Applied and Computational Mathematics
2015; 4(4): 225-231
Published online June 24, 2015 (http://www.sciencepublishinggroup.com/j/acm)
doi: 10.11648/j.acm.20150404.11
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

Existence Theorem for Abstract Measure Delay Integro-Differential Equations

S. S. Bellale[1], S. B. Birajdar[2], D. S. Palimkar[3]

[1]Mathematics Research Centre, Dayanand Science College, Maharashtra, India
[2]Department of Mathematics, Bidve Engineering College, Latur, Maharashtra, India
[3]Department of Mathematics, Vasantrao Naik College, Nanded, Maharashtra, India

Email address:
sidhesh.bellale@gmail.com (S. S. Bellale), lsbidave@gmail.com (S. B. Birajdar), dspalimkar@rediffmail.com (D. S. Palimkar)

To cite this article:
S. S. Bellale, S. B. Birajdar, D. S. Palimkar. Existence Theorem for Abstract Measure Delay Integro-Differential Equations. *Applied and Computational Mathematics*. Vol. 4, No. 4, 2015, pp. 225-231. doi: 10.11648/j.acm.20150404.11

Abstract: In this paper, we have proved the existence and uniqueness results for an abstract measure delay integro-differential equation by using Leray-Schauder nonlinear alternative under certain Caratheodory conditions. The various aspects of the solutions of the abstract measure integro-differential equations have been studied in the literature using the various fixed point techniques such as Schauder,s fixed point principle and Banach contraction mapping principal etc. In this paper we have proved existence and uniqueness condition for Abstract Measure delay integro-differential equations.

Keywords: Time Scale, Abstract Measure Integro-Differential Equation, Abstract Measure Delay Integro-Differential Equation, Existence Theorem and Extermal Solutions

1. Introduction

The concept of stability has been widely used by many scientists under various model formulations. Absolute stability was originally formulated by Lur'e and Postnikov and is connected between with engineering and mathematical considerations. From a mathematical point of view, one arrives at this concept from of continuity. In our study we shall be concerned with a view of such physical phenomena.

Functional integro-differential equations with delay is a hereditary system in which the rate of charge or the derivative of the unknown function or set function depends upon the past history. The functional integro–differential equations of neutral type is a hereditary system in which the derivative of the unknown function is determined by the values of a state variable as well as the derivative of the state variable over some past interval in the phase space. Although the general theory and the basic result for integro-differential equations have now been thoroughly investigated, the study of functional integro-differential equations has not been complete yet. In recent years, this has been an increasing interest for such equations among the mathematicians all over the word.

The study of abstract measure integro-differential equations is initiated and developed at length in a series of papers by Dhage [1,2,3]. The study of abstract measure delay differential equations was initiated by Joshi [9,10], Shendge and Joshi [14] and Bellale [4,5,6].

Using the approach of above mentioned papers, in this paper, we prove the existence and stability results for a abstract measure delay integro-differential equations.

2. Preliminaries

Let R denote the real line, R^n an Euclidean space with respect to the norm $|\cdot|_n$ defined by

$$|x|_n = \max\{|x_1|,.........,|x_n|\} \tag{2.1}$$

For $x = (x_1,.........,x_n) \in R^n$.

Let X be a real Banach space with any convenient norm $\|\cdot\|$. For any two points x, y in X, the segments $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \,|\, z = x + r(y-x), 0 \le r \le 1\}.$$

Let x_0 and y_0 be two fixed points in X, such that $\overline{0y_0} \subset \overline{0x_0}$, where 0 is the zero vector of X. Let z be a point of

X, such that $\overline{0x_0} \subset \overline{0z}$. For this z and $x \in \overline{y_0 z}$, define the sets S_x and $\overline{S}_x$ as follows.

$$S_x = \{rx : -\infty < r < 1\}$$
$$\overline{S}_x = \{rx : -\infty < r \le 1\}$$

For $x_1, x_2 \in \overline{y_0 z}$, we write $x_1 < x_2 \, (or\, x_2 > x_1)$ if $\overline{y_0 x_1} \subset \overline{y_0 x_2}$. Let the positive number $\|x_0 - y_0\|$ be denoted by ω. For each $x \in \overline{x_0 z}, z > x_0$, let x_ω denote that element of $\overline{y_0 z}$ which

$$x_\omega < x, \|x - x_\omega\| = \omega.$$

Note that, x_ω and ωx are not the same points unless $\omega = 0$ and $x = 0$.

Let M denote the σ-algebra of all subsets of X so that (X, M) becomes a measurable space. Let $ca(X, M)$ by

$$\|p\| = |p|_n (X) \tag{2.2}$$

Where $|p|$ is a total variation measure of p and is given by

$$|p|_n (X) = \sum_{i-1}^{\infty} |p(E_i)|_n \tag{2.3}$$

For all $E_i \subset X$ with $X = \bigcup_{i=1}^{\infty} E_i, E_i \cap E_i = \theta$ for $i \ne j$. It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (2.2). Let μ be a σ finite measure on X and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for all $E \in M$. In this case, we write $p << \mu$.

For a fixed $x_0 \in X$, let M_0 be the smallest σ-algebra on $\overline{S}_{x0}$, containing $\{x_0\}$ and the sets $S_x, x \in \overline{y_0 x_0}$. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $\overline{S}_x$ for $x \in \overline{x_0 z}$. We define the set B_H and C_H by

$$B_H = \{\mu \in R \,|\, |\mu| < H\},$$

$$B_H = \left\{p \in ca\left(\overline{S}_z, M_z\right) \middle| \, \|q\| + C < H\right\},$$

Where $H > 0$ is given (large enough) and $C > 0$.

Finally let, $L^1_\mu (S_z, R)$ denote the space of all μ integrable nonnegative real valued functions h on S_z with the norm $\|\cdot\|_{L^1_\mu}$, defined by

$$\| h \|_{L^1_\mu} = \int_{S_z} |h(x)| d\mu$$

3. Statement of the Problem

Let μ be a σ finite real measure on X. Given a $p \in AC^1(1(X, M))$ with $p << \mu$ consider the following abstract measure delay integro differentiate equation involving the delay ω,

$$\frac{dp}{d\mu} = \int_{\overline{S}x_\omega} f\left(t, p\left(\overline{S}_t\right), \int_{\overline{S}x_\omega} k\left(t, p(\overline{S}_{t_\omega}\right) d\mu\right) d\mu \tag{3.1}$$

$a.e.\{\mu\}$ on $\overline{x_o z}$

$$p(E) = q(E) \quad E \in M_0,$$

Where q is a given known vector measure, $\frac{dp}{d\mu}$ is Radon – Nikodym derivative of p with respect to μ and the function $F : S_z \times R \times R \to R$ is such that $x \to f\left(t, p\left(\overline{S}_t\right), \int_{\overline{S}t_\omega} k\left(t, p\left(\overline{S}_{t_\omega}\right)\right) d\mu\right)$ is μ integrable for each $p \in AC^1(S_z, M_z)$.

Definition 3.1 :-

Given an initial real measure q on M_0, a vector $p \in AC^1(S_z, M_z)(z > x)$ is said to be a solution of delay (1.1) if

$$p(E) = q(E), E \in M_0$$

$p << \mu$ on $\overline{x_0 z}$

Remark 3.1 :- The delay integro-Differential equation (3.1) is equivalent to the abstract measure delay integro-differential equation.

and

$$p(E) = \begin{cases} \int_E \left(\int_{\overline{S}x_\omega} f\left(t, p\left(\overline{S}_t\right), \int_{\overline{S}_{t_\omega}} k\left(t, p\left(\overline{S}_{t_\omega}\right)\right) d\mu\right) d\mu; \ E \in M_z, E \subset \overline{x_0 z} = \{q(E) \quad ; \ E \in M_0 \end{cases}$$

A solution p of delay integro-differential equation (3.1) on $\overline{x_0 z}$ will be denoted by $p\left(\overline{S}x_0, q\right)$

We apply the Schauder's fixed point theorem foe formulating the main existence result for the delay integro-differential equation (3.1). Before stating this result, we recall definition.

Definition 3.2. :-

An operator Q on a Banach space X into it self is called compact if for any bounded subset of S of X. Q (X) is a relatively compact subset of X. Q is called totally bounded if Q (S) is a totally bounded subset of X for each bounded subset S of X. If Q is continuous and totally bounded, then it is called completely continuous on X.

Every compact operator is totally bounded, but the converse may not be true however both notions coincide on bounded subset of X.

Theorem 3.1 :-

Let S be a non-empty, closed convex and bounded subset of the Banach space X and let $Q{:}S \to S$ be a continuous and compact operator. Then the operator equation $Qx = x$ has a solutions.

Now we shall prove the main existence theorem for the delay (3.1) under suitable conditions of the function R.

4. Existence and Uniqueness Theorem

Definition 4.1:-

A function $\beta : S_z \times R^n \times R^n \to R^n$ is said to satisfy conditions of Caratheodory or simply it is Caratheodory if $x \to \beta(x,y,z)$ is μ measurable for each $(y,z) \in R^n \times R^n$, and $(y,z) \to \beta(x,y,z)$ is continuous for almost everywhere μ on $x \in \overline{x_0 z}$.

Definition 4.2:-

A Caratheodory function f is called L^1_μ Caratheodory if

For each given real number $p > 0$ there exists a function $h_p \in L^1_\mu(S_z, R)$ such that

$$\left|\beta(x,y,z)\right|_n \le h_p(x) \ a.e.[\mu] x \in \overline{x_0 z},$$

for all $y, z \in R$ with $|y|_n \le p, |z|_n \le p$.

Definition 4.3 :-

A function $\beta : S_z \times R^n \times R^n \to R^n$ is called $L^1_\mu(R^n)$ Caratheodory if there exists a function $h \in L^1_\mu(S_z, R)$ such that.

$$\left|\beta(x,y,z)\right|_n \le h(x) \ a.e.[\mu] x \in \overline{x_0 z}$$

for all $y, z \in R^n$

Consider the following set of assumptions.

(A_0) $\mu(\{x_0\}) = 0$, (A_1) For any $\delta > x_0$, the σ -algebra M_z is compact with respect to topology generated by the pseudo-metric defined by

$$d(E_1 E_2) = |\mu|_n (E_1 \Delta E_2), \qquad E_1, E_2 \in M_z$$

(A_2) q is continuous on M_z with respect to the pseudo-metric d defined in (B_1)

(A_3) The function $f(x,y,z)$ is $L^1_\mu(R^n)-$ Caratheodory.

(A_4) The function f is continuous and there exist functions $l_1, l_2 \in L^1_\mu(S_z R^+)$ such

$$\left|f(x,y,z) - f(x,y_2,z_2)\right|_n \le l_1(x)|y_1 - y_2|_n + l_2(x)|z_1 - z_2|_n$$

for all $y_1, z_1\, y_2\, z_2 \in R^n$

Theorem 4.1:- Suppose the assumption $(B_1)-(B_4)$ hold. Then for a given initial measure $q \in C_H$, the delay (3.1) admits a solutions $p(\overline{S}_{xo}, q)$ on $\overline{x_0 x_1}$ for some $x_1 \in \overline{x_0 x_1}$.

Proof:-Let $\{r_n\},(r_n > 1)$ be a decreasing sequence of real numbers such that $r_n \to 1$ as $n \to \infty$ as

$$S_{ro\,xo} > S_{r_1 x_0} > \ldots\ldots > S_{r_n x_0} > \ldots\ldots\ldots$$

Then we have $\lim_{n \to \infty} \mu(\{Sr_n\, x_0 - Sx_0\}) = 0$

There fore, these exists a real number r and a point $x_1 = rx_0$ such that

$$\int w(x)\, d\mu < H - \|q\| - \|h_k\| L^1_\mu m_1$$

Where $m_1 = \mu(\overline{x_0 x_1})$. This is possible by virtue of (A_1) and the positiveness of μ.

How in the Banach space $AC^1(Sx_1, Mn_1)$ we define a subset S by

$$S = \{p \in AC^1(S_z, M_z) | p(E) = q(E) \tag{4.2}$$

if $E \in M_0$ and $\|p\| \le k\}$

Where, the constant k is given by

$$k = \|q\| + \|h_k\|_{L^1_\mu} m_1 \tag{4.3}$$

From (4.2) and (4.3) it follows that $\|p\| < H$ for all $p \in S$.

Define an operator T from S into $AC^1(Sx_1, Mx_1)$ by

$$Tp(E) = \begin{cases} \int\limits_E \left(\int_{S_{x_\omega}} f\left(t, p(\overline{S}_t), \int_{St_\omega} k\left(t, p(\overline{S}_{t_\omega})\right) d\mu\right) d\mu, \text{if } E \in M_z, E \subset \overline{x_0 z} \\ q(E), \quad \text{if } E \in M_0 \end{cases} \tag{4.4}$$

We shall show that the operator T satisfies all the conditions of theorem (3.1) on S.

Step I:- We show that + continuously maps S into itself. First show that the operator T maps S into itself. Let p t S be arbitrary and let $E \in Mx_1$. Then there are sets $F \in M_0$ and $G \in Mx_1$, $G \subset \overline{x_0 x_1}$ such that $E = FUG$.

Then

$$\left|Tp(E)\right|_n \le \left|q(F)\right|_n + \int_G \left(\int_{\bar{S}_{x\omega}} f\left(t, p\left(\bar{S}_t\right), \int_{\bar{S}_{t\omega}} k\left(t, p\left(\bar{S}_{t\omega}\right)\right) d\mu\right) d\mu \le \|q\| + \int_E \left(\int_{\bar{S}_{x\omega}} h_k(t)\, d\mu \right) d\mu \le \|q\| + \int_{\overline{x_0 x_1}} \|h\|_{L^1_\mu}\, d\mu \le \|q\| + \|h_k\|_{L^1_\mu} m_1$$

for all $E \in Mx_1$. From the definition of the norm in $AC^1(Sx_1, Mx_1)$, one has $\|Tp\| \le \|q\| + \|h_k\|_{L^1_\mu} m_1 = K$.

This show that T maps S into itself. Now we show that T is continuous on S. Let $\{p_n\}$ be a sequence of vector measures in S converging to vector measure p, that is, $\lim_{n \to \infty} \|p_n - p\| = 0$. Then for $\eta > 0$, by hypothesis (A_4), these exists a $\delta > 0$ such that

$$\|p_n - p\| < \delta \Rightarrow \left| f\left(x, p_n\left(\bar{S}_x\right), \int_{\bar{S}_{x\omega}} k\left(t, p_n\left(\bar{S}_t\right)\right) d\mu - f\left(x, p\left(\bar{S}_x\right), \int_{\bar{S}_{x\omega}} k\left(t, p\left(\bar{S}_t\right)\right) d\mu \right|\right. < \frac{\eta}{m_1^2}$$

Therefore, fore any $E \in Mx_1$,

$$\left|Tp_n(E) - Tp(E)\right|_n \le \int_E \left(\int_{S_{x\omega}} \left| t\left(t, p_n\left(\bar{S}_x\right), \int_{S_{t\omega}} k\left(t, p_n\left(\bar{S}_t\right)_\omega\right)\right| d\mu - F\left(t, p\left(\bar{S}_x\right), \int_{\bar{S}_{t\omega}} k\left(t, p\left(\bar{S}_{t\omega}\right)\right) d\mu \right| d\mu \right) d\mu$$

$$\le \int_E \left(\int_{S_{x\omega}} \frac{\eta}{m_1^2}\, d\mu \right) d\mu < \eta$$

$\|p_n - p\| < \delta.$ This shows that T is a continuous operator on S.

Step II:- Next, we show that T(S) is a uniformly bounded and set in $AC^1(Sx_1, Mx_1)$. Now as in step I, it is proved that T (S) is a subset of S and hence it is uniformly bounded set in $AC^1(Sx_1, Mx_1)$. Now by the definition of the map T are has

$$Tp(E) = \begin{cases} \int_{\bar{S}_{x\omega}} \left(\int_E f\left(t, p\left(\bar{S}_t\right), \int_{S_{t\omega}} k\left(t, p\left(\bar{S}_{t\omega}\right)\right) d\mu\right) d\mu, \text{if } E \in M_z,\ E \subset \overline{x_0 x_1} \\ q(E) \quad , \text{ if } E \in M_0 \end{cases}$$

Further we show that T (S) is an equi - continuous set in $Ac^1(Sx_1, Mx_1)$.

Let $E_1, E_2 \in M_z$ then there are sets $F_1, F_2 \in M_0$ and $G_1, G_2 \in Mx_1$ with $G_1, G_2 \subset \overline{x_0 x_1}$, and
$Fi \cap Gi = \phi, \quad i = 1, 2.$

We know that the set – identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)$$
$$G_2 = (G_2 - G_1) \cup (G_2 \cap G_1)$$

Therefore, we have

$$Tp(E_1) - Tp(E_2) = q(F_1) - q(F_2) + \int_{G_1 - G_2} \left(\int_{\bar{S}_{x\omega}} t\left(t, p(S_t), \int_{\bar{S}_{t\omega}} k\left(t, p\left(\bar{S}_{t\omega}\right)\right) d\mu\right) d\mu - \int_{G_2 - G_1} \left(\int_{\bar{S}_{x\omega}} f\left(t, p(S_t), \int_{\bar{S}_{t\omega}} k\left(t, p\left(\bar{S}_{t\omega}\right)\right) d\mu\right) d\mu$$

Since $f(x, y, z)$ is L^1_μ- Caratheodory, we have that

$$\left|Tp(E_1) - Tp(E_2)\right|_n \le \left|q(F_1) - q(F_2)\right|_n + \int_{G_1 \Delta G_2} \left(\int_{\bar{S}_{x\omega}} \left| f\left(x, p\left(\bar{S}_x\right), \int_{\bar{S}_{x\omega}} k\left(x, p_n\left(\bar{S}_{x\omega}\right)\right|_n d\mu \right) d\mu$$
$$\le \left|q(F_1) - q(F_2)\right| + \int_{G_1 \Delta G_2} \left(\int_{\bar{S}_{x\omega}} h_r(x)\, d\mu \right) d\mu$$
$$\le \left|q(F_1) - q(F_2)\right| + \int_{G_1 \Delta G_2} \|h_r\|_{L^1_\mu} d\mu$$

Assume that $d(E_1, E_2) = |\mu|_n (E_1 \Delta E_2) \to 0$.

Then we have $E_1 \to E_2$ and consequently $F_1 \to F_2$ and $|\mu|_n (G_1 \Delta G_2) \to 0.$ From the continuity of on M_0 it follows that

$$\left|Tp(E_1) - Tp(E_2)\right|_n \le \left|q(F_1) - q(F_2)\right|_n + \int_{G_1 \Delta G_2} \|h_r\|_{L^1_\mu} d\mu \to 0 \quad as\ E_1 \to E_2$$

This show that T (S) is an equi-continuous set on $AC^1(S_z, M_z)$, thus T (s) is uniformly bounded and equi-continuous set in $AC'(S_z, M_z)$ so it is compact in the norm topology on $AC^1(Sx_1, Mx_1)$. Now an application of Arzela

Ascoli Theorem yields that T (S) is a compact subset of $AC^1(Sx_1, Mx_1)$. As a result, T is a continuous and compact operator on essences, an application of theorem 3.1 yields that the operator equation $p = Tp$ has a solution is S. As a result, the delay (3.1) has a solution on $\overline{x_0x_1}$. This completes the proof.

5. Extension of Stability

A solution of the delay (3.1) so obtained can be extended to the larger segment, whenever $\mu(\{x_1\}) = 0$. An existence result in this condition is.

Theorem 5.1. Under the hypothesis of theorem (4.1) let $p(\overline{S}x_0, q)$ be solution of the delay (4.3.1) on $\overline{x_0x_1}$. Then the larger segment if $\mu(\{x_1\}) = 0$

Proof :-

Definition 5.1:-

We consider the following assumptions.

(H_1) The function $f : S_z \times B_x \to R$ is μ-integrable and $f(x,0) = 0$ for all $x \in S_z$ for some $z > x_0$.

(H_2) Given $\delta > 0$, there exists $a \in > 0$ such that

$$\left|f(x,y,z) - f(x,y_z,z_z)\right| \le \delta\left[|y_2 - z_1| + |y_2 - z_2|\right]$$

For all $y_1, y_2, z_1, z_2 \in R$ with

$$|y_1| \subseteq \varepsilon \quad , \quad |y_2| \le \varepsilon \, , |z_1| \le \varepsilon \, , \, |z_2| \le \varepsilon$$

Theorem 5.2:-

Let the assumptions $(H_1).(H_2)$ hold. Then for each $\varepsilon > 0$ and a fixed number $b \in (0,1)$, there exists a unique solution $p(\overline{S}x_0, q)$ of the delay (3.1) satisfying $\|p\| \le \varepsilon$ whenever $\|q\| \le b\varepsilon$.

Proof:- Let $\delta = \frac{1-b}{2m^2}$, where $m = \mu(\overline{x_0z})$.

Then corresponding this δ there is a number $\varepsilon > 0$ such that

$$\left|f(x,y_1,z_1) - f(x,y_2,z_2)\right| \le \delta\left[|y_1 - z_1| + |y_2 - z_2|\right] \quad (5.1)$$

For all $y_1, y_2, z_1, z_2 \in R$

With $|y_1| \le \varepsilon, |z_1| \le \varepsilon, |z_2| \le \varepsilon, |y_2| \le \varepsilon$

Define a subset

$\overline{S}(\varepsilon)$ of $AC^1(S_2, M_2)$ by

$$\overline{S}(\varepsilon) = \left\{p \in AC^1(S_2, M_2) \middle| \|p\| \le \varepsilon\right\} \quad (5.2)$$

Let $p \in \overline{S}(\varepsilon)$. Using (H_1) and (H_2) we obtain

$$\left|f\left(x, p\left(\overline{S}_x\right), \int_{S_{t_\omega}} k\left(t, p\left(\overline{S}_{t_\omega}\right)\right) d\mu\right)\right| \le 2\delta\varepsilon \quad (5.3)$$

Define an operator T from $\overline{S}(\varepsilon)$ into $AC^1(S_2, M_2)$ by

$$Tp(E) = \begin{cases} \int_E \left(\int_{S_{x\omega}} t\left(t, p\left(\overline{S}_t\right), \int_{S_{t\omega}} k\left(t, p\left(\overline{S}_{t_\omega}\right) d\mu\right) d\mu\right) d\mu & if\ E \in M_z, E \subset \overline{x_0z} \\ q(E);\ if\ E \in M_0 \end{cases} \quad (5.4)$$

Now if $E \in M_z$, then there are two disjoint sets F and G in M_z such that

$$E = F \cup G,\ F \in M_0,\ G \subset \overline{x_0z}.$$

Hence for $E \in M_z$, from (5.2) and (5.3) it follows that

$$|Tp(E)| \le |q|(E) + \int_G \left(\int_{\overline{S}_{x\omega}} \left| f\left(t, p\left(\overline{S}_t\right), \int k\left(t, p\left(\overline{S}_{t_\omega}\right) d\mu\right)\right| d\mu \right) d\mu \le \|q\| + 2\int_G \left(\int_{\overline{S}_{x\omega}} \delta \in d\mu \right) d\mu \le \|q\| + 2m^2\delta\varepsilon \le b\varepsilon + (1-b)\varepsilon = \varepsilon$$

For all $p \in \overline{S}(\varepsilon)$, where $\|q\| \le b\varepsilon$ and $\delta = \frac{1-b}{2m^2}$

Therefore $\|Tp\| \le \varepsilon$ for all $p \in \overline{S}(\varepsilon)$. This shows that T maps S (e) into itself. Next, we show that T is a contraction operator on $\overline{S}(\varepsilon)$. Let $p_1, p_2 \in \overline{S}(\varepsilon)$. Then by (H_2),

$$|Tp_1(E) - Tp_2(E)| \le \left| \int_E \left(\int_{\overline{S}_{x\omega}} \left(t, p_1\left(\overline{S}_t\right), \int k\left(t, p\left(\overline{S}_{t_\omega}\right)\right) d\mu\right) d\mu\right) d\mu - \int_E \left(\int_{S_{x\omega}} f\left(t, p_2\left(\overline{S}_t\right), \int k\left(t, p_2\left(\overline{S}_{t_\omega}\right)\right) d\mu\right) d\mu \right) d\mu \right|$$

$$\leq \int_E \left(\int_{S_{x_\omega}} \left| f\left(t, p_1\left(\overline{S}_t\right), \int k\left(t, p_1\left(\overline{S}_{t_\omega}\right)\right) - f\left(t, p_2\left(\overline{S}_t\right), \int k\left(t, p_2\left(\overline{S}_{t_\omega}\right)\right) d\mu\right) \right| d\mu \right) d\mu$$

$$\leq \int_E \left(\int_{S_{x_\omega}} \delta \left[\left| p_1\left(\overline{S}_t\right) - p_2\left(\overline{S}_t\right) \right| + \left| \int k\left(t, p_1\left(\overline{S}_{t_\omega}\right)\right) d\mu - \int k\left(t, p_2\left(\overline{S}t_\omega\right)\right) d\mu \right| \right] d\mu \right) d\mu \leq 2 \int_E \left(\int_{S_{x_\omega}} \delta \left\| p_1 - p_2 \right\| d\mu \right) d\mu$$

$$\leq 2m^2 \delta \left\| p_1 - p_2 \right\| \leq (1-b) \left\| p_1 - p_2 \right\| \tag{5.5}$$

For all $E \in M_z$, $E \subset \overline{x_0 z}$. This further implies that.

$$\left\| Tp_1 - Tp_2 \right\| \leq \alpha \left\| p_1 - p_2 \right\|$$

Where, $\alpha = (1-b) < 1$. This shows that T is a contraction operator on $\overline{S}(\varepsilon)$ with the contraction constants α. Therefore, by an application of contraction mapping principle, there is a unique solution $p\left(\overline{S}x_{(0)}q\right)$ of the delay (3.1) satisfying $\|p\| \leq \varepsilon$ whenever $\|q\| \leq b\varepsilon$. This completes the proof.

Example 5.1:- Let $X = R\mu$ the Lebesgue measure on R, $\overline{S}_x = [0, x] x > 0$ and $q(E) = \mu(E), E \subset [0,2]$. Consider the delay AMIGDE

$$\frac{dp}{d\mu} = 6 \int_{\overline{S}_{x-\frac{1}{2}}} p\left(\overline{S}_{t-1}\right) d\mu \tag{5.6}$$

and

$$p(E) = q(E), E \subset \left[0, \frac{1}{2}\right] \tag{5.7}$$

Here $\omega = \frac{1}{2}$ for $0 \leq x \leq 2$, we observe that

$$p\left(\overline{S}_x\right) = p(0, x) = q([0, x]) = x$$

If $x \in [1,2]$ then we have

$$p\left(\overline{S}_x\right) = q\left(\overline{S}_1\right) + \int_{[1,x]} \left(\int_{\left[x-t-\frac{1}{2}\right]} 6p\left(\overline{S}_{s-\frac{1}{2}}\right) ds \right) dt = 1 + 6 \int_1^x \left(\int_1^{t-\frac{1}{2}} \left(s - \frac{1}{2}\right) ds \right) dt$$

$$= 1 + \frac{6}{2} \int_1^x \left[(t-1)^2 - \frac{1}{4} \right] dt = 1 + 3 \left[\left(\frac{t-1}{3} \right)^3 - \frac{1}{4} t \right]_1^x = (x-1)^3 - \frac{3}{4} x + \frac{7}{4}$$

In this way the solution p for the linear delay AMIGDE (5.6) can be found recursively on $[0, \infty]$.

Acknowledgments

The authors are thankful to the referee for their valuable suggestions. The author S. S. Bellale is also thankful to UGC, New Delhi (MRP-Major, R.N.41-750, 2012) for their financial support.

References

[1] B. C. Dhage "On abstract measure integro-differential equations", J. Math. Phy. Sci.20 (1986), 367 – 380

[2] B. C. Dhage "On system of abstract measure integro – differential inequations and applications", Bull. Inst. Math. Acad. Sinica18 (1989), 65 – 75

[3] B. C. Dhage "Mixed monotonicity theorems for a system of abstract of measure delay integro – differential equations", An. Stint. Univ. "Al. I. Cuza" IasiXLII (1996) 355 – 366

[4] B. C. Dhage and S. S. Bellale, "Abstract measure integro – differential equations", Global Jour Math. Anal.1 (2007), 91 – 108.

[5] B. C. Dhage and S. S. Bellale, "Existence theorem for perturbd abstract measure differential equations", Nonlinear Analysis, 71(2009),e319-e328

[6] S. S. Bellale, "Hybrid Fixed Point Theorem For Abstract Measure Differential Equation", World Academy Of Science, Engineering and Technology, 73(2013) 782-785 ISSN-e2010-3778

[7] B. C. Dhage, D. N. Chate and S. K. Ntouyas, "Abstract measure differential equations", Dynamic Systems & Appl. 13 (2004) 105 – 108.

[8] J. Dagudji and A. Granas, "Fixed point Theory", Monograhie Math. PNW. Warsaw 1982.

[9] S. R. Joshi, "A system of abstract measure delay differential equations", J. Math. Phy. Sci. 13 (1979), 496 – 506.

[10] S. R. Joshi and S. G. Deo, "On abstract measure delay differential equations", An. Stint. Univ. Al I. CuzaIassiXXVI (1980), 327 – 335.

[11] W. Rudin, "Real and Complex, Analysis, McGraw", – Hill Inc.New York, 1966.

[12] R. R. Sharma, "An abstract measure differential equation", Proc, Amer, Math. Soc. 32 (1972) 503 – 510.

[13] R. R. Sharma, "A measure differential inequality with applications", Proc. Amer. Math. Soc. 48 (1975) 87 – 97.

[14] G. R. Shendge and S. R. Joshi, “Abstract measure differential inequalities and applications”, Acta Math Hung. 41 (1983), 53 – 54.

[15] D. R. Smart, “Fixed point Theorems”, Cambridge Unive. Press, Cambridge 1974.