$$

where  $q = -((\mu_A + \eta_A) - \eta_A \varphi)$

**3.2. The Basic Reproduction Number, ' $R_0$ '**

The basic reproduction number of the model (1)  $R_0$  is calculated by using the next generation matrix of an ODE [14]. Using the approach of [14].  $R_0$  is obtaining by taking the largest (dominant) Eigen value (spectral radius) of

$$\left[ \frac{\partial F_i(E_0)}{\partial X_j} \right] \left[ \frac{\partial V_i(E_0)}{\partial X_j} \right]^{-1},$$

where,  $F_i$  is the rate of appearance of new infection in compartment  $i$ ,  $V_i^+$  is the transfer of individuals out of the

$$F = \begin{pmatrix} 0 & \left( \frac{B_1(1-\pi)(\mu_h + \theta_2) + \theta_2\pi}{\mu_h + \theta_2} + B_2 \frac{\pi\mu_h}{\mu_h + \theta_2} \right) \beta_{mh} \\ \frac{B_3\beta_{hm}kq}{\varphi\mu_m} & 0 \end{pmatrix}$$

The transfer of individuals out of the compartment  $i$  is given by

$$V_i = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (\mu_h + \eta_h + a)I_h \\ \mu_m I_m \end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by,

compartment  $i$  by all other means and  $E_0$  is the disease free equilibrium.

$$F_i = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} (B_1S_{h_1} + B_2S_{h_2})\beta_{mh} \frac{I_m}{N_h} \\ B_3\beta_{hm} \frac{I_h}{N_h} S_m \end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by

$$F = \begin{pmatrix} \frac{\partial F_1}{\partial I_h}(E_0) & \frac{\partial F_1}{\partial I_m}(E_0) \\ \frac{\partial F_2}{\partial I_h}(E_0) & \frac{\partial F_2}{\partial I_m}(E_0) \end{pmatrix}$$

This implies that

$$F = \begin{pmatrix} 0 & \frac{(B_1S_{h_1} + B_2S_{h_2})\beta_{mh}}{N_h} \\ B_3\beta_{hm} \frac{S_m}{N_h} & 0 \end{pmatrix}$$

With

$$S_{h_1} = \frac{(1-\pi)N_h(\mu_h + \theta_2) + \theta_2\pi N_h}{\mu_h + \theta_2},$$

$$S_{h_2} = \frac{\pi\mu_h N_h}{\mu_h + \theta_2}, \quad S_m = \frac{kN_h q}{\varphi\mu_m}$$

we have

$$V = \begin{pmatrix} \frac{\partial V_1}{\partial I_h}(E_0) & \frac{\partial V_1}{\partial I_m}(E_0) \\ \frac{\partial V_2}{\partial I_h}(E_0) & \frac{\partial V_2}{\partial I_m}(E_0) \end{pmatrix}$$

This gives

$$V = \begin{pmatrix} \mu_h + \eta_h + a & 0 \\ 0 & \mu_m \end{pmatrix}$$

With

Therefore

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \frac{1}{\mu_m} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} 0 & \left( \frac{B_1(1-\pi)(\mu_h + \theta_2) + \theta_2\pi}{\mu_h + \theta_2} + \frac{B_2\pi\mu_h}{\mu_h + \theta_2} \right) \beta_{mh} \\ \frac{B_3\beta_{hm}kq}{\varphi\mu_m} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \frac{1}{\mu_m} \end{pmatrix} \tag{2}$$

Then eigenvalues of the equation (2) is given by

$$\det(FV^{-1} - \lambda I) = \det \begin{pmatrix} 0 - \lambda & \left( \frac{B_1\beta_{mh}(1-\pi)(\mu_h + \theta_2) + \theta_2\pi\beta_{mh} + B_2\beta_{mh}\pi\mu_h}{(\mu_h + \theta_2)\mu_m} \right) \\ \frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h + \eta_h + a)} & 0 - \lambda \end{pmatrix}$$

This gives

$$\lambda^2 = \left( \frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h + \eta_h + a)} \right) \left( \frac{B_1\beta_{mh}(1-\pi)(\mu_h + \theta_2) + \theta_2\pi\beta_{mh} + B_2\beta_{mh}\pi\mu_h}{(\mu_h + \theta_2)\mu_m} \right)$$

consequently

$$\lambda = \sqrt{\left( \frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h + \eta_h + a)} \right) \left( \frac{B_1\beta_{mh}(1-\pi)(\mu_h + \theta_2) + \theta_2\pi\beta_{mh} + B_2\beta_{mh}\pi\mu_h}{(\mu_h + \theta_2)\mu_m} \right)}$$

or

$$\lambda = -\sqrt{\left( \frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h + \eta_h + a)} \right) \left( \frac{B_1\beta_{mh}(1-\pi)(\mu_h + \theta_2) + \theta_2\pi\beta_{mh} + B_2\beta_{mh}\pi\mu_h}{(\mu_h + \theta_2)\mu_m} \right)}$$

It follows that the Basic Reproductive number which is given by the largest Eigen value for model system (1) denoted by  $R_0$  is given as

$$R_0 = \sqrt{\left( \frac{kqB_3\beta_{hm}}{\varphi\mu_m(\mu_h + \eta_h + a)} \right) \left( \frac{B_1\beta_{mh}(1-\pi)(\mu_h + \theta_2) + \theta_2\pi\beta_{mh} + B_2\beta_{mh}\pi\mu_h}{(\mu_h + \theta_2)\mu_m} \right)}$$

But

$$q = -((\mu_A + \eta_A)\mu_m - \eta_A\varphi)$$

It is follows that

$$R_0 = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}((\mu_A + \eta_A)\mu_m - \eta_A\varphi)(B_1(1-\pi)(\mu_h + \theta_2) + \theta_2\pi + B_2\pi\mu_h)}{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)}}$$

or

where

$$R_0 = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)}} \quad t = ((\mu_A + \eta_A)\mu_m - \eta_A\varphi)(B_1(1-\pi)(\mu_h + \theta_2) + \theta_2\pi + B_2\pi\mu_h)$$

Model System (1) has infection-free equilibrium  $E_0$  if  $R_0 < 1$ , otherwise endemic equilibrium.

**3.3. Sensitivity Analysis of Model Parameters**

In this subsection we would like to know difference factors for disease transmission where this helps to reduce mortality and morbidity due to dengue fever disease.

In order to determine how best human mortality and morbidity due to dengue fever disease is reduced, we calculate the sensitivity indices of the reproduction number  $R_0$  to each parameter in the model using the approach of [14]. These indices tell us which parameters have high impact on  $R_0$  and should be targeted by intervention strategies [13].

$$R_0 = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}((\mu_A + \eta_A)\mu_m - \eta_A\varphi)(B_1(1-\pi)(\mu_h + \theta_2) + \theta_2\pi + B_2\pi\mu_h)}{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)}}$$

Then analytical expression for the sensitivity of  $R_0$  with respect to each parameter can be calculated using a set of reasonable parameter values. Parameter values are obtained from the different literatures like [12], [13] and [17] (<http://www.wavuti.com/2014/05/wizara-ya-afya-kitengo-cha.html>). Other parameter values are estimated to vary within realistic means and given as  $\beta_{hm} = 0.375$ ,  $\beta_{mh} = 0.45$ ,  $\pi = 0.96$ ,  $B_1 = 0.5$ ,  $B_2 = 0.9$ ,  $B_3 = 0.7$ ,  $\mu_m = \frac{1}{11}$ ,  $k = 3$ ,  $\eta_A = 0.35$ ,  $\mu_A = 0.25$ ,  $\mu_h = \frac{1}{78 \times 365}$ ,  $\eta_h = 1/3$ ,  $\varphi = 5$ ,  $\theta_1 = 0.01$ ,  $\theta_2 = 0.6$ ,  $a = 0.001$ ,  $\delta_h = 0.98$ .

The sensitivity indices of  $R_0$  with respect to  $\eta_A$  and  $\mu_m$  are given by  $X_{\eta_A}^{R_0} = \frac{\partial R_0}{\partial \eta_A} \times \frac{\eta_A}{R_0} = +0.506702261$  and  $X_{\mu_m}^{R_0} = \frac{\partial R_0}{\partial \mu_m} \times \frac{\mu_m}{R_0} = -1.016085624$  respectively.

Other indices  $X_{B_3}^{R_0}$ ,  $X_{\beta_{hm}}^{R_0}$ ,  $X_k^{R_0}$ ,  $X_{\beta_{mh}}^{R_0}$ ,  $X_\pi^{R_0}$ ,  $X_{\mu_h}^{R_0}$ ,  $X_{\eta_h}^{R_0}$ ,  $X_\varphi^{R_0}$ ,  $X_{B_1}^{R_0}$ ,  $X_{B_2}^{R_0}$ ,  $X_{\theta_2}^{R_0}$ ,  $X_a^{R_0}$ ,  $X_a^{R_0}$  and  $X_{\mu_A}^{R_0}$  are obtained following the same method and tabulated as follows:

**Table 1.** Sensitivity Indices of Model Parameters to  $R_0$ .

S/N	Parameter Symbol	Sensitivity index
1	$\eta_A$	+0.506702261
2	$\beta_{hm}$	+0.500000191
3	$B_3$	+0.499999846
4	$\beta_{mh}$	+0.499999617
5	$k$	+0.499998468
6	$\pi$	+0.244896308
7	$\varphi$	+0.016085811
8	$B_1$	+0.010204166
9	$B_2$	+0.00002580447057

Definition 1: The normalised forward sensitivity index of a variable ‘p’ that depends differentiable on a parameter ‘q’ is defined as:

$$X_q^p = \frac{\partial p}{\partial q} \times \frac{q}{p} \text{ in [16].} \tag{3}$$

Having an explicit formula for  $R_0$  in equation (3), we derive an analytical expression for the sensitivity of  $R_0$  as

$$X_q^{R_0} = \frac{\partial R_0}{\partial q} \times \frac{q}{R_0}$$

where

S/N	Parameter Symbol	Sensitivity index
10	$\theta_2$	+0.000002867004084
11	$\mu_h$	-0.00005539116931
12	$a$	-0.001495356938
13	$\mu_A$	-0.006702415127
14	$\eta_h$	-0.498452312
15	$\mu_m$	-1.016085624

The parameters are ordered from most sensitive to the least.

**3.3.1. Interpretation**

By analysing sensitivity indices of model parameters to  $R_0$ , it is observed that the following parameters  $\eta_A$ ,  $\beta_{hm}$ ,  $B_3$ ,  $\beta_{mh}$ ,  $k$ ,  $\pi$ ,  $\varphi$ ,  $B_1$ ,  $B_2$  and  $\theta_2$  when each one increases keeping the other parameters constant they increase the value of  $R_0$  implying that they increase the endemicity of the disease as they have positive indices. While parameters such as,  $a$ ,  $\mu_A$ ,  $\eta_h$  and  $\mu_m$  when each one increases while keeping the other parameters constant they decrease the value of  $R_0$  implying that they decrease the endemicity of the disease as they have negative indices.

But individually, the most sensitive parameter is maturation rate from larvae to adult (per day)  $\eta_A$ , followed by the transmission probability from  $I_h$  (per bite)  $\beta_{hm}$ , average daily biting (per day) for mosquito susceptible  $B_3$ , transmission probability from  $I_m$  (per bite)  $\beta_{mh}$ , number of larvae per human  $k$ , Fraction of subpopulation recruited into the population  $\pi$ , number of eggs at each deposit per capita (per day)  $\varphi$ , average daily biting (per day) for careful human susceptible  $B_1$ , average daily biting (per day) for careless human susceptible  $B_2$ , Positive change in behaviour of Careless individuals  $\theta_2$ , average lifespan of humans (per day)  $\mu_h$ , Per capita disease induced death rate for humans  $a$ , natural mortality of larvae (per day)  $\mu_A$ , mean viremic period (per day)  $\eta_h$  and finally the least sensitive parameter is

average lifespan of adult mosquitoes (Per day)  $\mu_m$ .

**3.4. Local Stability of Disease Free Equilibrium Point**

Local stability of the disease free equilibrium is determined

$$J_{E_0} = \begin{pmatrix} -\mu_h & \theta_2 & 0 & 0 & \theta_1 & 0 & 0 & A \\ 0 & -\theta_2 - \mu_h & 0 & 0 & 0 & 0 & 0 & \frac{\pi B_2 \beta_{mh} \mu_h}{\theta_2 + \mu_h} \\ 0 & 0 & -a - \eta_h - \mu_h & 0 & 0 & 0 & 0 & B \\ 0 & 0 & \eta_h & -\delta_h - \mu_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_h & -\theta_1 - \mu_h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta_A - \mu_A - \frac{q}{\mu_m} & \varphi - \frac{q}{\eta_A} & \varphi - \frac{q}{\eta_A} \\ 0 & 0 & -\frac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & \eta_A & -\mu_m & 0 \\ 0 & 0 & \frac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & 0 & 0 & -\mu_m \end{pmatrix} \tag{4}$$

where  $A = -\frac{B_1\beta_{hm}(\theta_2 - (-1 + \pi)\mu_h)}{\theta_2 + \mu_h}$ ,  
 $B = \frac{\beta_{mh}(\pi B_2\mu_h + B_1(\theta_2 - (-1 + \pi)\mu_h))}{\theta_2 + \mu_h}$

Thus the stability of the disease free equilibrium point is clarified by studying the behaviour of  $J_{E_0}$  in which for local stability of DFE we seek for its all eigenvalues to have negative real parts. It follows that, the characteristic function of the matrix (4) with  $\lambda$  being the eigenvalues of the Jacobian matrix,  $J_{E_0}$ . The Jacobian matrix has the following eigenvalues:

$$\lambda_1 = -\mu_h, \lambda_2 = -\theta_2 - \mu_h$$

The other eigenvalues are given as

$$\lambda_3 = -\frac{1}{2} \left( a + \eta_h + \mu_h + \mu_m + \frac{1}{\sqrt{\varphi}\sqrt{\theta_2 + \mu_h}\sqrt{\mu_m}} (\sqrt{\sigma}) \right)$$

where  $\sigma = 4k\pi q B_2 B_3 \beta_{hm} \beta_{mh} \mu_h + 4kq B_1 B_3 \beta_{hm} \beta_{mh} (\theta_2 - (-1 + \pi)\mu_h) + \varphi(\theta_2 + \mu_h)(a + \eta_h + \mu_h - \mu_m)^2 \mu_m$

$$\alpha = q^2 + 2q(\eta_A + \mu_A)\mu_m + (-6q + \eta_A^2 + \mu_A^2 + 2\eta_A(2\varphi + \mu_A))\mu_m^2 - 2(\eta_A + \mu_A)\mu_m^3 + \mu_m^4$$

Therefore the system is stable since all the eight eigenvalues are negative. This implies that at  $R_0 < 1$  the Disease-free Equilibrium point is locally asymptotically stable.

**3.5. Global Stability of Disease Free Equilibrium Point**

In this subsection, we adopt the idea of [8], to analyse the global behaviour of the equilibria for system (1). The following theorem provides the global property of the disease

free equilibrium  $E_0$  of the system. The results are obtained by means of Lyapunov function.

Theorem 1: If  $R_0 \leq 1$ , then the infection-free equilibrium is globally asymptotically stable in the interior of  $\Omega$

Proof: To determine the global stability of the disease-free equilibrium point, we construct the following Lyapunov function:

by the variation matrix  $J_{E_0}$  of the model system (1) corresponding to the disease free  $E_0$  as

when  $\sqrt{\sigma}$  is not a real number

$$\lambda_4 = -\delta_h - \mu_h, \lambda_5 = -\theta_1 - \mu_h$$

$$\lambda_6 = -\frac{1}{2} \left( a + \eta_h + \mu_h + \mu_m - \frac{1}{\sqrt{\varphi}\sqrt{\theta_2 + \mu_h}\sqrt{\mu_m}} (\sqrt{\sigma}) \right)$$

when  $\sqrt{\sigma}$  is not a real number

$$\lambda_7 = -\frac{1}{2\mu_m} (q + \mu_m(\eta_A + \mu_A + \mu_m) + \sqrt{\alpha})$$

when  $\sqrt{\alpha}$  is not a real number and finally

$$\lambda_8 = -\frac{1}{2\mu_m} (q + \mu_m(\eta_A + \mu_A + \mu_m) - \sqrt{\alpha})$$

when  $\sqrt{\alpha}$  is not a real number

$$L(t) = -kB_3\beta_{hm}tI_h(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m I_m(t) \quad (5) \quad \dot{L}(t) = -kB_3\beta_{hm}tI'_h(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m I'_m(t)$$

Calculating the time derivative of  $L$  along (5), we obtain

Then we substitute  $I'_h(t)$  &  $I'_m(t)$  from system (1) to obtain

$$\dot{L}(t) = -kB_3\beta_{hm}t \left( (B_1S_{h_1} + B_2S_{h_2})\beta_{mh} \frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h \right) + \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m \left( B_3\beta_{hm} \frac{I_h}{N_h} S_m - \mu_m I_m \right)$$

Consequently

$$\dot{L}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m \mu_m I_m \left( \frac{-kB_3\beta_{hm}t(B_1S_{h_1} + B_2S_{h_2})\beta_{mh}}{N_h\varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m\mu_m} - 1 \right) \\ (\mu_h + \eta_h + a)B_3\beta_{hm} \left( ktI_h + \varphi(\mu_h + \theta_2)\mu_m \frac{I_h}{N_h} S_m \right)$$

But  $R_0^2 = \frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m(\mu_h + \theta_2)\mu_m(\mu_h + \eta_h + a)}$  it follows that

$$\dot{L}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m \mu_m I_m \left( \frac{(B_1S_{h_1} + B_2S_{h_2})}{N_h} R_0^2 - 1 \right) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h + \theta_2)\mu_m R_0^2} \left( ktI_h + \varphi(\mu_h + \theta_2)\mu_m \frac{I_h}{N_h} S_m \right)$$

implying that

$$\dot{L}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m \mu_m I_m (\sqrt{f}R_0 + 1)(\sqrt{f}R_0 - 1) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h + \theta_2)\mu_m R_0^2} \left( ktI_h + \varphi(\mu_h + \theta_2)\mu_m \frac{I_h}{N_h} S_m \right)$$

Therefore

$$\dot{L}(t) = -\varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m \mu_m I_m (\sqrt{f}R_0 + 1)(1 - \sqrt{f}R_0) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m(\mu_h + \theta_2)\mu_m R_0^2} \left( ktI_h + \varphi(\mu_h + \theta_2)\mu_m \frac{I_h}{N_h} S_m \right)$$

where  $f = \frac{(B_1S_{h_1} + B_2S_{h_2})}{N_h}$

Thus,  $L'(t)$  is negative if  $R_0 \leq 1$  and  $L' = 0$  if and only if  $I_h = I_m = 0$  is reduced to the DFE. Consequently, the largest compact invariant set in  $\{(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m) \in \Omega, L' = 0\}$  when  $R_0 \leq 1$  is the singleton  $\{E_0\}$ . Hence, by LaSalle's invariance principle it implies that " $E_0$ " is globally asymptotically stable in  $\Omega$  [18]. This completes the proof.

### 3.6. Existence and Stability of Endemic Equilibrium

Since we are dealing with presence of dengue fever disease in human population, we can reduce system (1) to a 4-dimensional system by eliminating  $T_h, R_h, A_m$  &  $S_m$  respectively, in the feasible region  $\Omega$ . The values of  $S_m$  can be determined by setting  $S_m = mN_h - I_m$  to obtain

$$S_{h_1}^* = (m + B_3N_h^2\beta_{hm}\beta_{mh}\mu_h(B_2(2\theta_2 + (2 - \pi)\mu_h) - B_1(\theta_2 + (1 - \pi)\mu_h))) + \\ N_h\beta_{mh}((a + \eta_h + \mu_h)(B_2(2\theta_2 + \mu_h) - B_1(\theta_2 + \mu_h))\mu_m + m + (2B_2 - B_1))$$

#### 3.6.1. The Endemic Equilibrium and Its Stability

Here, we study the existence and stability of the endemic equilibrium points. If  $R_0 > 1$ , then the host-vector model system (6) has a unique endemic equilibrium given by

$$E^* = (S_{h_1}^*, S_{h_2}^*, I_h^*, I_m^*) \text{ in } \Omega \text{ with}$$

$$\frac{dS_{h_1}}{dt} = (1 - \pi)\mu_h N_h - B_1\beta_{mh} \frac{I_m}{N_h} S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2}$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh} \frac{I_m}{N_h} S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} \quad (6)$$

$$\frac{dI_h}{dt} = (B_1S_{h_1} + B_2S_{h_2})\beta_{mh} \frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h$$

$$\frac{dI_m}{dt} = B_3\beta_{hm} \frac{I_h}{N_h} S_m - \mu_m I_m$$

$$B_3\beta_{hm}\theta_1(\theta_2 + \mu_h)R_h^* + \sqrt{\left(N_h^2\beta_{mh}^2(4B_1B_2\mu_h(a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m\right. \\ \left. + \frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h + \theta_2)\mu_m R_0^2} + m + B_3\beta_{hm}(N_h\mu_h + \theta_1 R_h^*)\right) + (B_2\mu_h(-m\pi - B_3N_h\beta_{hm}\mu_h \\ + (a + \eta_h + \mu_h)\mu_m) + B_1((a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m - m - B_3\beta_{hm}(N_h\mu_h(\theta_2 - \\ (-1 + \pi)\mu_h) + \theta_1(\theta_2 + \mu_h)R_h^*)))^2\right)} / (2m + (B_2 - B_1)B_3N_h\beta_{hm}\beta_{mh}\mu_h(\theta_2 + \mu_h)),$$

$$S_{h_2}^* = \left(m + B_3N_h^2\beta_{hm}\beta_{mh}\mu_h\pi B_2\mu_h + \frac{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)R_0^2}{k\beta_{mh}t} N_h^2\beta_{mh}\mu_h B_1\right.$$

$$\left.(\theta_2 + (1 + \pi)\mu_h) + N_h\beta_{mh}\left((a + \eta_h + \mu_h)\left(B_2\mu_h + B_1\frac{kB_3\beta_{hm}\beta_{mh}t}{\mu_m(\mu_h + \eta_h + a)\varphi\mu_m R_0^2}\right)\mu_m + m + \right.$$

$$\left. \frac{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)R_0^2}{k\beta_{mh}t} B_1\theta_1(\theta_2 + \mu_h)R_h^*\right) + \sqrt{\left(N_h^2\beta_{mh}^2(4B_1B_2\mu_h(a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h + \theta_2)\mu_m R_0^2} + m + B_3\beta_{hm}(N_h\mu_h + \theta_1 R_h^*)\right) + (B_2\mu_h(-m\pi - \right.$$

$$B_3N_h\beta_{hm}\mu_h + (a + \eta_h + \mu_h)\mu_m) + B_1((a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m - m - B_3\beta_{hm}(N_h\mu_h(\theta_2 -$$

$$\left.(-1 + \pi)\mu_h) + \theta_1(\theta_2 + \mu_h)R_h^*\right))^2\right)} / (2m + (B_2 - B_1)B_3N_h\beta_{hm}\beta_{mh}\mu_h(\theta_2 + \mu_h))$$

$$I_h^* = \frac{\mu_m k \beta_{mh} t}{k \beta_{mh} t m + \varphi \mu_m^2 (\mu_h + \eta_h + a) (\mu_h + \theta_2) R_0^2}$$

$$I_m^* = \left(m + \frac{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)R_0^2}{k\beta_{mh}t} N_h^2\beta_{mh}\mu_h(\pi B_2\mu_h + B_1(\theta_2 + (1 - \pi)\mu_h)) + N_h\beta_2\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h + \theta_2)\mu_m R_0^2}(B_2\mu_h + B_1(\theta_2 + \mu_h)) + m + \frac{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)R_0^2}{k\beta_{mh}t}\right.\right.$$

$$\left. B_1\theta_1(\theta_2 + \mu_h)R_h^*\right) + \sqrt{\left(N_h^2\beta_{mh}^2\left(4B_1B_2\mu_h(a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h + \theta_2)\mu_m R_0^2}\right.\right.\right.$$

$$\left. + m + B_3\beta_{hm}(N_h\mu_h + \theta_1 R_h^*) + (B_2\mu_h(-m\pi - B_3N_h\beta_{hm}\mu_h + (a + \eta_h + \mu_h)\mu_m) + \right.$$

$$B_1((a + \eta_h + \mu_h)(\theta_2 + \mu_h)\mu_m - m - B_3\beta_{hm}(N_h\mu_h(\theta_2 - (-1 + \pi)\mu_h) +$$

$$\left. \theta_1(\theta_2 + \mu_h)R_h^*\right))^2\right)} / (2B_1B_2\beta_{mh}^2((a + \eta_h + \mu_h)\mu_m + m(\sqrt{T}R_0 + 1)(\sqrt{T}R_0 - 1)))$$

where  $T = \frac{\varphi\mu_m^2(\mu_h + \eta_h + a)(\mu_h + \theta_2)(N_h\mu_h + \theta_1 R_h^*)}{mk\beta_{mh}t}$

$$R_0^2 = \frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m(\mu_h + \theta_2)\mu_m(\mu_h + \eta_h + a)}$$

the additive compound matrices approach is used, using the idea of [19]. Local stability of the endemic equilibrium point is determined by the variational matrix  $J(E^*)$  of the nonlinear system (6) corresponding to  $E^*$  and get the matrix

### 3.6.2. Local Stability of the Endemic Equilibrium

In order to analyse the stability of the endemic equilibrium,

$$J(E^*) = \begin{pmatrix} -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} & \theta_2 & 0 & -\frac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} & 0 & -\frac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ \frac{B_1\beta_{mh}I_m^*}{N_h} & \frac{B_2\beta_{mh}I_m^*}{N_h} & -a - \eta_h - \mu_h & \frac{\beta_{mh}(B_1S_{h_1}^* + B_2S_{h_2}^*)}{N_h} \\ 0 & 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I_m^*}{N_h} & -\mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h} \end{pmatrix} \tag{7}$$



From (7) the second additive compound matrix is given by

$$J_{(E^*)}^{[2]} = \begin{bmatrix} a_{11} & 0 & -\frac{B_2\beta_{mh}S_{h_2}^*}{N_h} & 0 & \frac{B_1\beta_{mh}S_{h_1}^*}{N_h} & 0 \\ \frac{B_2\beta_{mh}I_m^*}{N_h} & a_{22} & a_{23} & \theta_2 & 0 & \frac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & mB_3\beta_1 - \frac{B_3\beta_{hm}I_m^*}{N_h} & a_{33} & 0 & \theta_2 & 0 \\ -\frac{B_1\beta_{mh}I_m^*}{N_h} & 0 & 0 & a_{44} & a_{45} & \frac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ 0 & 0 & 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I_m^*}{N_h} & a_{55} & 0 \\ 0 & 0 & \frac{B_1\beta_{mh}I_m^*}{N_h} & 0 & \frac{B_2\beta_{mh}I_m^*}{N_h} & a_{66} \end{bmatrix} \text{ in [11]}$$

where

$$a_{11} = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h}, a_{22} = -\frac{B_1\beta_{mh}I_m^*}{N_h} - a - \eta_h - 2\mu_h, a_{23} = \frac{\beta_{mh}(B_1S_{h_1}^* + B_2S_{h_2}^*)}{N_h}, a_{33} = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h}$$

$$a_{44} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - a - \eta_h - \mu_h, a_{45} = \frac{\beta_{mh}(B_1S_{h_1}^* + B_2S_{h_2}^*)}{N_h}, a_{55} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h},$$

$$a_{66} = -a - \eta_h - \mu_h - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h}$$

The following lemma was stated and proved by [20] to demonstrate the local stability of endemic equilibrium point  $E^*$ .

Lemma 3.2:

Let  $J(E^*)$  be a  $4 \times 4$  real matrix. If  $tr(J(E^*))$ ,  $\det(J(E^*))$  and  $\det(J^{[2]}(E^*))$  are all negative, then all

eigenvalues of  $J(E^*)$  have negative real parts.

Using the above Lemma, we will study the stability of the endemic equilibrium.

Theorem 3.3: If  $R_0 > 1$ , the endemic equilibrium  $E^*$  of the model (6) is locally asymptotically stable in  $\Omega$

Proof:

From the Jacobian matrix  $J(E^*)$  in (7), we have

$$tr(J(E^*)) = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - a - \eta_h - \mu_h - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h} < 0$$

$$\det(J(E^*)) = -\frac{1}{N_h^3} \left( \frac{kB_3\beta_{hm}\beta_{mh}t}{\phi\mu_m^2(\mu_h + \theta_2)R_0^2} (N_h\mu_m + B_3\beta_{hm}I_h^*)(N_h\mu_h + B_1\beta_{mh}I_m^*) \right.$$

$$\left. (N_h(\theta_2 + \mu_h) + B_2\beta_{mh}I_m^*) + B_3\beta_{hm}\beta_{mh}\mu_h(mN_h - I_m^*)(B_2N_h(\theta_2 + \mu_h)S_{h_2}^* + B_1(N_h(\theta_2 + \mu_h)S_{h_1}^* + B_2\beta_2I_m^*(S_{h_1}^* + S_{h_2}^*))) \right) < 0$$

$$\det(J^{[2]}(E^*)) = \frac{1}{N_h^6} \left( B_3\beta_{hm}\beta_{mh}(mN_h - I_m^*)(fR_0^2(mN_h - I_m^*)F + (N_h(a + \eta_h + 2\mu_h) + B_1\beta_{mh}I_m^*)(B_2\beta_{mh}I_m^*V + (N_h(a + \eta_h + \mu_h + \mu_m) + B_3\beta_{hm}I_h^*))(N_h(\mu_h + \mu_m) + B_3\beta_{hm}I_h^* + B_1\beta_{mh}I_m^*)(N_h(\theta_2 + 2\mu_h) + (B_1 + B_2)\beta_{mh}I_m^*)(B_1S_{h_1}^* + B_2S_{h_2}^*)) + \right.$$

$$\left. (\beta_{mh}B_1I_m^*B_1(N_h(\mu_h + \mu_m) + B_3\beta_{hm}I_h^*)S_{h_1}^*(\sqrt{\delta}R_0 + 1)(\sqrt{\delta}R_0 - 1) + \beta_{mh}B_1I_m^*B_2N_h\theta_2S_{h_2}^*)) \right) + (N_h(\theta_2 + \mu_h + \mu_m) + B_3\beta_{hm}I_h^* + B_2\beta_{mh}I_m^*)(G + N_h(a + \eta_h + \mu_h + \mu_m)(\sqrt{y}R_0 + 1)(\sqrt{y}R_0 - 1)(fR_0^2B_1B_2N_h\beta_{mh}\theta_2I_m^*(mN_h - I_m^*)S_{h_2}^* +$$

$$\left( N_h (a + \eta_h + \theta_2 + 2\mu_h) + B_2 \beta_{mh} I_m^* \right) \left( \left( N_h (a + \eta_h + 2\mu_h) + B_1 \beta_{mh} I_m^* \right) Q + \left( B_2^2 B_3 \beta_{hm} \beta_{mh} (mN_h - I_m^*) \beta_{mh} I_m^* S_{h_2}^* + fR_0^2 (mN_h - I_m^*) J \right) \right) < 0$$

where  $f = \frac{(\mu_h + \eta_h + a)\varphi\mu_m(\mu_h + \theta_2)\mu_m}{kt}$ ,  $\delta = \frac{B_1 I_m^* S_{h_1}^* f}{(N_h(\mu_h + \mu_m) + B_3 \beta_{hm} I_h^*) S_{h_1}^* B_3 kt \beta_{hm}}$

$$F = (B_1 (N_h (a + \eta_h + \mu_h + \mu_m) + B_3 \beta_{hm} I_h^*) S_{h_1}^* + B_1^2 \beta_{mh} I_m^* S_{h_1}^* + B_2 (N_h (a + \eta_h + \mu_h + \mu_m) + B_3 \beta_{hm} I_h^* + B_2 \beta_{mh} I_m^*) S_{h_2}^* (B_2 N_h (\theta_2 + 2\mu_h) S_{h_2}^* + B_1 (N_h (\theta_2 + 2\mu_h) S_{h_1}^* + B_2 \beta_{mh} I_m^* (S_{h_1}^* + S_{h_2}^*)))$$

$$G = B_1 B_3 \beta_{hm} \beta_{mh}^2 (mN_h - I_m^*) I_m^* (N_h (\theta_2 + 2\mu_h) + (B_1 + B_2) \beta_{mh} I_m^*) (B_1 (N_h (a + \eta_h + \theta_2 + 2\mu_h) + B_2 \beta_{mh} I_m^*) S_{h_1}^* + B_2 N_h \theta_2 S_{h_2}^*)$$

$$J = (N_h (\theta_2 + 2\mu_h) + (B_1 + B_2) \beta_{mh} I_m^*) (B_1 S_{h_1}^* + B_2 S_{h_2}^*)$$

$$V = (N_h (\theta_2 + 2\mu_h) + (B_1 + B_2) \beta_{mh} I_m^*) (B_2 (N_h (\mu_h + \mu_m) + B_3 \beta_{hm} I_h^*) + B_1 (N_h \theta_2 + B_2 \beta_{mh} I_m^*)) S_{h_2}^*$$

$$Q = (N_h (\mu_h + \mu_m) + B_3 \beta_{hm} I_h^* + B_1 \beta_{mh} I_m^*) (N_h (\theta_2 + 2\mu_h) + (B_1 + B_2) \beta_{mh} I_m^*)$$

$$y = \frac{f I_h^*}{N_h (a + \eta_h + \mu_h + \mu_m) kt \beta_{mh}}$$

$$R_0^2 = \frac{-k B_3 \beta_{hm} \beta_{mh} t}{\varphi \mu_m (\mu_h + \theta_2) \mu_m (\mu_h + \eta_h + a)}$$

**3.6.3. Global Stability of Endemic Equilibrium Point (EEP)**

Theorem 3: If  $R_0 > 1$  the endemic equilibrium  $E^*$  of the model system (1) is globally asymptotically stable

Proof: To establish the global stability of endemic equilibrium  $E^*$  we construct the following positive Lyapunov function  $V$  as follows;

Thus, from the lemma 1, the endemic equilibrium  $E^*$  of the model system (6) is locally asymptotically stable in  $\Omega$ .

$$V (S_{h_1}^*, S_{h_2}^*, I_h^*, T_h^*, R_h^*, A_m^*, S_m^* \& I_m^*) = (S_{h_1} - S_{h_1}^* \ln S_{h_1}) + (S_{h_2} - S_{h_2}^* \ln S_{h_2}) + (I_h - I_h^* \ln I_h) (T_h - T_h^* \ln T_h) + (R_h - R_h^* \ln R_h) + (A_m - A_m^* \ln A_m) + (S_m - S_m^* \ln S_m) + (I_m - I_m^* \ln I_m) \tag{8}$$

Direct calculation of the derivative of  $V$  along the solutions of (8) gives,

$$\frac{dV}{dt} (S_{h_1}^*, S_{h_2}^*, I_h^*, T_h^*, R_h^*, A_m^*, S_m^* \& I_m^*) = \left( 1 - \frac{S_{h_1}^*}{S_{h_1}} \right) \frac{dS_{h_1}}{dt} + \left( 1 - \frac{S_{h_2}^*}{S_{h_2}} \right) \frac{dS_{h_2}}{dt} + \left( 1 - \frac{I_h^*}{I_h} \right) \frac{dI_h}{dt} + \left( 1 - \frac{T_h^*}{T_h} \right) \frac{dT_h}{dt} + \left( 1 - \frac{R_h^*}{R_h} \right) \frac{dR_h}{dt} + \left( 1 - \frac{A_m^*}{A_m} \right) \frac{dA_m}{dt} + \left( 1 - \frac{S_m^*}{S_m} \right) \frac{dS_m}{dt} + \left( 1 - \frac{I_m^*}{I_m} \right) \frac{dI_m}{dt}$$

Consequently

$$\frac{dV}{dt} = \left( \frac{S_{h_1} - S_{h_1}^*}{S_{h_1}} \right) \frac{dS_{h_1}}{dt} + \left( \frac{S_{h_2} - S_{h_2}^*}{S_{h_2}} \right) \frac{dS_{h_2}}{dt} + \left( \frac{I_h - I_h^*}{I_h} \right) \frac{dI_h}{dt} + \left( \frac{T_h - T_h^*}{T_h} \right) \frac{dT_h}{dt} + \left( \frac{R_h - R_h^*}{R_h} \right) \frac{dR_h}{dt} + \left( \frac{A_m - A_m^*}{A_m} \right) \frac{dA_m}{dt} + \left( \frac{S_m - S_m^*}{S_m} \right) \frac{dS_m}{dt} + \left( \frac{I_m - I_m^*}{I_m} \right) \frac{dI_m}{dt}$$

which gives

$$\frac{dV}{dt} = A - B \tag{9}$$

where

$$\begin{aligned}
 A = & \mu_h N_h + \pi \mu_h N_h \frac{S_{h_1}^*}{S_{h_1}} + \frac{B_1 \beta_{mh} I_m^* (S_{h_1} - S_{h_1}^*)^2}{N_h S_{h_1}} + \theta_1 R_h + \theta_1 R_h^* \frac{S_{h_1}^*}{S_{h_1}} + \theta_2 S_{h_2} + \theta_2 S_{h_2}^* \frac{S_{h_1}^*}{S_{h_1}} \\
 & + \pi \mu_h N_h + \frac{B_2 \beta_{mh} I_m^* (S_{h_2} - S_{h_2}^*)^2}{N_h S_{h_2}} + \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1} + \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1}^* \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1} \frac{I_h^*}{I_h} \\
 & + \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1}^* + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* \\
 & + \eta_h I_h + \eta_h I_h^* \frac{T_h^*}{T_h} + \delta_h T_h + \delta_h T_h^* \frac{R_h^*}{R_h} + \varphi S_m + \varphi S_m^* \frac{A_m^*}{A_m} + \varphi I_m + \frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} S_m^* + \\
 & \varphi I_m^* \frac{A_m^*}{A_m} + \frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} I_m^* + \eta_A A_m + \eta_A A_m^* \frac{S_m^*}{S_m} + \frac{B_3 \beta_{hm} I_h^* (S_m - S_m^*)^2}{N_h S_m} + \\
 & \frac{B_3 \beta_{hm} I_h}{N_h} S_m + \frac{B_3 \beta_{hm} I_h^*}{N_h} S_m \frac{I_m^*}{I_m} + \frac{B_3 \beta_{hm} I_h^*}{N_h} S_m^* + \frac{B_3 \beta_{hm} I_h}{N_h} S_m^* \frac{I_m^*}{I_m} \\
 B = & -\mu_h N_h \frac{S_{h_1}^*}{S_{h_1}} - \pi \mu_h N_h - \frac{B_1 \beta_{mh} I_m (S_{h_1} - S_{h_1}^*)^2}{N_h S_{h_1}} - \mu_h \frac{(S_{h_1} - S_{h_1}^*)^2}{S_{h_1}} - \\
 & \theta_1 R_h^* - \theta_1 R_h \frac{S_{h_1}^*}{S_{h_1}} - \theta_2 S_{h_2}^* - \theta_2 S_{h_2} \frac{S_{h_1}^*}{S_{h_1}} - \pi \mu_h N_h \frac{S_{h_2}^*}{S_{h_2}} - \frac{B_2 \beta_{mh} I_m (S_{h_2} - S_{h_2}^*)^2}{N_h S_{h_2}} - \\
 & \mu_h \frac{(S_{h_2} - S_{h_2}^*)^2}{S_{h_2}} - \theta_2 \frac{(S_{h_2} - S_{h_2}^*)^2}{S_{h_2}} - \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1}^* - \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1} - \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* - \\
 & \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h} - (\mu_h + \eta_h + a) \frac{(I_h - I_h^*)^2}{I_h} - \eta_h I_h^* - \\
 & \eta_h I_h \frac{T_h^*}{T_h} - (\mu_h + \delta_h) \frac{(T_h - T_h^*)^2}{T_h} - \delta_h T_h \frac{R_h^*}{R_h} - \delta_h T_h^* - (\mu_h + \theta_1) \frac{(R_h - R_h^*)^2}{R_h} - \\
 & \varphi S_m \frac{A_m^*}{A_m} - \varphi S_m^* - \frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} S_m - \varphi I_m \frac{A_m^*}{A_m} - \varphi I_m^* - \frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} I_m - \\
 & (\mu_A + \eta_A) \frac{(A_m - A_m^*)^2}{A_m} - \eta_A A_m \frac{S_m^*}{S_m} - \eta_A A_m^* - \frac{B_3 \beta_{hm} I_h (S_m - S_m^*)^2}{N_h S_m} - \mu_m \frac{(S_m - S_m^*)^2}{S_m} - \frac{B_3 \beta_{hm} I_h}{N_h} S_m \frac{I_m^*}{I_m} - \frac{B_3 \beta_{hm} I_h}{N_h} S_m^* - \frac{B_3 \beta_{hm} I_h}{N_h} S_m^* - \\
 & \frac{B_3 \beta_{hm} I_h^*}{N_h} S_m^* \frac{I_m^*}{I_m} - \mu_m \frac{(I_m - I_m^*)^2}{I_m}
 \end{aligned}$$

Thus from equation (9), if  $A < B$  then  $\frac{dV}{dt}$  will be negative definite, meaning that  $\frac{dV}{dt} < 0$ . It follows that

$\frac{dV}{dt} = 0$  if and only if  $S_{h_1} = S_{h_1}^*, S_{h_2} = S_{h_2}^*, I_h = I_h^*, T_h = T_h^*, R_h = R_h^*, A_m = A_m^*, S_m = S_m^*$  and  $I_m = I_m^*$ . Therefore the largest compact invariant set in  $\left\{ S_{h_1}^*, S_{h_2}^*, I_h^*, T_h^*, R_h^*, A_m^*, S_m^*, I_m^* \in \Omega : \frac{dV}{dt} = 0 \right\}$  is the singleton  $\{E^*\}$  where  $E^*$  is the endemic equilibrium of the model

system (1). By LaSalle's invariant principle, then it implies that  $E^*$  is globally asymptotically stable in  $\Omega$  if  $A < B$ . This completes the proof.

### 4. Numerical Simulations

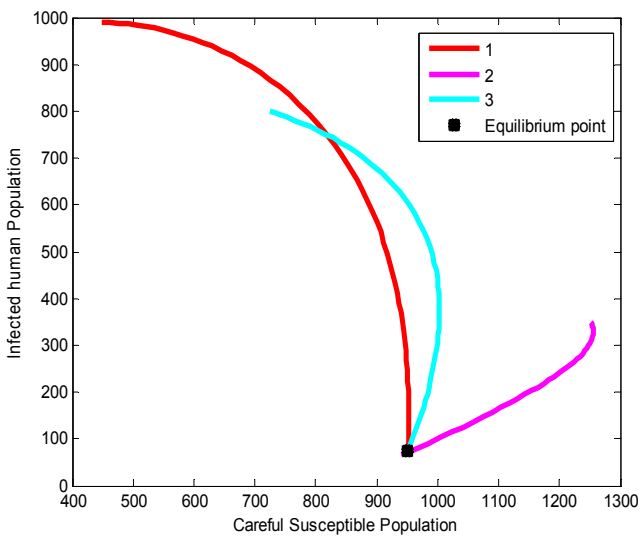
In this section, we illustrate the analytical results of the study by carrying out numerical simulations of the model system (1) using a set of reasonable parameter values. Parameter values are obtained from the different literatures like [12], [13] and [17] (<http://www.wavuti.com/2014/05/wizara-ya-afya-kitengo-cha.html>). Other parameter values are estimated to vary within realistic means and given

as shown below.

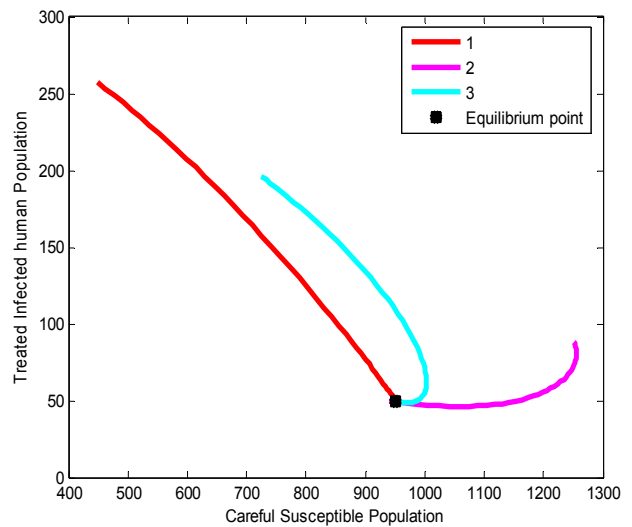
$$\begin{aligned} \beta_{hm} &= 0.375, \beta_{mh} = 0.45, \pi = 0.96, B_1 = 0.5, B_2 = 0.9, \\ B_3 &= 0.7, \mu_m = \frac{1}{11}, k = 3, \eta_A = 0.35, \mu_A = 0.25, \\ \mu_h &= \frac{1}{78 \times 365}, \eta_h = 1/3, \varphi = 5, \theta_1 = 0.01, \theta_2 = 0.6, \\ a &= 0.001, \delta_h = 0.98 \end{aligned} \quad (10)$$

1.  $S_h(0) = 951, S_{h_2}(0) = 950, I_h(0) = 63, T_h(0) = 50, R_h(0) = 49, A_m(0) = 15000, S_m(0) = 10000$  and  $I_m(0) = 5000$ .
2.  $S_h(0) = 940, S_{h_2}(0) = 945, I_h(0) = 65, T_h(0) = 50, R_h(0) = 49, A_m(0) = 10000, S_m(0) = 2000$  and  $I_m(0) = 1000$
3.  $S_h(0) = 949, S_{h_2}(0) = 950, I_h(0) = 65, T_h(0) = 50, R_h(0) = 49, A_m(0) = 20000, S_m(0) = 10000$  and  $I_m(0) = 3000$

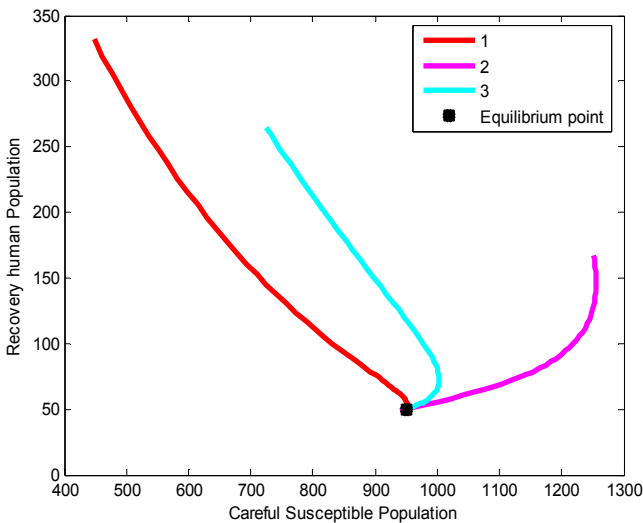
Figures 2 (i)-(vi) show the proportion of Dengue fever disease infectives, treated and recovery proportion all plotted against the proportion of susceptible population. This shows the dynamic behaviour of the endemic equilibrium of the model system (1) using the parameter values in (10) for different initial starting values in three cases as shown below [16].



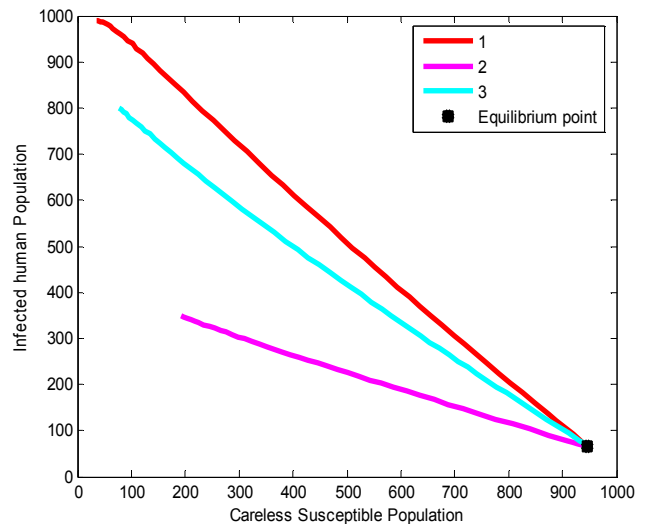
(i)



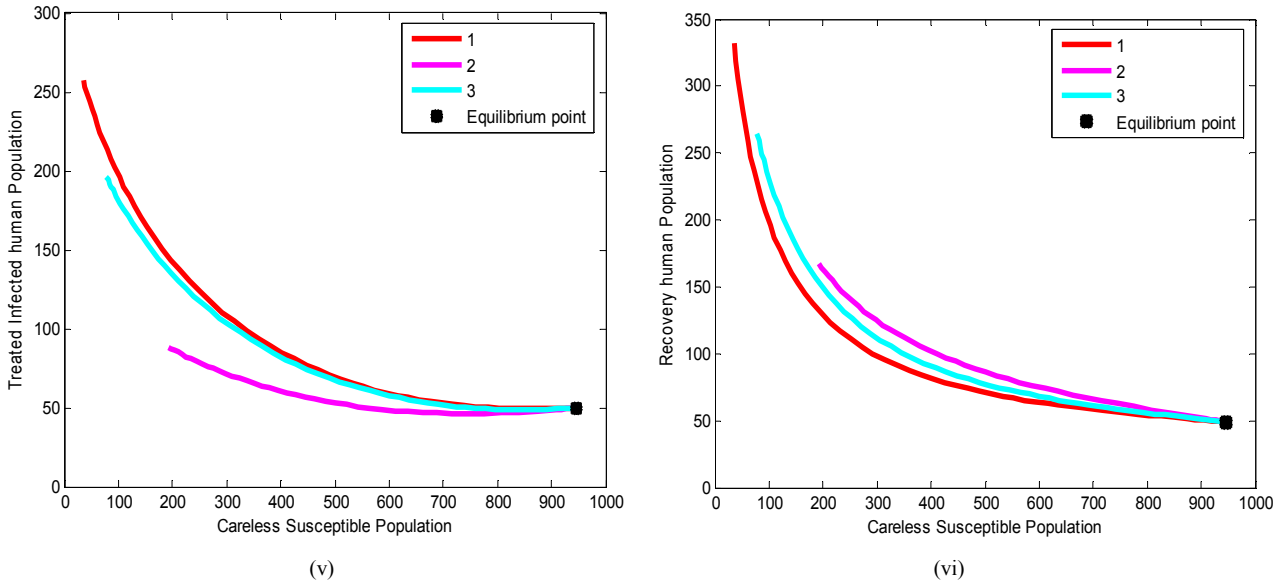
(ii)



(iii)



(iv)



Figures 2. (i)-(vi): Variation of proportion of Dengue fever disease infectives, treated and recovery proportion all plotted against the proportion of susceptible population.

The equilibrium point of the endemic equilibrium  $E^*$  was obtained as  $S_{h_1}^* = 950$ ,  $I_h^* = 75$ ,  $T_h^* = 50$  and  $R_h^* = 50$  and then  $S_{h_2}^* = 946$ ,  $I_h^* = 66$ ,  $T_h^* = 50$  and  $R_h^* = 49$

It is observed from figures 2(i)-(vi) that for any starting initial value, the solution curves tend to the equilibrium  $E^*$ . Therefore we conclude that the model system (1) is globally stable about this endemic equilibrium point  $E^*$  for the parameters displayed in (10).

Figure 3 shows the variation of population in different classes.

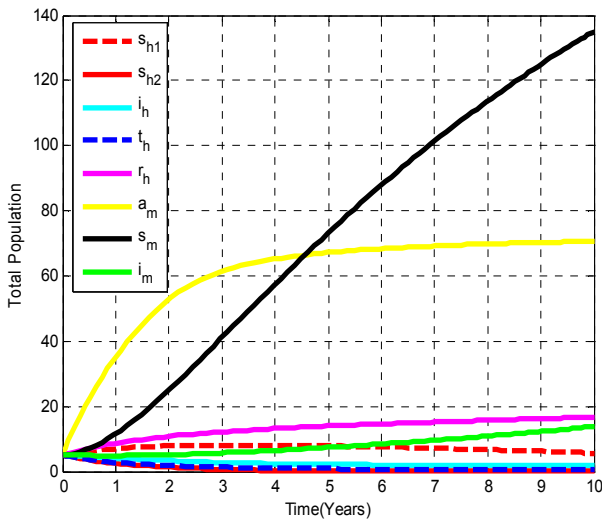
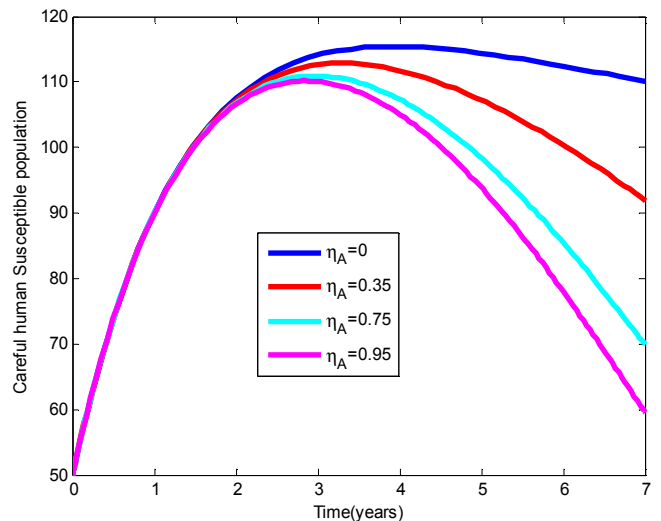


Figure 3. Distribution of population with time in all classes of human and mosquito.

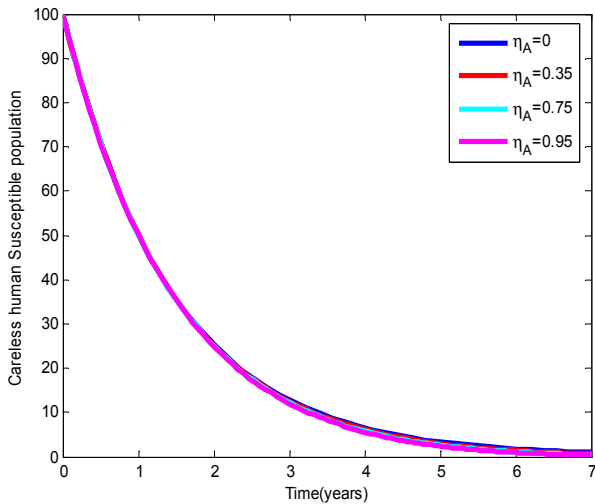
From figure 3, it is observed that careful human susceptible population increases in time reaching its equilibrium position due to treatment and change of behaviour of careless susceptible. Moreover, careless human susceptible population decreases with time, due to careless individual moving to

other classes. Dengue fever disease infected population decreases in time then reaches equilibrium due to the increase in the number of population changing behaviour and become careful and increase of recovered population. Treated infected population decrease due to the increase of the recovered population. Furthermore aquatic phase increases and then reach the equilibrium point due to its short life span and other move to susceptible class. Mosquito susceptible increases with time and reaches its equilibrium point due to its short life span and others move to infected class.

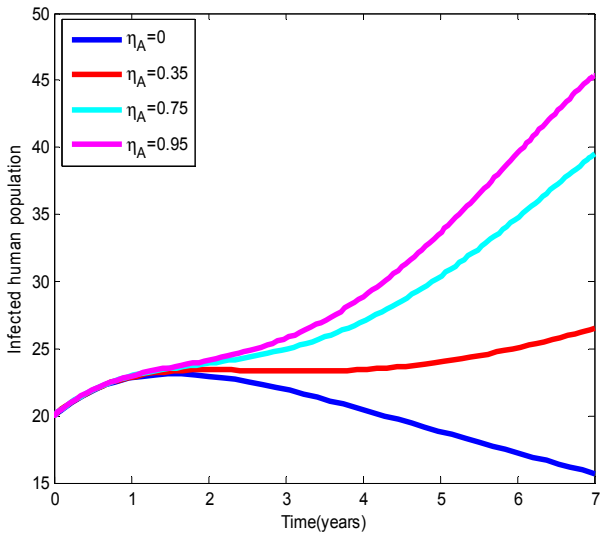
Figures 4(i)-(iv) show the variation of careful and careless human susceptible, infected human and infected mosquito population for different values of maturation rate from larvae to adult (per day)  $\eta_A$



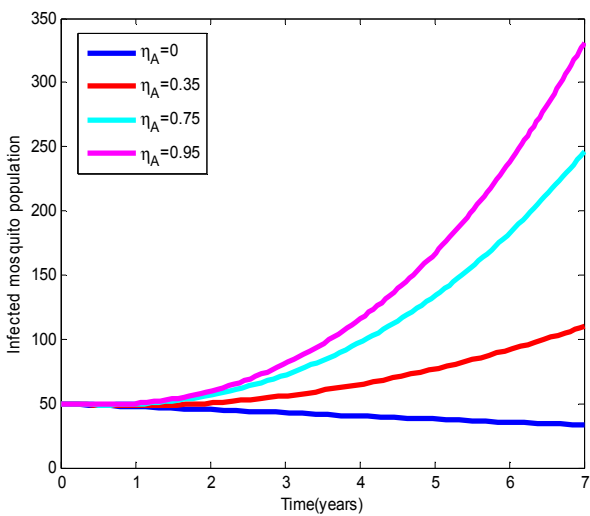
(i)



(ii)



(iii)



(iv)

**Figure 4.** (i)-(iv) variation of careful and careless human susceptible, infected human and infected mosquito population for different values of maturation rate from larvae to adult (per day)  $\eta_A$ .

From figure 4(i)-(iv) the maturation rate from larvae to adult (per day)  $\eta_A$  is varied, and it is observed that when maturation rate from larvae to adult (per day) increases, careful human susceptible increases and then decreases with time due to the increase of production of infected mosquito. Moreover careless human susceptible decrease with time while infected human and mosquito population increase.

### 5. Conclusion

A compartmental model for Dengue fever disease was presented, based on two populations, humans (with temporary immunity, careful and careless susceptible) and mosquitoes with treatment. Sensitivity analysis revealed that the most sensitive parameter is maturation rate from larvae to adult (per day). Simulation shows that when maturation rate from larvae to adult (per day) increase, the number of infected individual increase while careful and careless susceptible decrease. This indicates that on the reduction of maturation rate from larvae to adult (per day), it is possible to maintain the basic reproduction number below unity and the disease can be eradicated from the community.

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