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FD-RBF for Partial Integro-Differential Equations with a Weakly Singular Kernel

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Abstract: Finite Difference Method and Radial Basis Functions are applied to solve partial integro-differential equations with a weakly singular kernel. The product trapezoidal method is used to compute singular integrals that appear in the discretization process. Different RBFs are implemented and satisfactory results are shown the ability and the usefulness of the proposed method.

Keywords: Partial Integro-Differential Equations (PIDE), Weakly Singular Kernel, Radial Basis Functions (RBF), Finite Difference Method (FDM), Product Trapezoidal Method

1. Introduction

Mathematical modeling of some scientific and engineering problems lead to partial integro-differential equations (PIDEs). In this paper, the following PIDE, with a weakly singular kernel is considered.

$$u_t(x,t)=\mu u_{xx}(x,t)+\int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s)\,ds,\ 0<x<1, 0\le t\le T \quad (1)$$

where $\mu \ge 0$, and is subject to the following boundary and initial conditions:

$$u(0,t)=u(1,t)=0,\ 0\le t\le T,$$
$$u(x,0)=g(x),\ 0\le x\le 1.$$

This type of integro-differential equations appears in some phenomena such as heat conduction in materials with memory [12, 22], population dynamics, and viscoelasticity [25, 6]. The numerical solution of PIDEs is considered by many authors [1, 2, 19, 20, 23, 27, 30].

PIDEs with weakly singular kernels have been studied in some papers. Numerical solution of a parabolic integro-differential equation with a weakly singular kernel by means of the Galerkin finite element method is discussed in [5]. A finite difference scheme and a compact difference scheme are presented for PIDEs, with a weakly singular kernel, in [28] and [21], respectively. A spectral collocation method is considered in [17] for weakly singular PIDEs. Also Quintic B-spline collocation method [31] and Crank–Nicolson/quasi- wavelets method [32] are used for solving fourth order partial integro-differential equation with a weakly singular kernel and some others [10, 14].

In recent years, meshless methods, as a class of numerical methods, are used for solving functional equations. Meshless methods just use a scattered set of collocation points, regardless any relationship between the collocation points. This property is the main advantage of these techniques in comparison with the mesh dependent methods, such as finite difference and finite element. Since 1990, radial basis function methods (RBF) [13] are used as a well-known family of meshless methods to approximate the solutions of various types of linear and nonlinear functional equations, such as Partial Differential Equations (PDEs), Ordinary Differential Equations (ODEs), Integral Equations (IEs), and Integro-Differential Equations (IDEs) [7, 8, 11, 13, 16, 18, 24]. In the present work, for the first time derivatives, we use the Finite Difference (FD) scheme to discretize the equation which it makes a system of partial integro-differential equations. Then we use radial basis functions (RBFs) to solve this system. Recently FD-RBF method is used to solve some problems like nonlinear parabolic-type Volterra partial integro-differential equations [1], fractional-diffusion inverse heat conduction problems [33], and wave equation with an integral condition [34].

In this paper, FD-RBF methods are applied for numerical solution of PIDEs with a weakly singular kernel. Singular integrals, which appear in the method, are computed by the product trapezoidal integration rule.

The paper is organized as follows. In Section 2, the RBFs are introduced. Section 3, as the main part, is devoted to solving weakly singular PIDEs, by finite difference and RBFs. An illustrative example is included in Section 4. A conclusion is presented in Section 5.

2. Radial Basis Functions

Interpolation of a function $u:\mathbb{R}^d \to \mathbb{R}$ by RBF can be presented as the following [4]

$$s_N(\mathbf{x}) := \sum_{i=0}^{N} \lambda_i \phi(\| \mathbf{x}-\mathbf{x}_i \|), \quad \mathbf{x}\in \mathbb{R}^d. \tag{2}$$

Where $\phi:[0,\infty)\to\mathbb{R}$ is a fixed univariate function, the coefficients $(\lambda_i)_{i=0}^{N}$ are real numbers, $(\mathbf{x}_i)_{i=0}^{N}$ is a set of interpolation points in $\mathbb{R}^d$, and $\|\cdot\|$ is the Euclidean norm.

Eq. (2) can be written as follows

$$s_N(\mathbf{x}) := \sum_{i=0}^{N} \lambda_i \phi_i(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x})\mathbf{\Lambda}, \tag{3}$$

where

$$\phi_i(\mathbf{x}) = \phi(\| \mathbf{x}-\mathbf{x}_i \|),$$
$$\mathbf{\Phi}(\mathbf{x}) = [\phi_0(\mathbf{x}), \phi_1(\mathbf{x}), \dots, \phi_N(\mathbf{x})]^T,$$
$$\mathbf{\Lambda} = [\lambda_0, \lambda_1, \dots, \lambda_N]^T.$$

Consider $N+1$ distinct support points $(\mathbf{x}_j, u(\mathbf{x}_j))$, $j=0,1,\dots,N$. One can use interpolation conditions to find λ_i s by solving the following linear system

$$\mathbf{A\Lambda} = \mathbf{u},$$

in which

$$\mathbf{A} = [\phi(\| \mathbf{x}_j - \mathbf{x}_i \|)]_{i,j=0}^{N},$$

and

$$\mathrm{u} = [\mathrm{u}(\mathrm{x}_0), \mathrm{u}(\mathrm{x}_1), \dots, (\mathrm{x}_N)]^{\mathrm{T}}$$

Table 1. *Some well-known RBFs.*

Name of the RBF	Definition
Gaussian	$\phi(r) = e^{-(cr)^2}$
Inverse Quadric	$\phi(r) = \frac{1}{r^2+c^2}$
Hardy Multiquadric	$\phi(r) = \sqrt{r^2+c^2}$
Inverse Multiquadric	$\phi(r) = \frac{1}{\sqrt{r^2+c^2}}$
Cubic	$\phi(r) = r^3$
Thin Plate Spline	$\phi(r) = r^2\log(r)$
Hyperbolic Secant	$\phi(r) = \mathrm{sech}(\frac{r}{c})$

Some well-known RBFs are listed in Table 1, where the Euclidian distance r is real and non-negative, and c is a positive scalar, called the shape parameter.

Also the generalized Thin Plate Splines (TPS) are defined as the following:

$$\phi(\mathrm{r}) = \mathrm{r}^{2\mathrm{m}}\log(\mathrm{r}), \mathrm{m} = 1,2, \dots$$

Some of RBFs are unconditionally positive definite (e.g. Gaussian or Inverse Multiquadrics) to guarantee that the resulting system is solvable, and some of them are conditionally positive definite. Although, some of RBFs are conditionally positive definite functions, polynomials are augmented to Eq. (2) to guarantee that the outcome interpolation matrix is invertible. Such an approximation can be expressed as follows

$$s(\mathbf{x}) = \sum_{i=0}^{N} \lambda_i \phi(\| \mathbf{x}-\mathbf{x}_i \|) + \sum_{i=1}^{l} \lambda_{N+i} p_i(\mathbf{x}), \quad \mathbf{x}\in \mathbb{R}^d. \tag{4}$$

where $p_i(x), i=1,\dots,l$, are polynomials on $\mathbb{R}^d$ of degree at most $m-1$, and $l = \binom{m-1}{d}$. Here l is the dimension of the linear space Π_{m-1}^{d} of polynomials of total degree less than or equal to m with d variables.

Collocation method is used to determinate the coefficients $(\lambda_0, \lambda_1, \dots, \lambda_N)$ and $(\lambda_{N+1}, \lambda_{N+2}, \dots, \lambda_{N+l})$. This will produce $N+1$ equations at $N+1$ points. l additional equations is usually written in the following form

$$\sum_{i=0}^{N} \lambda_i p_j(\mathbf{x}_i) = 0, \quad \mathrm{j} = 1, \dots, l \tag{5}$$

3. Application of FD-RBF Method

In this section we explain the process of solving PIDEs, with a weakly singular kernel, in the following form

$$u_t(x,t) = \mu u_{xx}(x,t) + \int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s)\,ds, \quad 0<x<1, \quad 0\le t\le T, \tag{6}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $\mu \ge 0$, with the following boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \quad 0\le t\le T, \tag{7}$$

$$u(x,0) = g(x), \quad 0\le x\le 1. \tag{8}$$

At first we introduce grid points $t_i = a+ih$, $i=0,1,\dots,M$, where $M\ge 1$ is an integer, and $h=(T-0)/M$.

Considering (6) at point (x, t_i), we have

$$u_t(x,t_i) = \mu u_{xx}(x,t_i) + \int_0^{t_i} (t_i-s)^{-\frac{1}{2}} u_{xx}(x,s)\,ds, \quad i = 1, \dots, M \tag{9}$$

As the finite difference technique, we have

$$u_t(x,t_i) \approx \frac{u(x,t_{i+1})-u(x,t_{i-1})}{2h}.$$

Discretizing (9) by the θ-weighted method leads to

$$\frac{u(x,t_{i+1})-u(x,t_{i-1})}{2h} = \mu\left(\theta u_{xx}(x,t_{i+1})+(1-\theta)u_{xx}(x,t_i)\right)$$
$$+\int_0^{t_i}(t_i-s)^{-\frac{1}{2}}(\theta u_{xx}(x,s+h)+(1-\theta)u_{xx}(x,s))ds,$$

$i = 1,\dots m-1$, and $\theta\in[0,1]$. By using the notation $u^i(x)=u(x,t_i)$ we have

$$u^{i+1}(x)=u^{i-1}(x)+2h\mu\left(\theta u_{xx}^{i+1}(x)+(1-\theta)u_{xx}^{i}(x)\right)$$
$$+2h\int_0^{t_i}(t_i-s)^{-\frac{1}{2}}\left(\theta u_{xx}(x,s+h)+(1-\theta)u_{xx}(x,s)\right)ds. \tag{10}$$

We now use the product trapezoidal integration technique, well addressed in [9], to approximate the integrals

$$\int_0^{t_i}(t_i-s)^{-\frac{1}{2}}u_{xx}(x,s+h)ds \approx \sum_{j=0}^{i} w_{ij}u_{xx}^{j+1}(x), \tag{11}$$

and

$$\int_0^{t_i}(t_i-s)^{-\frac{1}{2}}u_{xx}(x,s)ds \approx \sum_{j=0}^{i} w_{ij}u_{xx}^{j}(x), \tag{12}$$

where

$$w_{i0}=\frac{1}{h}\int_{t_0}^{t_1}\frac{t_1-s}{\sqrt{t_i-s}}ds,$$
$$w_{ij}=\frac{1}{h}\left(\int_{t_{j-1}}^{t_j}\frac{s-t_{j-1}}{\sqrt{t_i-s}}ds+\int_{t_j}^{t_{j+1}}\frac{t_{j+1}-s}{\sqrt{t_i-s}}\right),\quad j=1,2,\dots,i\text{-}1$$
$$w_{ii}=\frac{1}{h}\int_{t_{i-1}}^{t_i}\frac{s-t_{i-1}}{\sqrt{t_i-s}}ds.$$

Substituting (11) and (12) into (10), results in

$$u^{i+1}(x)=u^{i-1}(x)+2h\mu(\theta u_{xx}^{i+1}(x)+(1-\theta)u_{xx}^{i}(x))$$
$$+2h\theta\sum_{j=0}^{i}w_{ij}u_{xx}^{j+1}(x)+2h(1-\theta)\sum_{j=0}^{i}w_{ij}u_{xx}^{j}(x),$$

or

$$u^{i+1}(x)-2h\theta(\mu+w_{ii})u_{xx}^{i+1}=u^{i-1}(x)+2h\mu(1-\theta)u_{xx}^{i}(x)+$$
$$2h(1-\theta)w_{i0}u_{xx}^{0}(x)+2h\sum_{j=1}^{i}(\theta w_{i,j-1}+(1-\theta)w_{ij})u_{xx}^{j}(x), \tag{13}$$

for i=1,…m-1. Let's approximate the function $u^i(x)$ in terms of RBFs, as follows

$$u^i(x)=\sum_{n=0}^{N}\lambda_{i,n}\phi_n(x)+\lambda_{i,N+1}x+\lambda_{i,N+2}=\mathbf{\Phi}^T(x)\mathbf{\Lambda}^i, \tag{14}$$

where

$$\phi_n(x)=\phi(\|x-x_n\|),$$

$x_n=a+nk$, $n=0,1,\dots,N$, are center points, $k=(b-a)/N$ and $N\ge 1$ is an integer,

$$\boldsymbol{\Phi}(x) = [\phi_0(x),\dots,\phi_N(x)\ x\ 1]^T \tag{15}$$

and $\mathbf{\Lambda}^i=[\lambda_{i,0}\ \dots\ \lambda_{i,N}\ \lambda_{i,N+1}\ \lambda_{i,N+2}]^T$ is an unknown vector.

Collocating Eq. (14) at $(x_j)_{j=0}^N$ leads to the following system

$$u_j^i := u^i(x_j)=\sum_{n=0}^{N}\lambda_{i,n}\phi_n(x_j)+\lambda_{i,N+1}x_j+\lambda_{i,N+2},\ \text{j=0,…,N}$$

Two additional conditions, as mentioned in section 2, can be written as

$$\sum_{j=0}^{N}\lambda_{i,j}=0, \tag{1}$$

$$\sum_{j=0}^{N}\lambda_{i,j}x_j=0. \tag{2}$$

Eqs. (14), (16), and (17) can be written in the following matrix form

$$\mathbf{u}^i=\mathbf{A}\mathbf{\Lambda}^i,$$

where $u^i=[u_0^i,\dots,u_N^i\ 0\ 0]^T$, $\mathbf{\Lambda}^i=[\lambda_{i,0}\ \dots\ \lambda_{i,N}\ \lambda_{i,N+1}\ \lambda_{i,N+1}]^T$, and

$$A=\begin{bmatrix}\phi_0(x_0) & \phi_1(x_0) & \dots & \phi_N(x_0) & x_0 & 1\\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_N(x_1) & x_1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \phi_0(x_N) & \phi_1(x_N) & \dots & \phi_N(x_N) & x_N & 1\\ x_0 & x_1 & \dots & x_N & 0 & 0\\ 1 & 1 & \dots & 1 & 0 & 0\end{bmatrix} \tag{18}$$

By two times differentiation from Eq. (14), with respect to x, we obtain

$$u_{xx}^i=\sum_{n=0}^{N}\lambda_{i,n}\phi_n(x)=\mathbf{\Psi}^T(x)\mathbf{\Lambda}^i, \tag{19}$$

Where

$$\Psi(x) = [\emptyset''_0(x)\emptyset''_1(x)\dots\emptyset''_N(X)00]^T$$

Substituting Eqs. (14) and (19) in Eq. (13), leads to

$$\begin{aligned}&\mathbf{\Phi}^T(x)\mathbf{\Lambda}^{i+1}-2h\theta(\mu+w_{ii})\mathbf{\Psi}^T(x)\mathbf{\Lambda}^{i+1}\\&=\mathbf{\Phi}^T(x)\mathbf{\Lambda}^{i-1}+2h\mu(1-\theta)\mathbf{\Psi}^T(x)\mathbf{\Lambda}^{i}+2h(1-\theta)w_{i0}\mathbf{\Psi}^T(x)\mathbf{\Lambda}^{0}\\&+2h\sum_{j=1}^{i}(\theta w_{i,j-1}+(1-\theta)w_{ij})\mathbf{\Psi}^T(x)\mathbf{\Lambda}^{j},\end{aligned} \tag{3}$$

for $i=1,\dots,M-1$. So, we consider collocation points, x_n, $n=0,1,\dots,N$, to obtain the entries of the vectors of the coefficients $\mathbf{\Lambda}^i$, $i=1,\dots,M$ in Eq. (20). This leads to

$$\begin{aligned}&\mathbf{A}\mathbf{\Lambda}^{i+1}-2h\theta(\mu+w_{ii})\mathbf{T}\mathbf{\Lambda}^{i+1}=\mathbf{A}\mathbf{\Lambda}^{i-1}+2h\mu(1-\theta)\mathbf{T}\mathbf{\Lambda}^{i}\\&+2h(1-\theta)w_{i0}\mathbf{T}\mathbf{\Lambda}^{0}+2h\sum_{j=1}^{i}(\theta w_{i,j-1}+(1-\theta)w_{ij})\mathbf{T}\mathbf{\Lambda}^{j},\end{aligned} \tag{21}$$

where $i=1,\dots,M-1$, and

$$T=\begin{bmatrix}0 & \cdots & 0 & 0 & 0\\ \phi''_0(x_1) & \cdots & \phi''_N(x_1) & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ \phi''_0(x_{N-1}) & \cdots & \phi''_N(x_{N-1}) & 0 & 0\\ 0 & \cdots & 0 & 0 & 0\\ 0 & \cdots & 0 & 0 & 0\\ 0 & \cdots & 0 & 0 & 0\end{bmatrix}$$

As the first step we must determine $\mathbf{\Lambda}^0$ and $\mathbf{\Lambda}^1$. Obviously, $\mathbf{\Lambda}^0$ can be obtained by the initial condition

$$u^0(x)=u(x,0)=g(x)$$
$$\Rightarrow \mathbf{A}\mathbf{\Lambda}^0=\mathbf{u}_0=[g(x_0)\quad g(x_1)\quad \dots\quad g(x_N)\quad 0\quad 0]^T.$$

To approximate $\mathbf{\Lambda}^1$, we use

$$u_t(x,t_1)\approx\frac{u(x,t_1)-u(x,t_0)}{h}. \tag{22}$$

Substituting (22) into (9), similarly leads to

$$(\mathbf{A}-v\mathbf{T}-h\mu\mathbf{T})\mathbf{\Lambda}^1=(\mathbf{A}+w\mathbf{T})\mathbf{\Lambda}^0, \tag{23}$$

where

$$v=\int_0^{t_1}\frac{s-t_1}{\sqrt{t_2-s}}ds,.$$

and

$$w=\int_0^{t_1}\frac{t_2-s}{\sqrt{t_2-s}}ds.$$

The convergence of RBF interpolation has been addressed by Buhmann [3, 4], and other researchers [15, 26, 29].

4. Numerical Example

In this section, an example is provided to illustrate the efficiency of this approach. For the sake of comparison purposes, we use the two norm and infinity norm of errors.

Consider the following weakly singular PIDE [21, 28]

$$u_t(x,t)=\int_0^t(t-s)^{-\frac{1}{2}}u_{xx}(x,s)\,ds,\quad 0<x<1,\quad 0\le t\le T,$$

with the boundary and initial conditions

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T,$$
$$u(x,0)=\sin(x),\quad 0\le x\le 1.$$

The exact solution is $u(x,t)=M(\pi^{\frac{5}{2}}t^{\frac{3}{2}})\sin(\pi x)$, where M denotes the function

$$M(z)=\sum_{n=0}^{\infty}(-1)^n\Gamma(\frac{3}{2}n+1)^{-1}z^n.$$

We will use $\theta=\frac{1}{2}$, $h=0.001$, $T=1$, and $N=25$.

Errors of the numerical solutions for TPS-RBF ($m=4$), IMQ-RBF ($c=0.1$), and Sech-RBF ($c=0.1$) are shown in the Table 2 and are plotted in Figures 1, 2, and 3, respectively.

Table 2. *Errors.*

	TPS-RBF		IMQ-RBF		Sech-RBF	
t	$\Vert E\Vert_\infty$	$\Vert E\Vert_2$	$\Vert E\Vert_\infty$	$\Vert E\Vert_2$	$\Vert E\Vert_\infty$	$\Vert E\Vert_2$
0.1	5.98e-04	2.16e-03	8.92e-03	2.71e-02	5.31e-03	1.60e-02
0.2	1.83e-03	6.34e-03	6.46e-03	2.48e-02	3.16e-03	1.25e-02
0.3	2.47e-03	8.53e-03	5.36e-03	1.70e-02	2.29e-03	7.10e-03
0.4	2.17e-03	7.51e-03	3.65e-03	7.75e-03	2.26e-03	5.58e-03
0.5	1.22e-03	4.22e-03	4.06e-03	1.72e-02	2.51e-03	1.18e-02
0.6	1.33e-04	4.46e-04	5.77e-03	2.31e-02	3.52e-03	1.41e-02
0.7	6.64e-04	2.31e-03	5.64e-03	2.09e-02	3.15e-03	1.18e-02
0.8	9.82e-04	3.41e-03	3.78e-03	1.34e-02	1.92e-03	6.83e-03
0.9	8.72e-04	3.02e-03	1.41e-03	4.64e-03	5.35e-04	1.72e-03
1.0	5.19e-04	1.80e-03	4.89e-04	2.17e-03	4.76e-04	1.92e-03

5. Conclusion

Three different RBFs are implemented in a FD method for solving a PIDE with a weakly singular kernel successfully. The results of applying the method on the illustrative example confirm the ability and the usefulness of the proposed app−roach. In comparison with those results reported in [21, 28], this method achieved more accurate results with less data grid points.

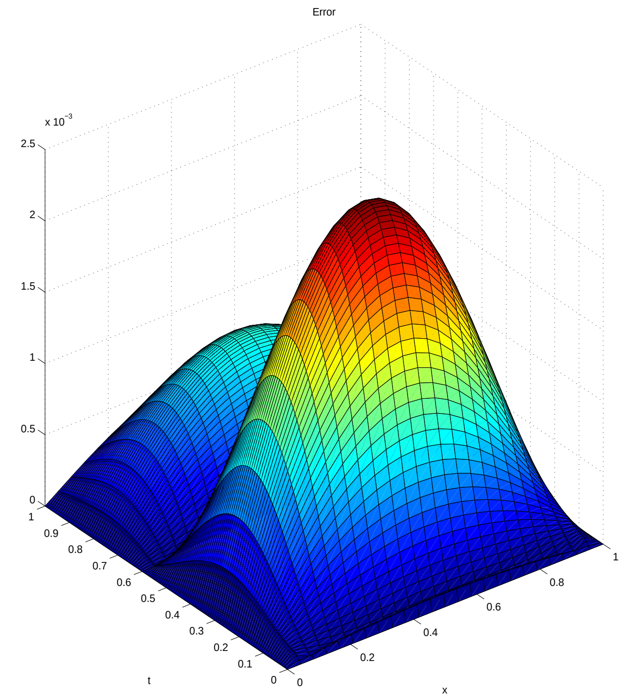

Figure 1. *TPS-RBF Error.*

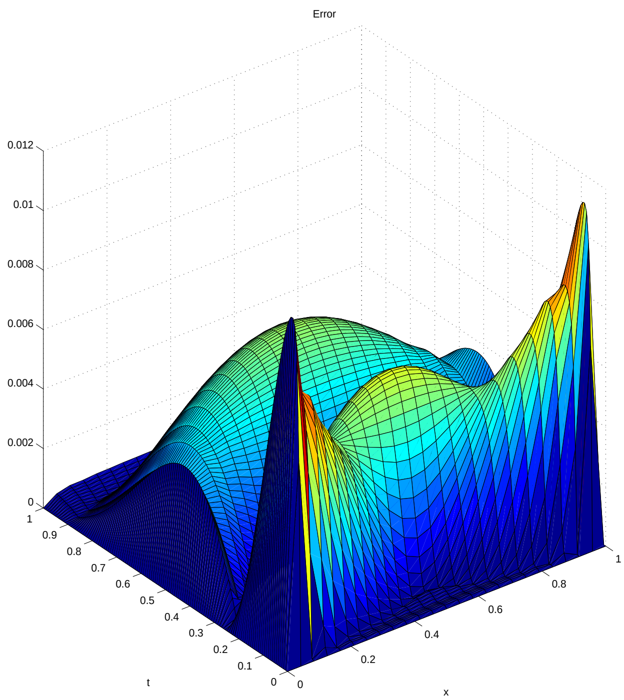

Figure 2. *IMQ-RBF Error.*

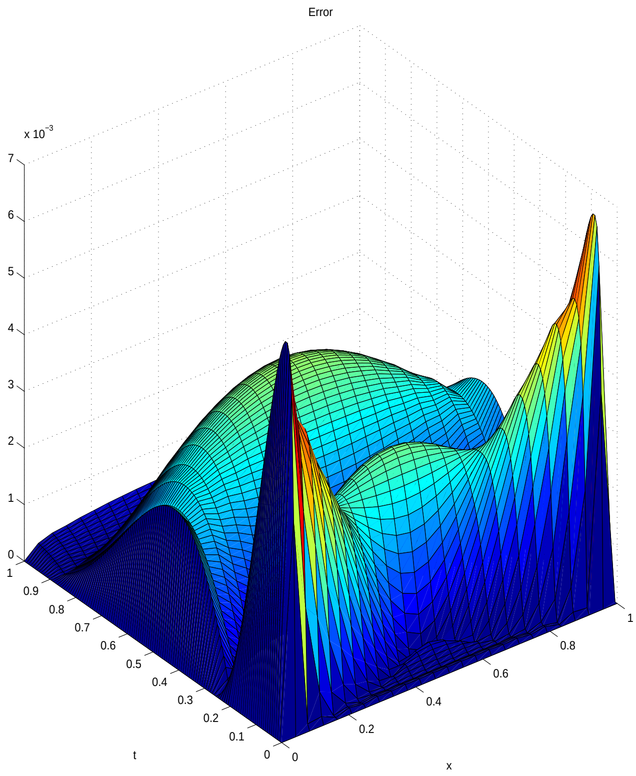

Figure 3. *Sech-RBF Error.*

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