

On a Subclass of Close-to-Convex Functions Associated with Fixed Second Coefficient

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Abstract: We consider a subclass of univalent functions $f(z)$ for which there corresponds a convex function $g(z)$ of order α such that $\operatorname{Re}\{zf'(z) / g(z)\} \geq \beta$. We investigate the influence of the second coefficient of $g(z)$ on this class. We also prove distortion, covering, and radius of convexity theorems.

Keywords: Analytic Function, Univalent Function, Convex Function of Order α , Close-to-Convexity, Fixed Second Coefficient, Radius of Convexity

1. Introduction

Let A be the class of regular functions on the unit disk $E = \{z : |z| < 1\}$. Let S be the class of univalent functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $C_b(\alpha)$ denote the subclass of S consisting of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

such that $\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \geq \alpha$, $z \in E$, $|b_2| = b$, and

$0 \leq \alpha < 1$. $C_b(\alpha)$ is the class of convex functions of order α . It is known that $0 \leq b \leq 1$ [2]. Moreover, $g(z) \in C1(\alpha)$ if and only if $g(z) = \frac{z}{1 - \epsilon z}$, $|\epsilon| = 1$.

Definition 1.1: A normalized regular function of the form (1.1) is said to belong to the class $C'_b(\alpha, \beta)$ if there exists a function $g(z) \in C_b(\alpha)$ of the form (1.2) such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \geq \beta, \quad \beta \geq 0.$$

It is clear that for $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we have $C'_b(\alpha', \beta) \subset C'_b(\alpha, \beta)$ and $C'_b(\alpha, \beta') \subset C'_b(\alpha, \beta)$.

Note that $C'_b(0, \beta) \subset C'_b(\beta)$. In [14] the author has defined a subclass C' consisting of functions $f(z)$ of the form (1.1) for which $\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0$ where $g(z) \in C$, C being the class of convex functions and obtained the inclusion relation $K^* \subset C' \subset K \subset S^*$, K^* is the subclass known as the class of quasi-convex functions introduced and studied by K.I. Noor [12]. Thus $C'_b(\alpha, \beta) \subset C'$ and hence every member of $C'_b(\alpha, \beta)$ is univalent. By specializing α and β we obtain some important subclasses. If $f(z) \in C'_b(0, 0)$ then $f(z)$ is close-to-convex; if $f(z) \in C'_b(1, \beta)$, then $\operatorname{Re}\{f'(z)\} \geq \beta$ that is in $R(\beta)$ as in [13]; if $f(z) \in C'_b(\alpha, 1)$ then $f(z) = g(z)$ so that $f(z)$ is convex of order α . If $f(z) \in C'_b(1, 1)$ then $\operatorname{Re}\{f'(z)\} \geq 0$ so $g(z) = z$.

In this paper we prove distortion, covering and radius of convexity theorems for the class $C'_b(\alpha, \beta)$.

In what follows, we assume $f(z)$ is in $C'_b(\alpha, \beta)$ with

$g(z)$ its associated function in $C_b(\alpha)$.

First we give an example for $f(z) \in C'_b(\alpha, \beta)$ by using the following.

Lemma: [16] Let $Q(z)$ be analytic for $z \in \mathbb{C}$ with $Q(0)=1$. Then $\text{Re } Q(z) \geq \beta$ if and only if

$$Q(z) = \frac{1+(1-2\beta)G(z)}{1-G(z)}, \text{ where } G(z) \text{ is analytic,}$$

$G(0) = 0$, and $|G(z)| < 1$ for $z \in E$.

Example: Let

$$f(z) = \frac{1}{2} \int_0^z \frac{1}{t} \left[t\psi(t) + \int_0^t \psi(t)dt \right] \left(\frac{1+(1-2\beta t)}{1-t} \right) dt \text{ and}$$

$$g(z) = \frac{1}{2} \left[z\psi(z) + \int_0^z \psi(t)dt \right] = \frac{1}{2} \left(z \int_0^z \psi(t)dt \right)' \text{ where}$$

$$\psi(t) = \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^{\frac{2b}{3}}.$$

Since

$$\frac{zf'(z)}{g(z)} = \frac{1+(1-2\beta)z}{1-z} \tag{1.3}$$

has real part $\geq \beta$, it suffices to show that $g(z) \in C_b(\alpha)$.

If $\phi(z) = \int_0^z \psi(t)dt$, then we have

$$1 + \frac{z\phi''(z)}{\phi'(z)} = \frac{1 + \frac{4b}{3}z + (1-2\alpha)z^2}{1-z^2} = \frac{1+(1-2\alpha)G(z)}{1-G(z)}.$$

Solving for $G(z)$, we obtain

$$G(z) = \frac{z \left[z + \frac{2b}{3(1-\alpha)} \right]}{\left[1 + \frac{2b}{3(1-\alpha)}z \right]} = zh(z).$$

Since $\alpha + b \leq 1$, $h(z)$ maps $E \rightarrow E$, and $|G(z)| \leq |z| < 1$ for $z \in E$. Since $G(z)$ satisfies the conditions of Lemma, $\phi(z) \in C_b(\alpha)$. So,

$$g(z) = \frac{1}{2} \left[z\psi(z) + \int_0^z \psi(t)dt \right] = \frac{1}{2} \left(z \int_0^z \psi(t)dt \right)'$$

is convex for $|z| < \frac{1}{2}$ by Livingston [12]. Thus existence of $f(z) \in C'_b(\alpha, \beta)$ is asserted.

2. Distortion Theorem for $C'_b(\alpha, \beta)$

Theorem 2.1: Let $f(z) \in C'_b(\alpha, \beta)$. Then

$$|f'(z)| \leq \frac{1+(1-2\beta)r}{r(1-r)} \int_0^r \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt \tag{2.1}$$

and

$$|f'(z)| \geq \frac{1-(1-2\beta)r}{r(1+r)} \int_0^r \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{1-\alpha} dt \tag{2.2}$$

where the integrand on the right hand side of (2.2) is taken to be 1 for $\alpha=1$.

Equality holds in (2.1) for the function

$$f_1(z) = \int_0^z \frac{1+(1-2\beta)s}{s(1-s)} \int_0^s \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt ds$$

and equality holds in (2.2) for the function

$$f_2(z) = \int_0^z \frac{1-(1-2\beta)s}{s(1+s)} \int_0^s \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{1-\alpha} dt ds.$$

Proof: From (1.3), by Lemma we obtain

$$\frac{zf'(z)}{g(z)} = \frac{1+(1-2\beta)G(z)}{1-G(z)} \tag{2.3}$$

where $G(0) = 0$ and $|G(z)| < 1$ for $z \in E$. Since $G(z)$ satisfies the conditions of Schwarz's lemma, (1.6) yields

$$\frac{1-(1-2\beta)r}{1+r} \leq \left| \frac{zf'(z)}{g(z)} \right| \leq \frac{1+(1-2\beta)r}{1-r} \tag{2.4}$$

That is, $\frac{1-(1-2\beta)r}{r(1+r)} \leq \left| \frac{f'(z)}{g(z)} \right| \leq \frac{1+(1-2\beta)r}{r(1-r)}$.

We have [2, p. 105],

$$\int_0^r \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{\alpha-1} dt \leq |g(z)| \leq \int_0^r \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt. \tag{2.5}$$

Combining (1.7) and (1.8), the result follows. Clearly $f_1(z)$ and $f_2(z) \in C_b(\alpha, \beta)$ with

$$g_1(z) = \int_0^z \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt \text{ and}$$

$$g_2(z) = \int_0^z \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{\alpha-1} dt.$$

Theorem 2.2. Let $f(z) \in C'_b(\alpha, \beta)$. Then

$$\begin{aligned} \int_0^r \frac{1-(1-2\beta)s}{s(1-s)} \left[\int_0^s \left(1 + \frac{2bt}{1-\alpha} + t^2 \right)^{\alpha-1} dt \right] ds &\leq |f(z)| \\ &\leq \int_0^r \frac{1+(1-2\beta)s}{s(1+s)} \left[\int_0^s \left(\frac{1}{1-t^2} \right)^{1-\alpha} \left(\frac{1+t}{1-t} \right)^b dt \right] ds. \end{aligned}$$

Equality holds on the right-hand side for $f_1(z)$ in Theorem 2.1 and on the left-hand side for $f_2(z)$ in Theorem 2.1.

Proof: Integrating along the straight line segment from the origin to $z = re^{i\theta}$ and applying Theorem 2.1 we obtain

$$|f(z)| \leq \int_0^r |f'(se^{i\theta})| ds \leq \int_0^r \frac{1+(1-2\beta)s}{s(1+s)} \left[\int_0^s \left(\frac{1}{1-t^2}\right)^{1-\alpha} \left(\frac{1+t}{1-t}\right)^b dt \right] ds,$$

which proves the right-hand inequality.

To prove the left-hand inequality, for every r we choose $z_0, |z_0|=r$, such that

$$|f(z_0)| = \min_{|z|=r} |f(z)|.$$

If $L(z_0)$ is the pre-image of segment $[0, f(z_0)]$, then

$$|f(z)| \geq |f(z_0)| \geq \int_{L(z_0)} |f'(z)| |dz| \geq \int_0^r \frac{1-(1-2\beta)s}{s(1-s)} \left[\int_0^s \left(1 + \frac{2bt}{1-\alpha} + t^2\right)^{\alpha-1} dt \right] ds.$$

This completes the proof.

3. Covering Theorem for $C'_b(\alpha, \beta)$

Theorem 3.1: Let $f(z) \in C'_b(\alpha, \beta)$ with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \text{ If } f(z) \neq w \text{ for } z \in E, \text{ then}$$

$$|w| \geq 2/(6-2\beta+b).$$

Proof: First we establish that $|a_2| \leq 1-\beta + \frac{b}{2}$.

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, for

$$\phi(z) = \frac{1}{1-\beta} \left[\frac{zf'(z)}{g(z)} - \beta \right] = 1 + \frac{(2a_2 - b_2)z}{1-\beta} + \sum_{n=2}^{\infty} c_n z^n,$$

we have [9, p. 15],

$$\left| \frac{2a_2 - b_2}{1-\beta} \right| \leq 2, \text{ so } |a_2| \leq 1-\beta + \frac{b}{2}. \tag{3.1}$$

Since $f(z)$ does not assume the value w ,

$$\frac{wf(z)}{w-f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \dots$$

is in the class S . Therefore

$$\left| a_2 + \frac{1}{w} \right| \leq 2. \tag{3.2}$$

Now, using (3.1) and (3.2),

$$\left| \frac{1}{w} \right| = \left| a_2 + \frac{1}{w} - a_2 \right| \leq 2 + 1 - \beta + \frac{b}{2},$$

and this completes the proof.

4. A Radius of Convexity Theorem for $C'_b(\alpha, \beta)$

Lemma 4.1: If $f \in C'_b(\beta)$, then $f(z)$ maps the disk $|z| < r_1$ onto a convex domain, where r_1 is the least positive root of the equation $\gamma(r, b, \beta) = 0$, where

$$\gamma(r, b, \beta) = [1+(1-2\beta)r](1+r)^b + 2(1-\beta)r(1-r)^b.$$

Proof: Let $f(z) \in C'_b(\beta)$. Then for $g(z) \in C_b(0)$, $\frac{zf'(z)}{g(z)} = Q(z)$ with $\text{Re}\{\phi(z)\} \geq \beta$.

From $zf'(z) = g(z)Q(z)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zQ'(z)}{Q(z)}. \tag{4.1}$$

So, the radius of convexity of $f(z)$ is at least equal to the smallest positive root of

$$\min \text{Re} \left(\frac{zg'(z)}{g(z)} \right) + \min \text{Re} \left(\frac{zQ'(z)}{Q(z)} \right) = 0.$$

For $\alpha = 0$, from the inequalities in [2, p. 104] and [14, p. 384] we obtain

$$\text{Re} \left(\frac{zg'(z)}{g(z)} \right) \geq - \left| \frac{zg'(z)}{g(z)} \right| \geq - \frac{(1+r)^b}{(1-r)^{b+1}}. \tag{4.2}$$

Now, let $Q(z) = (1-\beta)P(z) + \beta$, where $P(z)$ is analytic, $P(0) = 1$, and $\text{Re}\{P(z)\} > 0$ in E .

Then $\frac{zQ'(z)}{Q(z)} = \frac{P'(z)}{P(z) + \frac{\beta}{1-\beta}}$. Using the lemma of Libera [11, p. 150] we obtain

$$\left| \frac{zQ'(z)}{Q(z)} \right| \leq \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}. \tag{4.3}$$

Using (4.2) and (4.3) in (4.1) we get

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &\geq -\frac{(1+r)^b}{(1-r)^{b+1}} - \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]} \\ &\geq -\frac{\gamma(r,b,\beta)}{(1-r)^{b+1}[1+(1-2\beta)r]}. \end{aligned}$$

Hence, $f(z)$ is convex in $|z| < r_1$ where r_1 is the least positive root of the equation $\gamma(r, b, \beta) = 0$ for given b, β . This lemma improves the result obtained in [16].

Theorem 4.2: If $f(z) \in C'_b(\alpha, \beta)$, then $f(z)$ maps the disk $|z| < R$ onto a convex domain, where R is the least positive root of the equation $\delta(r, b, \alpha, \beta) = 0$, where

$$\begin{aligned} \delta(r, b, \alpha, \beta) &= (1-2\alpha)(1-2\beta)r(1+r)^{b+\alpha-1}(1-r)^{\alpha-b} \\ &\quad + (1-2\alpha)(1+r)^{b+\alpha-1}(1-r)^{\alpha-b} \\ &\quad - 2(1-\beta)(1+r)^{2\alpha-1} + 2(1-\beta). \end{aligned}$$

Proof: Let $f(z) \in C'_b(\alpha, \beta)$. Then for $\frac{zf'(z)}{g(z)} = Q(z)$ with $\operatorname{Re}\{\phi(z)\} \geq \beta$.

From $zf'(z) = g(z)Q(z)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zQ'(z)}{Q(z)}. \tag{4.4}$$

So, the radius of convexity of $f(z)$ is at least equal to the smallest positive root of

$$\min \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) + \min \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 0.$$

Using the inequalities in [2, p. 104] and [14, p. 384] we obtain

$$\begin{aligned} \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) &\geq -\left|\frac{zg'(z)}{g(z)}\right| \\ &\geq -\frac{(2\alpha-1)r(1+r)^b}{(1-r^2)^{1-\alpha}(1-r)^b[(1+r)^{2\alpha-1}-1]} \end{aligned} \tag{4.5}$$

Now, let $Q(z) = (1-\beta)P(z) + \beta$, where $P(z)$ is analytic, $P(0) = 1$, and $\operatorname{Re}\{P(z)\} > 0$ in E .

Then $\frac{zQ'(z)}{Q(z)} = \frac{P'(z)}{P(z) + \frac{\beta}{(1-\beta)}}$. Using the lemma of Libera [11, p. 150] we obtain

$$\left|\frac{zQ'(z)}{Q(z)}\right| \leq \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]}. \tag{4.6}$$

Using (4.5) and (4.6) in (4.4) we get

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &\geq -\frac{(2\alpha-1)r(1+r)^b}{(1-r^2)^{1-\alpha}(1-r)^b[(1+r)^{2\alpha-1}-1]} \\ &\quad - \frac{2(1-\beta)r}{(1-r)[1+(1-2\beta)r]} \\ &\geq -\frac{r\delta(r,b,\alpha,\beta)}{(1-r)[(1+r)^{2\alpha-1}][1+(1-2\beta)r]}. \end{aligned}$$

Hence, $f(z)$ is convex in $|z| < R$ where R is the least positive root of the equation $\delta(r, b, \alpha, \beta) = 0$, for given b, α, β .

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