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Asymptotic Method of Krylov-Bogoliubov-Mitropolskii for Fifth Order Critically Damped Nonlinear Systems

Md. Nazrul Islam[1], Md. Mahafujur Rahaman[2, *], M. Abul Kawser[1]

[1]Department of Mathematics, Islamic University, Kushtia, Bangladesh
[2]Department of Computer Science & Engineering, Z. H. Sikder University of Science & Technology, Shariatpur, Bangladesh

Email address:
nazrul001bd@gmail.com (Md. N. Islam), mahfuz0809@gmail.com (Md. M. Rahaman), kawserma@yahoo.com (M. A. Kawser)

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Abstract: In oscillatory problems, the method of Krylov–Bogoliubov–Mitropolskii (KBM) is one of the most used techniques to obtain analytical approximate solution of nonlinear systems with a small non-linearity. This article modifies the KBM method to examine the solutions of fifth order critically damped nonlinear systems with four pairwise equal eigenvalues and one distinct eigenvalue, in which the latter eigenvalue is much larger than the former four pairwise eigenvalues. This paper suggests that the results obtained in this study correspond accurately to the numerical solutions obtained by the fourth order Runge-Kutta method. This paper, therefore, concludes that the modified KBM method provides highly accurate results, which can be applied for different kinds of nonlinear differential systems.

Keywords: KBM, Asymptotic Method, Critically Damped System, Nonlinearity, Runge-Kutta Method, Eigenvalues

1. Introduction

In oscillatory problems, the method of Krylov–Bogoliubov–Mitropolskii (KBM) [1, 2] is particularly convenient, and is the vastly used technique to obtain analytical approximate solution of nonlinear systems with a small non-linearity. The method was, in fact, developed by Krylov and Bogoliubov [2] for obtaining periodic solutions, which was amplified and justified by Bogoliubov and Mitropolskii [1], and later extended by Popov [3] and Meldelson [4] for damped nonlinear oscillations. Murty [5] developed a unified KBM method for solving second-order nonlinear systems. Sattar [6] studied a third-order over-damped nonlinear system. Bojadziev [7] studied the damped oscillations modeled by a three dimensional nonlinear system. Shamsul and Sattar [8] developed a method for third order critically-damped nonlinear equations. Rokibul and Akbar [9] investigated a new solution of third order more critically damped nonlinear systems. Shamsul and Sattar [10] presented a unified KBM method for solving third-order nonlinear systems. Akbar *et al*. [11] presented a method for solving the fourth-order over-damped nonlinear systems. Rokibul *et al*. [12] presented a new technique for fourth order critically damped nonlinear systems with some conditions. Rahaman and Rahman [13] suggested analytical approximate solutions of fifth order more critically damped systems in the case of smaller triply repeated roots. Rahaman and Kawser [14] also proposed asymptotic solutions of fifth order critically damped nonlinear systems with pairwise equal eigenvalues and another is distinct.

The aim of this article is to obtain the analytical approximate solutions of fifth order critically damped nonlinear systems by extending the KBM method. In this study, it is suggested that the results obtained by the perturbation solution have been compared with those obtained by the fourth order *Runge–Kutta* method.

2. The Method

We are going to propose a perturbation technique to solve fifth order non-linear differential systems of the form

$$x^{(v)} + k_1 x^{(iv)} + k_2 \dddot{x} + k_3 \ddot{x} + k_4 \dot{x} + k_5 x = -\varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}) \quad (1)$$

where $x^{(v)}$ and $x^{(iv)}$ stand for the fifth and fourth derivatives respectively, and over dots are used for the first, second and third derivatives of x with respect to t; k_1, k_2, k_3, k_4, k_5 are constants, ε is a sufficiently small parameter and $f\left(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}\right)$ is the given nonlinear function. As the unperturbed equation (1) has five real negative eigenvalues, where four eigenvalues are pairwise equal and the other one is distinct. Here, the distinct eigenvalue is much larger than the pairwise equal eigenvalues. Now, suppose that the eigenvalues are $-\lambda, -\lambda, -\mu, -\mu, -\nu$.

When $\varepsilon = 0$, the equation (1) becomes linear and the solution of the corresponding linear equation becomes

$$x(t,0) = (a_0 + b_0 t)e^{-\lambda t} + (c_0 + d_0 t)e^{-\mu t} + h_0 e^{-\nu t} \tag{2}$$

where a_0, b_0, c_0, d_0 and h_0 are constants of integration.

When $\varepsilon \neq 0$ following Shamsul [15], an asymptotic solution of the equation (1) is sought in the form

$$x(t,\varepsilon) = (a + bt)e^{-\lambda t} + (c + dt)e^{-\mu t} + he^{-\nu t} + \varepsilon u_1(a,b,c,d,h,t) + \dots \tag{3}$$

where *a*, *b*, *c*, *d* and *h* are the functions of *t* and they satisfy the first order differential equations

$$\dot{a} = \varepsilon A_1(a,b,c,d,h,t) + \dots \quad \dot{b} = \varepsilon B_1(a,b,c,d,h,t) + \dots$$
$$\dot{c} = \varepsilon C_1(a,b,c,d,h,t) + \dots \tag{4}$$
$$\dot{d} = \varepsilon D_1(a,b,c,d,h,t) + \dots \quad \dot{h} = \varepsilon H_1(a,b,c,d,h,t) + \dots$$

Now differentiating (3) five times with respect to *t*, substituting the value of *x* and the derivatives $x^{(v)}, x^{(iv)}, \dddot{x}, \ddot{x}, \dot{x}$ in the original equation (1) utilizing the relations presented in (4) and finally extracting the coefficients of ε, we obtain

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\begin{pmatrix}\dfrac{\partial A_1}{\partial t}+2B_1\\ +t\dfrac{\partial B_1}{\partial t}\end{pmatrix}+\\ &e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\begin{pmatrix}\dfrac{\partial C_1}{\partial t}+2D_1\\ +t\dfrac{\partial D_1}{\partial t}\end{pmatrix}+\\ &e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2 H_1+\\ &(D+\lambda)^2(D+\mu)^2(D+\nu)u_1 = -f^0(a,b,c,d,h,t)\end{aligned} \tag{5}$$

Where $f^{(0)}(a,b,c,d,h,t) = f(x,\dot{x},\ddot{x},\dddot{x},x^{iv})$

And $x(t,0) = (a_0 + b_0 t)e^{-\lambda t} + (c_0 + d_0 t)e^{-\mu t} + h_0 e^{-\nu t}$

We have expanded the function $f^{(0)}$ in the Taylor's series (Sattar [16], Shamsul [17, 18], Shamsul and Sattar [8]) about the origin in power of *t*. Therefore, we obtain

$$f^{(0)} = \sum_{q=0}^{\infty}\left\{t^q \sum_{i,j,k,l=0}^{\infty} F_{q,l}(a,b,c,d,h)\, e^{-(i\lambda+j\mu+k\nu)t}\right\} \tag{6}$$

Thus, using (6), the equation (5) becomes

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\begin{pmatrix}\dfrac{\partial A_1}{\partial t}+2B_1\\ +t\dfrac{\partial B_1}{\partial t}\end{pmatrix}+\\ &e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\begin{pmatrix}\dfrac{\partial C_1}{\partial t}+\\ 2D_1+t\dfrac{\partial D_1}{\partial t}\end{pmatrix}+\\ &e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2 H_1+\\ &(D+\lambda)^2(D+\mu)^2(D+\nu)u_1 =\\ &-\sum_{q=0}^{\infty}\left\{t^q \sum_{i,j,k=0}^{\infty}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}\end{aligned} \tag{7}$$

Following the KBM method, Murty and Deekshatulu [19], Sattar [16], Shamsul [18], and Shamsul and Sattar [8, 20] imposed the condition that u_1 does not contain the fundamental terms (the solution (2) is called the generating solution and its terms are called the fundamental terms) of $f^{(0)}$. Therefore, equation (7) can be separated for unknown functions A_1, B_1, C_1, D_1, H_1 and u_1 in the following way:

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\begin{pmatrix}\dfrac{\partial A_1}{\partial t}+2B_1\\ +t\dfrac{\partial B_1}{\partial t}\end{pmatrix}+\\ &e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\begin{pmatrix}\dfrac{\partial C_1}{\partial t}+2D_1\\ +t\dfrac{\partial D_1}{\partial t}\end{pmatrix}+\\ &e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2 H_1 =\\ &-\sum_{q=0}^{1}\left\{t^q \sum_{i,j,k,l=0}^{\infty} F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}\end{aligned} \tag{8}$$

$$\begin{aligned} &(D+\lambda)^2(D+\mu)^2(D+\nu)u_1 =\\ &-\sum_{q=2}^{\infty}\left\{t^q \sum_{i,j,k,l=0}^{\infty} F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}\end{aligned} \tag{9}$$

Now equating the coefficients of t^0, t^1 from equation (8), we obtain

$$\begin{aligned} &e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1\right)\\ &+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1\right)\\ &+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2 H_1 = -\\ &\sum_{i,j,k,l=0}^{\infty} F_{0,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\end{aligned} \tag{10}$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}+e^{-\mu t}$$
$$(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}=-\sum_{i,j,k,l=0}^{\infty}F_{1,1}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t} \tag{11}$$

Here, we have only two equations (10) and (11) for determining the unknown functions A_1, B_1, C_1 , D_1 and H_1 Thus, to obtain the unknown functions A_1, B_1, C_1, D_1 and H_1, we need to impose some conditions (Shamsul [18, 20, 21, 23]) between the eigenvalues. Different authors have imposed different conditions according to the behavior of the systems, such as Shamsul [21] imposed the condition

$$i_1\lambda_1+i_2\lambda_2+...+i_n\lambda_n\le(i_1+i_1+...+i_1)(\lambda_1+\lambda_2+...+\lambda_n)$$

In this study, we have investigated solutions for the cases $\lambda>>\mu$ and $\lambda<<\nu$. Therefore, we shall be able to separate the equation (11) for unknown functions B_1 and D_1; and solving them for B_1 and D_1 substituting the values of B_1 and D_1 into the equation (11) and applying the conditions $\lambda>>\mu$ and $\lambda<<\nu$, we can separate the equation (12) for three unknown functions A_1, C_1 and H_1; and solving them for A_1, C_1 and H_1. Since $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$ and $\dot{h}$ are proportional to small parameter, they are slowly varying functions of time t, and for first approximate solution, we may consider them as constants in the right side. This assumption was first made by Murty and Deekshatulu [19]. Thus, the solutions of the equation (4) become

$$\begin{aligned} a&=a_0+\varepsilon\int_0^t A_1(a,b,c,d,h,t)dt\\ b&=b_0+\varepsilon\int_0^t B_1(a,b,c,d,h,t)dt\\ c&=c_0+\varepsilon\int_0^t C_1(a,b,c,d,h,t)dt\\ d&=d_0+\varepsilon\int_0^t D_1(a,b,c,d,h,t)dt\\ h&=h_0+\varepsilon\int_0^t H_1(a,b,c,d,h,t)dt \end{aligned} \tag{12}$$

Equation (9) is a non-homogeneous linear ordinary differential equation; therefore, it can be solved by the well-known operator method. Substituting the values of *a, b, c, d, h* and u_1 in the equations (3), we shall get the complete solution of (1). Therefore, the determination of the first approximate solution is complete.

3. Example

As an example of the above method, we have considered the Duffing type equation of fifth order nonlinear differential system:

$$x^{(v)}+k_1x^{(iv)}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\varepsilon x^3 \tag{13}$$

Comparing equation (13) and equation (1), we obtain $f(x,\dot{x},\ddot{x},\dddot{x},x^{(iv)})=x^3$

Therefore,

$$\begin{aligned} f^{(0)}&=a^3e^{-3\lambda t}+3a^2ce^{-(2\lambda+\mu)t}+3a^2he^{-(2\lambda+\nu)t}\\ &+3ac^2e^{-(\lambda+2\mu)t}+6ache^{-(\lambda+\mu+\nu)t}+3ah^2\\ &e^{-(\lambda+2\nu)t}+c^3e^{-3\mu t}+3c^2he^{-(2\mu+\nu)t}+3ch^2\\ &e^{-(\mu+2\nu)t}+h^3e^{-3\nu t}+3t\{a^2be^{-3\lambda t}+2abce^{-(2\lambda+\mu)t}\\ &+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}+2acde^{-(\lambda+2\mu)t}\\ &+bc^2e^{-(\lambda+2\mu)t}+2adhe^{-(\lambda+\mu+\nu)t}+2bche^{-(\lambda+\mu+\nu)t}\\ &+bh^2e^{-(\lambda+2\nu)t}+c^2de^{-3\mu t}+2cdhe^{-(2\mu+\nu)t}+dh^2\\ &e^{-(\mu+2\nu)t}\}+3t^2\{ab^2e^{-3\lambda t}+b^2ce^{-(2\lambda+\mu)t}+2abd\\ &e^{-(2\lambda+\mu)t}+b^2he^{-(2\lambda+\nu)t}+ad^2e^{-(\lambda+2\mu)t}+2bcd\\ &e^{-(\lambda+2\mu)t}+2bdhe^{-(\lambda+\mu+\nu)t}+cd^2e^{-3\mu t}+d^2h\\ &e^{-(2\mu+\nu)t}\}+t^3\{b^3e^{-3\lambda t}+b^2de^{-(2\lambda+\mu)t}+bd^2\\ &e^{-(\lambda+2\mu)t}+d^3e^{-3\mu t}\} \end{aligned} \tag{14}$$

Now comparing equations (6) and (14), we obtain

$$\begin{aligned} \sum_{i,j,k,l=0}^{\infty}F_{0,1}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}&=a^3e^{-3\lambda t}\\ +3a^2ce^{-(2\lambda+\mu)t}+3a^2he^{-(2\lambda+\nu)t}+3ac^2&\\ e^{-(\lambda+2\mu)t}+6ache^{-(\lambda+\mu+\nu)t}+3ah^2e^{-(\lambda+2\nu)t}&\\ +c^3e^{-3\mu t}+3c^2h\,e^{-(2\mu+\nu)t}+3ch^2e^{-(\mu+2\nu)t}&\\ +h^3e^{-3\nu t}& \end{aligned}$$

$$\begin{aligned} \sum_{i,j,k,l=0}^{\infty}F_{1,1}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}&=3\{a^2be^{-3\lambda t}\\ +2abce^{-(2\lambda+\mu)t}+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}&\\ +2acde^{-(\lambda+2\mu)t}+bc^2e^{-(\lambda+2\mu)t}+2adhe^{-(\lambda+\mu+\nu)t}&\\ +2bche^{-(\lambda+\mu+\nu)t}+bh^2e^{-(\lambda+2\nu)t}+c^2de^{-3\mu t}&\\ +2cdhe^{-(2\mu+\nu)t}+dh^2e^{-(\mu+2\nu)t}\}& \end{aligned}$$

$$\begin{aligned} \sum_{i,j,k,l=0}^{\infty}F_{2,1}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}&=3\{ab^2e^{-3\lambda t}\\ +b^2ce^{-(2\lambda+\mu)t}+2abde^{-(2\lambda+\mu)t}+b^2he^{-(2\lambda+\nu)t}&\\ +ad^2e^{-(\lambda+2\mu)t}+2bcde^{-(\lambda+2\mu)t}+2bdhe^{-(\lambda+\mu+\nu)t}&\\ +cd^2e^{-3\mu t}+d^2he^{-(2\mu+\nu)t}\}& \end{aligned} \tag{15}$$

$$\begin{aligned} \sum_{i,j,k,l=0}^{\infty}F_{3,1}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}&=b^3e^{-3\lambda t}\\ +b^2de^{-(2\lambda+\mu)t}+bd^2e^{-(\lambda+2\mu)t}+d^3e^{-3\mu t}& \end{aligned}$$

For equation (13), the equations (9) to (11) respectively become

$$(D+\lambda)^2(D+\mu)^2(D+\nu)u_1=-3t^2\{ab^2e^{-3\lambda t}+b^2ce^{-(2\lambda+\mu)t}+2abde^{-(2\lambda+\mu)t}+b^2he^{-(2\lambda+\nu)t}+ad^2e^{-(\lambda+2\mu)t}+2bcde^{-(\lambda+2\mu)t}+2bdhe^{-(\lambda+\mu+\nu)t}+cd^2e^{-3\mu t}+d^2he^{-(2\mu+\nu)t}\}-t^3\{b^3e^{-3\lambda t}+b^2de^{-(2\lambda+\mu)t}+bd^2e^{-(\lambda+2\mu)t}+d^3e^{-3\mu t}\} \quad (16)$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1\right)+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1=-\{a^3e^{-3\lambda t}+3a^2ce^{-(2\lambda+\mu)t}+3a^2he^{-(2\lambda+\nu)t}+3ac^2e^{-(\lambda+2\mu)t}+6ache^{-(\lambda+\mu+\nu)t}+3ah^2e^{-(\lambda+2\nu)t}+c^3e^{-3\mu t}+3c^2he^{-(2\mu+\nu)t}+3ch^2e^{-(\mu+2\nu)t}+h^3e^{-3\nu t}\} \quad (17)$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}=-3\{a^2be^{-3\lambda t}+2abce^{-(2\lambda+\mu)t}+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}+2acde^{-(\lambda+2\mu)t}+bc^2e^{-(\lambda+2\mu)t}+2adhe^{-(\lambda+\mu+\nu)t}+2bche^{-(\lambda+\mu+\nu)t}+bh^2e^{-(\lambda+2\nu)t}+c^2de^{-3\mu t}+2cdhe^{-(2\mu+\nu)t}+dh^2e^{-(\mu+2\nu)t}\} \quad (18)$$

Since the relations $\lambda>>\mu$ and $\lambda<<\nu$ among the eigenvalues, then the equation (18) can be separated for the unknown functions B_1 and D_1 in the following way:

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}=-3\{a^2be^{-3\lambda t}+2abce^{-(2\lambda+\mu)t}+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}+2acde^{-(\lambda+2\mu)t}+bc^2e^{-(\lambda+2\mu)t}+2adhe^{-(\lambda+\mu+\nu)t}+2bche^{-(\lambda+\mu+\nu)t}+bh^2e^{-(\lambda+2\nu)t}+2cdhe^{-(2\mu+\nu)t}+dh^2e^{-(\mu+2\nu)t}\} \quad (19)$$

$$e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}=-3c^2de^{-3\mu t} \quad (20)$$

Solving equations (19) and (20), we get

$$B_1=l_1a^2be^{-2\lambda t}+l_2abce^{-(\lambda+\mu)t}+l_3a^2de^{-(\lambda+\mu)t}+l_4abhe^{-(\lambda+\nu)t}+l_5(2acd+bc^2)e^{-2\mu t}+l_6(adh+bch)e^{-(\mu+\nu)t}+l_7cdhe^{-(2\mu+\nu-\lambda)t}+l_8dh^2e^{-(\mu+2\nu-\lambda)t}+l_9bh^2e^{-(\lambda+2\nu)t} \quad (21)$$

$$D_1=mc^2de^{-2\mu t} \quad (22)$$

Where

$$l_1=\frac{3}{2\lambda(3\lambda-\mu)^2(\nu-3\lambda)},\quad l_2=\frac{3}{2\lambda^2(\lambda+\mu)(\nu-\mu-2\lambda)},$$
$$l_3=\frac{3}{4\lambda^2(\lambda+\mu)(\nu-\mu-2\lambda)},\quad l_4=\frac{-3}{\lambda(\lambda+\nu)(2\lambda+\nu-\mu)^2},$$
$$l_5=\frac{3}{2\mu(\lambda+\mu)^2(\nu-\lambda-2\mu)},\quad l_6=\frac{-6}{(\lambda+\mu)(\mu+\nu)(\lambda+\nu)^2},$$
$$l_7=\frac{-3}{\mu(\mu+\nu)^2(2\mu+\nu-\lambda)},\quad l_8=\frac{-3}{4\nu^2(\mu+\nu)(\mu+2\nu-\lambda)},$$
$$l_9=\frac{-3}{2\nu(\lambda+\nu)(\lambda+2\nu-\mu)^2},\quad m=\frac{3}{2\mu(\lambda-3\mu)^2(\nu-3\mu)}$$

Using the values of B_1 and D_1 in equation (17), we obtain

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial A_1}{\partial t}+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial C_1}{\partial t}+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1=-a^3e^{-3\lambda t}-3a^2ce^{-(2\lambda+\mu)t}-3a^2he^{-(2\lambda+\nu)t}-3ac^2e^{-(\lambda+2\mu)t}-6ache^{-(\lambda+\mu+\nu)t}-3ah^2e^{-(\lambda+2\nu)t}-c^3e^{-3\mu t}-3c^2he^{-(2\mu+\nu)t}-3ch^2e^{-(\mu+2\nu)t}-h^3e^{-3\nu t}-2(3\lambda-\mu)^2(\nu-3\lambda)l_1a^2be^{-3\lambda t}-8\lambda^2(\nu-\mu-2\lambda)l_2abce^{-(2\lambda+\mu)t}-8\lambda^2(\nu-\mu-2\lambda)l_3a^2de^{-(2\lambda+\mu)t}+4\lambda(\mu-\nu-2\lambda)^2l_4abhe^{-(2\lambda+\nu)t}-2(\lambda+\mu)^2(\nu-2\mu-\lambda)l_5(2acd+bc^2)e^{-(\lambda+2\mu)t}+2(\lambda+\nu)^2(\lambda+\mu)l_6(adh+bch)e^{-(\lambda+\mu+\nu)t}+4\mu(\mu+\nu)l_7cdhe^{-(2\mu+\nu)t}+8\nu^2(\mu+\nu)l_8dh^2e^{-(\mu+2\nu)t}+2(\lambda+2\nu-\mu)^2(\lambda+\nu)l_9bh^2e^{-(\lambda+2\nu)t}-2(\lambda-3\mu)^2(\nu-3\mu)mc^2de^{-3\mu t} \quad (23)$$

Again, applying the conditions $\lambda>>\mu$ and $\lambda<<\nu$ in equation (23), we obtain the following equations for unknown functions A_1, C_1 and H_1:

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial A_1}{\partial t}=-a^3e^{-3\lambda t}-3a^2ce^{-(2\lambda+\mu)t}-3ac^2e^{-(\lambda+2\mu)t}-2(3\lambda-\mu)^2(\nu-3\lambda)l_1a^2be^{-3\lambda t}-8\lambda^2(\nu-\mu-2\lambda)l_2abce^{-(2\lambda+\mu)t}-8\lambda^2(\nu-\mu-2\lambda)l_3a^2de^{-(2\lambda+\mu)t}-2(\lambda+\mu)^2(\nu-2\mu-\lambda)l_5(2acd+bc^2)e^{-(\lambda+2\mu)t} \quad (24)$$

$$e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial C_1}{\partial t} = -c^3e^{-3\mu t} - 2(\lambda-3\mu)^2(\nu-3\mu)mc^2de^{-3\mu t} \tag{25}$$

$$\begin{aligned} e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1 &= -3a^2he^{-(2\lambda+\nu)t} \\ &- 6ache^{-(\lambda+\mu+\nu)t} - 3ah^2e^{-(\lambda+2\nu)t} - 3c^2he^{-(2\mu+\nu)t} \\ &- 3ch^2e^{-(\mu+2\nu)t} - h^3e^{-3\nu t} + 4\lambda(\mu-\nu-2\lambda)^2 l_4abh \\ &e^{-(2\lambda+\nu)t} + 2(\lambda+\nu)^2(\lambda+\mu)l_6(adh+bch)e^{-(\lambda+\mu+\nu)t} \\ &+ 4\mu(\mu+\nu)l_7cdhe^{-(2\mu+\nu)t} + 8\nu^2(\mu+\nu)l_8dh^2e^{-(\mu+2\nu)t} \\ &+ 2(\lambda+2\nu-\mu)^2(\lambda+\nu)l_9bh^2e^{-(\lambda+2\nu)t} \end{aligned} \tag{26}$$

Solving equations (24), (25) and (26), we obtain

$$\begin{aligned} A_1 &= r_1a^3e^{-2\lambda t} + r_2a^2ce^{-(\lambda+\mu)t} + r_3ac^2e^{-2\mu t} \\ &+ r_4a^2be^{-2\lambda t} + r_5abce^{-(\lambda+\mu)t} + r_6a^2d \\ &e^{-(\lambda+\mu)t} + r_7(2acd+bc^2)e^{-2\mu t} \end{aligned} \tag{27}$$

$$C_1 = n_1c^3e^{-2\mu t} + n_2c^2de^{-2\mu t} \tag{28}$$

$$\begin{aligned} H_1 &= s_1a^2he^{-2\lambda t} + s_2ache^{-(\lambda+\mu)t} + s_3ah^2e^{-(\lambda+\nu)t} \\ &+ s_4c^2he^{-2\mu t} + s_5ch^2e^{-(\mu+\nu)t} + s_6h^3e^{-2\nu t} \\ &+ s_7abhe^{-2\lambda t} + s_8(adh+bch)e^{-(\lambda+\mu)t} \\ &+ s_9cdhe^{-2\mu t} + s_{10}dh^2e^{-(\mu+\nu)t} + s_{11}bh^2e^{-(\lambda+\nu)t} \end{aligned} \tag{29}$$

The solution of the equation (3.3.4) for u_1 is

$$\begin{aligned} u_1 &= ab^2e^{-3\lambda t}(w_1t^2+w_2t+w_3) + (b^2c+2abd) \\ &e^{-(2\lambda+\mu)t}(w_4t^2+w_5t+w_6) + b^2he^{-(2\lambda+\nu)t} \\ &(w_7t^2+w_8t+w_9) + (ad^2+2bcd)e^{-(\lambda+2\mu)t} \\ &(w_{10}t^2+w_{11}t+w_{12}) + bdhe^{-(\lambda+\mu+\nu)t}(w_{13}t^2 \\ &+ w_{14}t+w_{15}) + cd^2e^{-3\mu t}(w_{16}t^2+w_{17}t+w_{18}) \\ &+ d^2he^{-(2\mu+\nu)t}(w_{19}t^2+w_{20}t+w_{21}) + b^3e^{-3\lambda t} \\ &(w_{22}t^3+w_{23}t^2+w_{24}t+w_{25}) + b^2de^{-(2\lambda+\mu)t} \\ &(w_{26}t^3+w_{27}t^2+w_{28}t+w_{29}) + bd^2e^{-(\lambda+2\mu)t} \\ &(w_{30}t^3+w_{31}t^2+w_{32}t+w_{33}) + d^3e^{-3\mu t} \\ &(w_{34}t^3+w_{35}t^2+w_{36}t+w_{37}) \end{aligned} \tag{30}$$

where $w_1 = \dfrac{1}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}$,

$$w_2 = \frac{2}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left(\frac{1}{\lambda}+\frac{2}{3\lambda-\mu}+\frac{1}{3\lambda-\nu}\right),$$

$$w_3 = \frac{2}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{3}{4\lambda^2}+\frac{2}{\lambda(3\lambda-\mu)}+\frac{3}{(3\lambda-\mu)^2} +\frac{1}{\lambda(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)}\right\},$$

$$w_4 = \frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)},$$

$$w_5 = \frac{6}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\lambda}+\frac{1}{2\lambda+\mu-\nu}\right),$$

$$w_6 = \frac{3}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\lambda^2}+\frac{2}{\lambda(\lambda+\mu)} +\frac{1}{(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$w_7 = \frac{3}{2\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2},$$

$$w_8 = \frac{3}{\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2}\left(\frac{2}{\lambda+\nu}+\frac{2}{2\lambda+\nu-\mu}+\frac{1}{2\lambda}\right),$$

$$w_9 = \frac{3}{\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2}\left\{\frac{3}{(\lambda+\nu)^2}+\frac{3}{(2\lambda+\nu-\mu)^2} +\frac{4}{(\lambda+\nu)(2\lambda+\nu-\mu)}+\frac{1}{4\lambda^2}+\frac{1}{\lambda(\lambda+\nu)}+\frac{1}{\lambda(2\lambda+\nu-\mu)}\right\},$$

$$w_{10} = \frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)},$$

$$w_{11} = \frac{3}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\mu}+\frac{1}{\lambda+2\mu-\nu}\right),$$

$$w_{12} = \frac{3}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)} \left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\mu^2}+\frac{2}{\mu(\lambda+\mu)}+\frac{1}{(\lambda+2\mu-\nu)^2} +\frac{1}{\mu(\lambda+2\mu-\nu)}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$w_{13} = \frac{6}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2},$$

$$w_{14} = \frac{12}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2}\left(\frac{1}{\lambda+\mu}+\frac{2}{\mu+\nu}+\frac{2}{\lambda+\nu}\right),$$

$$w_{15} = \frac{12}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2} \left\{\frac{3}{(\lambda+\nu)^2}+\frac{3}{(\mu+\nu)^2}+\frac{4}{(\mu+\nu)(\lambda+\nu)} \frac{1}{(\lambda+\mu)^2}+\frac{2}{(\lambda+\mu)(\mu+\nu)}+\frac{2}{(\lambda+\mu)(\lambda+\nu)}\right\},$$

$$w_{16} = \frac{3}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)},$$

$$w_{17} = \frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left(\frac{1}{\mu}+\frac{2}{3\mu-\lambda}+\frac{1}{3\mu-\nu}\right),$$

$$w_{18} = \frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)} \left\{\frac{3}{4\mu^2}+\frac{2}{\mu(3\mu-\lambda)}+\frac{3}{(3\mu-\lambda)^2}+\frac{1}{\mu(3\mu-\nu)} +\frac{1}{(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)}\right\},$$

$$w_{19}=\frac{3}{2\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2},$$

$$w_{20}=\frac{3}{\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2}\left(\frac{2}{\mu+\nu}+\frac{2}{2\mu+\nu-\lambda}+\frac{1}{2\mu}\right),$$

$$w_{21}=\frac{3}{\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2}\left\{\frac{3}{(\mu+\nu)^2}+\frac{3}{(2\mu+\nu-\lambda)^2}+\frac{4}{(\mu+\nu)(2\mu+\nu-\lambda)}+\frac{1}{4\mu^2}+\frac{1}{\mu(\mu+\nu)}+\frac{1}{\mu(2\mu+\nu-\lambda)}\right\},$$

$$w_{22}=\frac{1}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)},$$

$$w_{23}=\frac{3}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left(\frac{1}{\lambda}+\frac{2}{3\lambda-\mu}+\frac{1}{3\lambda-\nu}\right),$$

$$w_{24}=\frac{3}{2\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{3}{4\lambda^2}+\frac{2}{\lambda(3\lambda-\mu)}+\frac{3}{(3\lambda-\mu)^2}+\frac{1}{\lambda(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)}\right\},$$

$$w_{25}=\frac{3}{2\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{1}{2\lambda^3}+\frac{3}{2\lambda^2(3\lambda-\mu)}+\frac{3}{\lambda(3\lambda-\mu)^2}+\frac{4}{(3\lambda-\mu)^3}+\frac{1}{\lambda(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)^2}+\frac{1}{(3\lambda-\nu)^3}+\frac{3}{4\lambda^2(3\lambda-\nu)}+\frac{2}{\lambda(3\lambda-\mu)(3\lambda-\nu)}+\frac{3}{(3\lambda-\mu)^2(3\lambda-\nu)}\right\},$$

$$w_{26}=\frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)},$$

$$w_{27}=\frac{9}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\lambda}+\frac{1}{2\lambda+\mu-\nu}\right),$$

$$w_{28}=\frac{9}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\lambda^2}+\frac{2}{\lambda(\lambda+\mu)}+\frac{1}{(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$w_{29}=\frac{9}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{4}{(\lambda+\mu)^3}+\frac{3}{2\lambda^3}+\frac{3}{\lambda(\lambda+\mu)^2}+\frac{3}{2\lambda^2(\lambda+\mu)}+\frac{1}{(2\lambda+\mu-\nu)^3}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)^2}+\frac{2}{(\lambda+\mu)^2(2\lambda+\mu-\nu)}+\frac{3}{4\lambda^2(2\lambda+\mu-\nu)}+\frac{2}{\lambda(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$w_{30}=\frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)},$$

$$w_{31}=\frac{9}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\mu}+\frac{1}{\lambda+2\mu-\nu}\right),$$

$$w_{32}=\frac{9}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\mu^2}+\frac{2}{\mu(\lambda+\mu)}+\frac{1}{(\lambda+2\mu-\nu)^2}+\frac{1}{\mu(\lambda+2\mu-\nu)}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$w_{33}=\frac{9}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left\{\frac{4}{(\lambda+\mu)^3}+\frac{3}{2\mu^2(\lambda+\mu)}+\frac{1}{(\lambda+2\mu-\nu)^3}+\frac{3}{2\mu^3}+\frac{3}{\mu(\lambda+\mu)^2}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)^2}+\frac{1}{\mu(\lambda+2\mu-\nu)^2}+\frac{3}{(\lambda+\mu)^2(\lambda+2\mu-\nu)}+\frac{3}{4\mu^2(\lambda+2\mu-\nu)}+\frac{2}{\mu(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$w_{34}=\frac{1}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)},$$

$$w_{35}=\frac{3}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left(\frac{1}{\mu}+\frac{2}{3\mu-\lambda}+\frac{1}{3\mu-\nu}\right),$$

$$w_{36}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left\{\frac{3}{4\mu^2}+\frac{2}{\mu(3\mu-\lambda)}+\frac{3}{(3\mu-\lambda)^2}+\frac{1}{\mu(3\mu-\nu)}+\frac{1}{(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)}\right\},$$

$$w_{37}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left\{\frac{1}{2\mu^3}+\frac{3}{2\mu^2(3\mu-\lambda)}+\frac{3}{\mu(3\mu-\lambda)^2}+\frac{4}{(3\mu-\lambda)^3}+\frac{1}{\mu(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)^2}+\frac{1}{(3\mu-\nu)^3}+\frac{3}{4\mu^2(3\mu-\nu)}+\frac{2}{\mu(3\mu-\lambda)(3\mu-\nu)}+\frac{3}{(3\mu-\lambda)^2(3\mu-\nu)}\right\}$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 from equations (27), (21), (28), (22) and (29) into equation (4), we obtain

$$\dot{a} = \varepsilon\{r_1a^3e^{-2\lambda t} + r_2a^2ce^{-(\lambda+\mu)t} + r_3ac^2e^{-2\mu t} + r_4a^2be^{-2\lambda t} + r_5abce^{-(\lambda+\mu)t} + r_6a^2de^{-(\lambda+\mu)t} + r_7(2acd+bc^2)e^{-2\mu t}\}$$

$$\dot{b} = \varepsilon\{l_1a^2be^{-2\lambda t} + l_2abce^{-(\lambda+\mu)t} + l_3a^2de^{-(\lambda+\mu)t} + l_4abhe^{-(\lambda+\nu)t} + l_5(2acd+bc^2)e^{-2\mu t} + l_6(adh+bch)e^{-(\mu+\nu)t} + l_7cdhe^{-(2\mu+\nu-\lambda)t} + l_8dh^2e^{-(\mu+2\nu-\lambda)t} + l_9bh^2e^{-(\lambda+2\nu)t}\}$$

$$\dot{c} = \varepsilon\{n_1c^3e^{-2\mu t} + n_2c^2de^{-2\mu t}\} \qquad (31)$$

$$\dot{d} = \varepsilon mc^2de^{-2\mu t}$$

$$\dot{h} = \varepsilon\{s_1a^2he^{-2\lambda t} + s_2ache^{-(\lambda+\mu)t} + s_3ah^2e^{-(\lambda+\nu)t} + s_4c^2he^{-2\mu t} + s_5ch^2 e^{-(\mu+\nu)t} + s_6h^3e^{-2\nu t} + s_7abhe^{-2\lambda t} + s_8(adh+bch)e^{-(\lambda+\mu)t} + s_9cdhe^{-2\mu t} + s_{10}dh^2e^{-(\mu+\nu)t} + s_{11}bh^2e^{-(\lambda+\nu)t}\}$$

Here, all of the equations (31) have no exact solutions. However, since $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$ and $\dot{h}$ are proportional to the small parameter ε, they are slowly varying functions of time *t*. Consequently, it is possible to replace *a*, *b*, *c*, *d* and *h* by their respective values obtained in linear case (i.e., the values of *a*, *b*, *c*, *d* and *h* obtained when $\varepsilon = 0$) in the right hand side of equations (31). This type of replacement was first introduced by Murty and Deekshatulu [24], and Mutry *et.al.* [19] to solve similar types of nonlinear equations. Therefore, the solutions of equation (31) are

$$a = a_0 + \varepsilon\Big\{r_1a_0^3\frac{1-e^{-2\lambda t}}{2\lambda} + r_2a_0^2c_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + r_3a_0c_0^2\frac{1-e^{-2\mu t}}{2\mu} + r_4a_0^2b_0\frac{1-e^{-2\lambda t}}{2\lambda} + r_5a_0b_0c_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + r_6a_0^2d_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + r_7(2a_0c_0d_0+b_0c_0^2)\frac{1-e^{-2\mu t}}{2\mu}\Big\}$$

$$b = b_0 + \varepsilon\Big\{l_1a_0^2b_0\frac{1-e^{-2\lambda t}}{2\lambda} + l_2a_0b_0c_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + l_3a_0^2d_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + l_4a_0b_0h_0\frac{1-e^{-(\lambda+\nu)t}}{\lambda+\mu} + l_5(2a_0c_0d_0+b_0c_0^2)\frac{1-e^{-2\mu t}}{2\mu} + l_6(a_0d_0h_0+b_0c_0h_0)\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + l_7c_0d_0h_0\frac{1-e^{-(2\mu+\nu-\lambda)t}}{2\mu+\nu-\lambda} + l_8d_0h_0^2\frac{1-e^{-(\mu+2\nu-\lambda)t}}{\mu+2\nu-\lambda} + l_9b_0h_0^2\frac{1-e^{-(\lambda+2\nu)t}}{\lambda+2\nu}\Big\}$$

$$c = c_0 + \varepsilon\Big\{n_1c_0^3\frac{1-e^{-2\mu t}}{2\mu} + n_2c_0^2d_0\frac{1-e^{-2\mu t}}{2\mu}\Big\}$$

$$d = d_0 + \varepsilon mc_0^2d_0\frac{1-e^{-2\mu t}}{2\mu} \qquad (32)$$

$$h = h_0 + \varepsilon\Big\{s_1a_0^2h_0\frac{1-e^{-2\lambda t}}{2\lambda} + s_2a_0c_0h_0\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + s_3a_0h_0^2\frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu} + s_4c_0^2h_0\frac{1-e^{-2\mu t}}{2\mu} + s_5c_0h_0^2\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + s_6h_0^3\frac{1-e^{-2\nu t}}{2\nu} + s_7a_0b_0h_0\frac{1-e^{-2\lambda t}}{2\lambda} + s_8(a_0d_0h_0+b_0c_0h_0)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + s_9c_0d_0h_0\frac{1-e^{-2\mu t}}{2\mu} + s_{10}d_0h_0^2\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + s_{11}b_0h_0^2\frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu}\Big\}$$

Hence, we obtain the first approximate solution of the equation (13) as:

$$x(t,\varepsilon) = (a+bt)e^{-\lambda t} + (c+dt)e^{-\mu t} + he^{-\nu t} + \varepsilon u_1 \qquad (33)$$

where *a*, *b*, *c*, *d* and *h* are given by the equations (32) and u_1 is given by (30).

4. Results and Discussion

The perturbation solution is usually compared to the numerical solution to test the accuracy of the approximate solution obtained by a certain perturbation method. Therefore, we have first considered the eigenvalues $\lambda = 0.75$, $\mu = 0.1$ and $\nu = 3.82$. We have computed $x(t,\varepsilon)$ using (33), in which a, b, c, d and h are obtained from (32) and u_1 is calculated from equation (30) together with initial conditions $a_0 = 0.25$, $b_0 = 0.1$, $c_0 = 0.4$, $d_0 = 0.35$ and $h_0 = 0.5$ when $\varepsilon = 0.1$. The result obtained from (33) for various values of *t*, and the corresponding numerical solution obtained by a fourth order *Runge-Kutta* method is plotted in the Fig. 1.

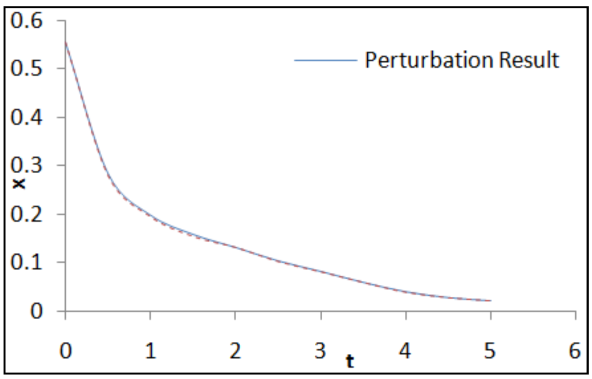

Fig. 1. *Perturbation results are plotted by continuous line and numerical results are plotted by dotted line.*

Again, we have computed $x(t, \varepsilon)$ from (33) by considering values of $\lambda = 0.70$, $\mu = 0.1$ and $\nu = 3.80$. We have computed $x(t, \varepsilon)$ using (33), in which a, b, c, d and h are obtained from (32) and u_1 is calculated from equation (30) together with initial conditions $a_0 = 0.35$, $b_0 = 0.15$, $c_0 = 0.35$, $d_0 = 0.35$ and $h_0 = 0.5$ when $\varepsilon = 0.1$. The result obtained from (33) for various values of *t*, and the corresponding numerical solution obtained by a fourth order *Runge-Kutta* method is plotted in the Fig. 2.

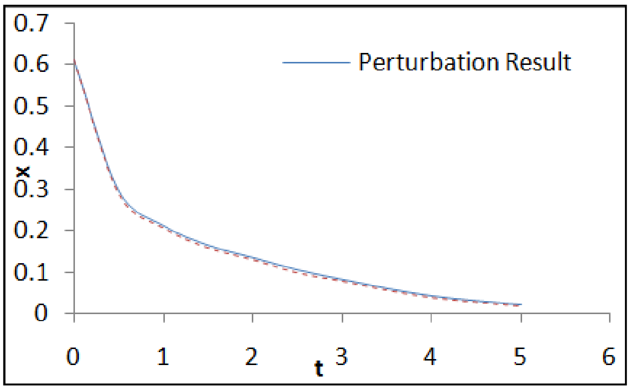

Fig. 2. *Perturbation results are plotted by continuous line and numerical results are plotted by dotted line.*

Finally, we have computed $x(t, \varepsilon)$ from (33) by considering values of $\lambda = 0.69$, $\mu = 0.15$ and $\nu = 3.79$. We have computed $x(t, \varepsilon)$ using (33), in which a, b, c, d and h are obtained from (32) and u_1 is calculated from equation (30) together with initial conditions $a_0 = 0.45$, $b_0 = 0.20$, $c_0 = 0.35$, $d_0 = 0.30$ and $h_0 = 0.5$ when $\varepsilon = 0.1$. The result obtained from (33) for various values of t, and the corresponding numerical solution obtained by a fourth order Runge-Kutta method is plotted in the Fig. 3.

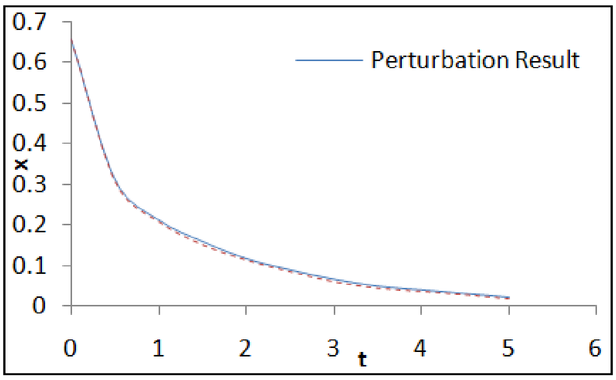

Fig. 3. *Perturbation results are plotted by continuous line and numerical results are plotted by dotted line.*

5. Conclusion

In conclusion, it is suggested that, in this article, the KBM method has been modified and applied successfully to the fifth order more critically damped nonlinear systems. In relation to the fifth order critically damped systems, the solutions are obtained in such circumstances where the four eigenvalues are pairwise equal and another eigenvalue is distinct. Normally, in the KBM method, it is noticed that much error occurs in the case of rapid changes of *x* with respect to time *t*. However, it is suggested that all the aforementioned results obtained in this paper correspond accurately to the numerical solutions obtained by the fourth order *Runge-Kutta* method. It is, therefore, concluded that the modified KBM method provides highly accurate results, which can be applied for different kinds of nonlinear differential systems.

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