The Stability Analysis of Two-Species Competition Model with Stage Structure and Diffusion Terms

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Abstract: In this paper, the author proposed and considered a reaction-diffusion equation with diffusion terms and stage structure. We discussed the stability of the positive equilibrium. By using the upper-lower solutions and monotone iteration technique, we obtained the zero steady state and the boundary equilibrium were linear unstable and the unique positive steady state was globally asymptotic stability. The traditional results are improved and this result applies to broader frameworks.

Keywords: Stage Structure, Reaction-Diffusion Equations, Equilibrium, Stability

1. Introduction

The growth of the species, often needs a process of development. Meanwhile, different species with different growth stage showed different characteristics. So, studying the population model with stage structure has practical significance. But, a single population is not too much, each species must be affected by the other populations and the environment, so, in recent years, the literature on the two phase structure of single population was more, see [1, 2, 3, 4, 5] etc. Among them, Chen Lansun in literature [1] listed some scholars’ research results. But, the stability of the equilibrium point on two population model was studied through ordinary differential equations in [2, 3, 4, 5]. Later, many scholars began to research the structure of three phase single population model. Gao Shujing [6] set up for a class of population model according to the young adults aged three stages. In 2005, Liang etc [7] established a class of population model which was divided into a pupa Larvae, adult three phase structure, Wu [8] studied the global asymptotic stability of a weakly-coupled reaction diffusion system in the three-species model. But in these papers, the authors did not notice the time delay. In fact, the development of population reproduction has some lag process, literature [9, 10, 11] considered the time delay on the basis of [6, 7], but, in these papers the author did not notice the free diffusion phenomena of the population. Literature [12] studied two species predator-prey with stage structure and diffusion terms which considered the effect of diffusion and phase structure. But the authors considered the species only spreading in the local scope. The local operator did not accurately describe the object of study behavior of space and time, therefore, we must introduce the convolution operator to describe the spatial diffusion process. On the basis of [12], we considered the following competition model with diffusion terms and stage structure:

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} - d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} &= -a_1 u_1(x,t) + \alpha u_2(x,t) \alpha(\{g_1 \ast u_2\})(x,t) \\
\frac{\partial u_2(x,t)}{\partial t} - d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} &= \alpha(\{g_1 \ast u_2\})(x,t) - b_1 u_2(x,t) - r_1 u_2^2(x,t) - c_1 u_2(x,t) u_1(x,t) \\
\frac{\partial u_3(x,t)}{\partial t} - d_3 \frac{\partial^2 u_3(x,t)}{\partial x^2} &= -b_2 u_1(x,t) - r_2 u_3^2(x,t) - c_2 u_2(x,t) u_3(x,t) + \beta (g_2 \ast u_3)(x,t)
\end{align*}
\]
Where, $u_i(x,t), u_2(x,t)$ respectively represent the population densities of the juvenile and adult of A at time t and location x, $u_i(x,t)$ represent the population densities of B at time t and location x, $d_1, d_2, d_3$ represent the diffusion rates, $a_1, b_1, b_2$ represent the death rate, $\alpha, \beta$ represent the birth rate, $r_1, r_2$ represent the density coefficient, $c_1, c_2$ represent the competition coefficient of adult A, B. The kernel $g_1(s), g_2(s)$ are inter-tribal and non-negative function satisfying

$$
\int_0^\infty g_1(s)ds = \int_0^\infty g_2(s)ds = 1, \quad (g_1 * u_1)(x,t) = \int_0^\infty g_1(s) e^{-rs} \frac{1}{\sqrt{4\pi ds}} e^{-(x-y)^2} u_1(t-s,y)dyds
$$

$$
(g_2 * u_2)(x,t) = \int_0^\infty g_2(s) e^{-rs} \frac{1}{\sqrt{4\pi ds}} e^{-(x-y)^2} u_2(t-s,y)dyds
$$

All the parameters are positive.

The paper is organized as follows: in section 2, we discuss the locally asymptotic stability between the zero balance and the boundary of equilibrium; in section 3, we use the upper-lower solutions and monotone iterative methods to discuss the global stability of the positive equilibrium point.

### 2. Equilibrium and Local Asymptotic Stability

The variables $u_i(x,t), u_2(x,t)$ of the system (1) have nothing with $u_i(x,t)$, so we need to consider the subsystems of the system (1):

In (1), let $u_2(x,t) = u(x,t), u_2(x,t) = v(x,t), \quad d_2, d_1, d_3, d_2$, then the system (1) can rewrite following:

$$
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - d_1 \frac{\partial^2 u(x,t)}{\partial x^2} &= \alpha (g_1 * u)(x,t) - h_1 u(x,t) - r_1 u^2(x,t) - c_1 u(x,t)v(x,t) \\
\frac{\partial v(x,t)}{\partial t} - d_2 \frac{\partial^2 v(x,t)}{\partial x^2} &= \beta (g_2 * v)(x,t) - b_1 v(x,t) - r_2 v^2(x,t) - c_2 u(x,t)v(x,t)
\end{align*}
$$

(2)

Lemma 2.1 Assume that $\alpha \int_0^\infty g_1(s)e^{-rs}ds > b_1, \quad \beta \int_0^\infty g_2(s)e^{-rs}ds > b_2, \quad \begin{cases} r_1 > c_2 k_1 \\ r_2 > c_2 k_2 \end{cases}$ or, $\begin{cases} r_1 < c_2 k_1 \\ r_2 < c_2 k_2 \end{cases}$, then the system (2) has four non-negative equilibrium:

$$
E_0 = (0, 0), \quad E_i = \left[ \frac{\alpha \int_0^\infty g_1(s)e^{-rs}ds - b_1}{r_1}, 0 \right] \Delta (k_i, 0), \quad E_2 = \left[ 0, \frac{\beta \int_0^\infty g_2(s)e^{-rs}ds - b_2}{r_2} \right] \Delta (0, k_2), \quad E = (u^*, v^*)
$$

Where,

$$
u^* = \frac{r_2 \left( \alpha \int_0^\infty g_1(s)e^{-rs}ds - b_1 \right) - c_1 \left( \beta \int_0^\infty g_2(s)e^{-rs}ds - b_2 \right)}{r_1 r_2 c_2}$$

$$
v^* = \frac{r_1 \left( \beta \int_0^\infty g_2(s)e^{-rs}ds - b_2 \right) - c_2 \left( \alpha \int_0^\infty g_1(s)e^{-rs}ds - b_1 \right)}{r_1 r_2 c_2}
$$

To study the asymptotic stability of the equilibrium by using of constant linear methods similar to [13], We introduce the transformation $U(t) = (u(x,t), v(x,t)) - E_i \quad (i = 1, 2)$ , $U_j = U(t + \theta)(-\tau \leq \theta \leq 0)$, Then the system (2) Can translate into PFDE of $C C\triangle C\left([-\tau, 0]; R^2\right):$

$$
\frac{d}{dt}U(t) = D\Delta U(t) + N(\tau)U(t) + f_0(U(t, \tau))
$$

(3)

Where, $D = \text{diag}(D_1, D_2), N(\tau): C \to R^2, \quad f_0: C \times R^2 \to R^2$. Then the characteristic equation for the linear part of
system (3) can become

\[
\begin{vmatrix}
\lambda + k^2 d_i - \alpha \int_0^+ g_i(s) e^{-k(\lambda + d_i+k^2)} ds + b_i + 2r_i u + c_i v & c_i u \\
& \lambda + k^2 d_i - \beta \int_0^+ g_i(s) e^{-k(\lambda + d_i+k^2)} ds + b_i + 2r_i v + c_i u \\
\end{vmatrix}
= 0
\]

Let

\[
g_i(\lambda, k^2) = \lambda + k^2 d_i - \alpha \int_0^+ g_i(s) e^{-k(\lambda + d_i+k^2)} ds + b_i + 2r_i u + c_i v
\]

\[
g(\lambda, k^2) = \lambda + k^2 d_i - \beta \int_0^+ g_i(s) e^{-k(\lambda + d_i+k^2)} ds + b_i + 2r_i v + c_i u
\]

Then the characteristic equation becomes

\[
g_1(\lambda, k^2)g_2(\lambda, k^2) - c_2 u v = 0.
\]

**Theorem 2.1** When

\[
\begin{cases}
k_1 r_i > c_1 k_2 \\
k_2 r_i < c_2 k_1
\end{cases}
\]

the equilibrium \( E_0 = (0,0) \) is unstable.

Proof: for the equilibrium \( E_0 = (0,0) \), the characteristic equation for the system (2) can become \( g_1(\lambda, k^2)g_2(\lambda, k^2) = 0 \)

If \( g_1(\lambda, k^2) = 0 \), which yields

\[
\lambda + k^2 d_i + b_i - \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds = 0.
\]

So, we get

\[
\lambda + k^2 d_i + b_i = \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds.
\]

If \( \Re \lambda > 0 \), then

\[
\lambda + k^2 d_i + b_i = \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds \leq \alpha \int_0^+ g_i(s) e^{-\lambda s} ds.
\]

In the same way, if \( g_2(\lambda, k^2) = 0 \), we can have \( \Re \lambda > 0 \).

Thus, \( E_0 = (0,0) \) is unstable.

**Theorem 2.2** When

\[
\begin{cases}
k_1 r_i > c_1 k_2 \\
k_2 r_i < c_2 k_1
\end{cases}
\]

the equilibrium \( E_1 \) is local asymptotic stability and \( E_2 \) is unstable.

Proof: for the equilibrium \( E_1 \), the characteristic equation for the system (2) can become \( g_1(\lambda, k^2)g_2(\lambda, k^2) = 0 \).

(i) If \( g_1(\lambda, k^2) = 0 \), which yields

\[
\lambda + k^2 d_i + b_i + 2r_i k_1 = \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds
\]

if \( \Re \lambda > 0 \), then

\[
\lambda + k^2 d_i + b_i + 2r_i k_1 = \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds \leq 2 \alpha \int_0^+ g_i(s) e^{-\lambda s} ds
\]

is contradiction with the suppose, so \( \Re \lambda \leq 0 \).

(ii) If \( g_2(\lambda, k^2) = 0 \), then

\[
\lambda + k^2 d_i - \beta \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds + b_i + c_i k_1 = 0.
\]

if \( \Re \lambda > 0 \), then

\[
\lambda + k^2 d_i + b_i + c_i k_1 = \beta \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds \leq \beta \int_0^+ g_i(s) e^{-\lambda s} ds
\]

is contradiction with the suppose, so \( \Re \lambda \leq 0 \).

Therefore, the equilibrium \( E_1 \) is local asymptotic stability.

By similar way, we can prove the balance \( E_2 \) of that the system (2) is not stable.

**Theorem 2.3** If

\[
\begin{cases}
k_1 r_i > c_1 k_2 \\
k_2 r_i > c_2 k_1
\end{cases}
\]

the equilibrium \( E^* \) is local asymptotic stability.

Proof: for the equilibrium \( E^* \), the characteristic equation for the system (2) can become

\[
(\lambda + d_i k^2 - \alpha \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds + b_i + 2r_i u + c_i v) (\lambda + d_i k^2 - \beta \int_0^+ g_i(s) e^{-(\lambda + d_i+k^2)} ds + b_i + 2r_i v + c_i u) - c_i u v = 0.
\]

Since \( r_i u + c_i v = \alpha \int_0^+ g_i(s) e^{-\lambda s} ds - b_i \), \( r_i v + c_i u = \beta \int_0^+ g_i(s) e^{-\lambda s} ds - b_i \), we have
\[(\lambda + d_1 k^2 + \alpha \int_{0}^\infty g_1(s) e^{-\gamma s} ds - \alpha \int_{0}^\infty g_2(s) e^{-\gamma s} ds + r_1 u^*) \cdot (\lambda + d_2 k^2 + \beta \int_{0}^\infty g_1(s) e^{-\gamma s} ds - \beta \int_{0}^\infty g_2(s) e^{-\gamma s} ds + r_2 v^*) \].

\[c_1 c_2 u^* v^* = 0\]

Let \(\lambda = a + bi\)

\[A_1 = a + d_1 k^2 + r_1 u^* + \alpha \int_{0}^\infty g_1(s) e^{-\gamma s} ds - \alpha \int_{0}^\infty g_2(s) e^{-\gamma s} ds \cos \text{bsds}\]

\[A_2 = a + d_1 k^2 + r_2 v^* + \beta \int_{0}^\infty g_1(s) e^{-\gamma s} ds - \beta \int_{0}^\infty g_2(s) e^{-\gamma s} ds \cos \text{bsds}\]

\[B_1 = b + \alpha \int_{0}^\infty g_1(s) e^{-\gamma s} ds \sin \text{bsds} \quad , \quad B_2 = b + \beta \int_{0}^\infty g_2(s) e^{-\gamma s} ds \sin \text{bsds}\]

So we can get \((A_1 + B_1 i)(A_2 + B_2 i) = c_1 c_2 u^* v^*\).

which yields \(A_1 A_2 - B_1 B_2 = c_1 c_2 u^* v^*\). Further, we get \(A_1 A_2 = c_1 c_2 u^* v^* + B_1 B_2\)

\(A_1 B_2 + A_2 B_1 = 0\).

Therefore, \(A_1 A_2 \leq c_1 c_2 u^* v^*\). If \(\Re \lambda \geq 0\), then \(A_1 \geq r_1 u^*\), \(A_2 \geq r_2 v^*\), so \(A_1 A_2 \geq r_1 r_2 u^* v^* > 0\).

Therefore, \(r_1 r_2 \leq c_1 c_2\) which is contradiction with the suppose.

Therefore, the equilibrium \(E^*\) is local asymptotic stability.

The methods are also appropriate for a class of food chain system with stage structure. Such as

\[x(t) = T e^{-\tau}(x - f) - U y(t) - a x(t) y(t)\]

\[y(t) = a x(t) y(t) - d y(t) - a y(t) z(t)\]

\[z(t) = a y(t) z(t) - d z(t)\]

Where, \(a(i = 1, 2, 3, 4), d(i = 1, 2), T, U, V\) are positive, \(x(t)\) represents the population densities of the juvenil e and adult at time \(t\) and location \(x\), \(y(t), z(t)\) respectively represent the middle and top predators.

### 3. Global Stability

Using the upper-lower solution method and the monotone iterative method to consider the stability the following equations with the initial-boundary value problem:

\[
\begin{align*}
\frac{\partial u_i(t,x)}{\partial t} &= d_i \frac{\partial^2 u_i(t,x)}{\partial x^2} + \alpha \left((g_1 * u_i)(t,x)\right) - b_i u_i(t,x) - c_i u_i(t,x) u_j(t,x) \\
\frac{\partial u_2(t,x)}{\partial t} &= d_2 \frac{\partial^2 u_2(t,x)}{\partial x^2} - b_2 u_2(t,x) - c_2 u_2(t,x) u_j(t,x) + \beta (g_2 * u_i)(t,x) \\
u_i(t,x) &= \phi_i(t,x), (t,x) \in [-\tau,0] \times [0,\pi] \\
\frac{\partial u_i(t,0)}{\partial x} &= 0, \quad t > 0, (i = 1,2)
\end{align*}
\]

**Definition 3.1** A pair of smooth function \(\tilde{u} = (\tilde{u}_1, \tilde{u}_2)\) and \(\hat{u} = (\hat{u}_1, \hat{u}_2)\) are said to be the coupled upper and lower solutions of problem (4),if \(\hat{u}_i \geq \tilde{u}_i\) (i = 1, 2) in \([-\tau,0] \times [0,\pi]\),and the following differential inequalities hold
Lemma 3.1 If there exists a pair of upper and lower \( \tilde{u}, \tilde{u} \) of problem (4) and \( \varphi(t, x) \) is Hölder continue in \([-\tau, 0] \times [0, \pi] \), then the system (4) has the unique solution \((u_i(t, x), u_2(t, x))\) satisfying \( u_i \leq u \leq \tilde{u}, (i = 1, 2) \).

Lemma 3.2 If \( \varphi(t, x) \) is Hölder continue in \([-\tau, 0] \times [0, \pi] \), and \( \varphi_i(t, x) \geq 0, \varphi_i(0, x) \geq 0 \) (i = 1, 2), then the system (4) has the unique solution.

Lemma 3.3 With the assuming of Lemma 3.2, if \( u(t, x) \) is the solution of the following problem

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \alpha \int_0^1 g(s)e^{-\gamma s}G(x, s, y)u(t, s, y)dyds - \beta u^2(t, x) +Au(t, x), t > 0, x \in [0, \pi]
\]

where, \( \alpha, \beta > 0, \int_0^1 g(s)ds = 1, A \geq 0 \), then \( \lim_{t \to +\infty} u(t, x) = \frac{\alpha \int_0^1 g(s)e^{-\gamma s}ds + A}{\beta} , x \in [0, \pi] \).

Theorem 3.1 If \( r_f \geq c_1c_2, \varphi_i(t, x) \geq 0, \varphi_i(0, x) \neq 0 \) (i = 1, 2), and let \((u_i(t, x), u_2(t, x))\) be the solution of the system (4), then \( \lim_{t \to +\infty} u_i(t, x), u_2(t, x) = (u_i^*, u_2^*) \), uniformly for \( x \in [0, \pi] \).

Proof: let \( K_i = \lim_{t \to +\infty} \sup_{x \in [0, \pi]} \max_{x \in [0, \pi]} u_i(t, x) \), \( K_2 = \lim_{t \to +\infty} \sup_{x \in [0, \pi]} \max_{x \in [0, \pi]} u_2(t, x) \)

Let \( u_1^{(i)}(t, x), u_2^{(i)}(t, x) \) be the solutions of

\[
\begin{align*}
\frac{\partial u_1^{(i)}(t, x)}{\partial t} & = d_i \frac{\partial^2 u_1^{(i)}(t, x)}{\partial x^2} + \alpha \left( g_i \ast u_1^{(i)} \right)(t, x) - b_i u_1^{(i)}(t, x) - r_i \left[ u_1^{(i)} \right]^2(t, x) \\
\frac{\partial u_2^{(i)}(t, x)}{\partial t} & = d_2 \frac{\partial^2 u_2^{(i)}(t, x)}{\partial x^2} - b_2 u_2^{(i)}(t, x) - r_2 \left[ u_2^{(i)} \right]^2(t, x) + \beta \left( g_2 \ast u_2^{(i)} \right)(t, x)
\end{align*}
\]

(5)

Clearly \( u_1^{(i)}(t, x), u_2^{(i)}(t, x) \) and \((0, 0)\) are the upper and lower solutions of problem (4), and by Lemma 3.1, we get \( 0 \leq u_i \leq u_i^{(i)}, (i = 1, 2) \).

On the other hand, by Lemma 3.3, we have

\[
\begin{align*}
\lim_{t \to +\infty} u_1^{(i)}(t, x) & = \frac{\alpha \int_0^1 g_i(s)e^{-\gamma s}ds - b_i}{r_i} \Delta \beta_i^{(i)}, \quad \lim_{t \to +\infty} u_2^{(i)}(t, x) = \frac{\beta \int_0^1 g_2(s)e^{-\gamma s}ds - b_2}{r_2} \Delta \beta_2^{(i)}
\end{align*}
\]

(6)

And thus for any sufficiently small \( \varepsilon > 0 \), there exists \( t_i > 0 \), such that \( t > t_i \),

\[
\max_{x \in [0, \pi]} u_i(t, x) < \beta_i^{(i)} + \varepsilon, \quad \max_{x \in [0, \pi]} u_i(t, x) < \beta_i^{(i)} + \varepsilon
\]

(7)
Let $u_1^{(i)}(t,x), u_2^{(i)}(t,x)$ be the solutions of

$$\frac{\partial u_1^{(i)}}{\partial t} = d_1 \frac{\partial^2 u_1^{(i)}}{\partial x^2} + \alpha \left((g_1 \ast u_1^{(i)}) - b_1 u_1^{(i)} - r_1 \left[u_1^{(i)}\right]^2 - c_1 u_1^{(i)} u_2^{(i)}\right)$$

$$\frac{\partial u_2^{(i)}}{\partial t} = d_2 \frac{\partial^2 u_2^{(i)}}{\partial x^2} - b_2 u_2^{(i)} - r_2 \left[u_2^{(i)}\right]^2 + \beta \left(g_2 \ast u_2^{(i)}\right) - c_2 u_1^{(i)} u_2^{(i)}$$

(8)

Then $(u_1^{(i)}(t,x), u_2^{(i)}(t,x))$ and $(u_1^{(i)}(t,x), u_2^{(i)}(t,x))$ are a pair of upper and lower solutions of problem (4), and by Lemma 3.1, we get $u_1^{(i)} \leq u \leq u_2^{(i)}(t = 1,2)$

By (7) and (8), we have

$$\frac{\partial u_1^{(i)}}{\partial t} \geq d_1 \frac{\partial^2 u_1^{(i)}}{\partial x^2} + \alpha \left((g_1 \ast u_1^{(i)}) - b_1 u_1^{(i)} - r_1 \left[u_1^{(i)}\right]^2 - c_1 u_1^{(i)} u_2^{(i)} + \epsilon\right)$$

$$\frac{\partial u_2^{(i)}}{\partial t} = d_2 \frac{\partial^2 u_2^{(i)}}{\partial x^2} - b_2 u_2^{(i)} - r_2 \left[u_2^{(i)}\right]^2 + \beta \left(g_2 \ast u_2^{(i)}\right) - c_2 u_1^{(i)} u_2^{(i)} + \epsilon)$$

(9)

By the comparison principle, we get $u_1^{(i)} \geq v_1^{(i)}, u_2^{(i)} \geq v_2^{(i)}$, where $v_1^{(i)}$ and $v_2^{(i)}$ are the upper and lower solutions of problem

$$\frac{\partial v_1^{(i)}}{\partial t} = d_1 \frac{\partial^2 v_1^{(i)}}{\partial x^2} + \alpha \left((g_1 \ast v_1^{(i)}) - b_1 v_1^{(i)} - r_1 \left[v_1^{(i)}\right]^2 - c_1 v_1^{(i)} v_2^{(i)} + \epsilon\right)$$

$$\frac{\partial v_2^{(i)}}{\partial t} = d_2 \frac{\partial^2 v_2^{(i)}}{\partial x^2} - b_2 v_2^{(i)} - r_2 \left[v_2^{(i)}\right]^2 + \beta \left(g_2 \ast v_2^{(i)}\right) - c_2 v_1^{(i)} v_2^{(i)} + \epsilon)$$

(10)

By Lemma 3.3, we have

$$\lim_{t \to +\infty} v_1^{(i)}(t,x) = \frac{\alpha \int_{s_1}^{x} g_1(s)e^{-\gamma_1 s}ds - \theta_1 - c_1 (\beta_2^{(i)} + \epsilon)}{r_1}$$

$$\lim_{t \to +\infty} v_2^{(i)}(t,x) = \frac{\beta \int_{s_2}^{x} g_2(s)e^{-\gamma_1 s}ds - \theta_2 - c_2 (\beta_2^{(i)} + \epsilon)}{r_2}$$

Therefore, we can conclude that

$$0 < \alpha^{(i)} \leq \liminf_{t \to +\infty} \min_{x \in [0,1]} u_1(t,x) \leq \limsup_{t \to +\infty} \max_{x \in [0,1]} u_1(t,x) \leq \beta^{(i)}$$

$$0 < \alpha^{(i)} \leq \liminf_{t \to +\infty} \min_{x \in [0,1]} u_2(t,x) \leq \limsup_{t \to +\infty} \max_{x \in [0,1]} u_2(t,x) \leq \beta^{(i)}$$

(11)

Where, $\alpha^{(i)} = \frac{\alpha \int_{s_1}^{x} g_1(s)e^{-\gamma_1 s}ds - \theta_1 - c_1 \beta_2^{(i)}}{r_1}, \alpha^{(i)} = \frac{\beta \int_{s_2}^{x} g_2(s)e^{-\gamma_1 s}ds - \theta_2 - c_2 \beta_2^{(i)}}{r_2}$

Furthermore, for any sufficiently small $\epsilon > 0$, there exists $t > t_2$, such that

$$\min_{x \in [0,1]} u_1^{(i)}(t,x) > \alpha^{(i)} - \epsilon, \min_{x \in [0,1]} u_2^{(i)}(t,x) > \alpha^{(i)} - \epsilon, \ t > t_2$$

(12)

Let $u_1^{(2)}(t,x), u_2^{(2)}(t,x)$ be the solutions of
\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= d_i \frac{\partial^2 u_i}{\partial x^2} + \alpha_i \left( g_i * u_i^{(1)} \right) - b_i u_i^{(1)} - r_i u_i^{(1)} - c_i u_i^{(1)} u_2^{(i)} \\
\frac{\partial u_2}{\partial t} &= d_i \frac{\partial^2 u_2}{\partial x^2} - b_i u_2^{(i)} - r_i u_2^{(i)} + \beta_i \left( g_i * u_2^{(i)} \right) - c_i u_2^{(i)} u_2^{(i)} \\
u_i^{(1)}(t,x) &= K_i, u_2^{(i)}(t,x) = K_2
\end{align*}
\] (13)

By definition 3.1, \((u_i^{(2)}(t,x), u_2^{(2)}(t,x))\) and \((u_i^{(1)}(t,x), u_2^{(1)}(t,x))\) are a pair of upper and lower solutions of problem (4), and by Lemma 3.1, we get \(u_i^{(1)} \leq u_i^{(2)}, (i = 1, 2)\)

By (12) and (13), we have
\[
\begin{align*}
\frac{\partial u_i}{\partial t} &\leq d_i \frac{\partial^2 u_i}{\partial x^2} + \alpha_i \left( g_i * u_i^{(1)} \right) - b_i u_i^{(1)} - r_i u_i^{(1)} - c_i u_i^{(1)} (\alpha_i^{(0)} - \varepsilon) \\
\frac{\partial u_2}{\partial t} &\leq d_i \frac{\partial^2 u_2}{\partial x^2} - b_i u_2^{(i)} - r_i u_2^{(i)} + \beta_i \left( g_i * u_2^{(i)} \right) - c_i u_2^{(i)} (\alpha_i^{(0)} - \varepsilon) \\
\end{align*}
\] (14)

By the comparison principle, we get \(u_i^{(2)} \leq w_i^{(1)}, u_2^{(2)} \leq w_2^{(i)}\), where \(w_i^{(1)}, w_2^{(i)}\) are the upper and lower solutions of problem
\[
\begin{align*}
\frac{\partial w_i^{(1)}}{\partial t} &= d_i \frac{\partial^2 w_i^{(1)}}{\partial x^2} + \alpha_i \left( g_i * w_i^{(1)} \right) - b_i w_i^{(1)} - r_i w_i^{(1)} - c_i w_i^{(1)} (\alpha_i^{(0)} - \varepsilon) \\
\frac{\partial w_2^{(i)}}{\partial t} &= d_i \frac{\partial^2 w_2^{(i)}}{\partial x^2} - b_i w_2^{(i)} - r_i w_2^{(i)} + \beta_i \left( g_i * w_2^{(i)} \right) - c_i w_2^{(i)} (\alpha_i^{(0)} - \varepsilon) \\
\end{align*}
\] (15)

By Lemma 3.3, we have
\[
\lim_{t \to +\infty} w_i^{(2)}(t,x) = \frac{\alpha_i^{(1)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(0)} - \varepsilon)}{r_i}, \quad \lim_{t \to +\infty} w_2^{(2)}(t,x) = \frac{\beta_i^{(1)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(0)} - \varepsilon)}{r_i}
\]

Therefore we can conclude that
\[
0 < \alpha_1^{(0)} \leq \liminf_{t \to +\infty} \min_{s \in [0,t]} u_i(t,x) \leq \limsup_{t \to +\infty} \max_{s \in [0,t]} u_i(t,x) \leq \beta_1^{(0)}
\]
\[
0 < \alpha_2^{(0)} \leq \liminf_{t \to +\infty} \min_{s \in [0,t]} u_i(t,x) \leq \limsup_{t \to +\infty} \max_{s \in [0,t]} u_i(t,x) \leq \beta_2^{(0)}
\]

(16)

where, \(\beta_1^{(0)} = \frac{\alpha_1^{(0)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(0)} - \varepsilon)}{r_i}, \beta_2^{(0)} = \frac{\beta_1^{(0)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(0)} - \varepsilon)}{r_i}\)

It is obvious that \(0 < \alpha_1^{(0)} \leq \beta_1^{(0)} , 0 < \alpha_2^{(0)} \leq \beta_2^{(0)}\).

Continue this process, we can get the following sequences
\[
\alpha_1^{(i+1)} = \frac{\alpha_1^{(i)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(i)} - \varepsilon)}{r_i}, \alpha_2^{(i+1)} = \frac{\beta_1^{(i)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(i)} - \varepsilon)}{r_i}
\]
\[
\beta_1^{(i+1)} = \frac{\alpha_1^{(i)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(i)} - \varepsilon)}{r_i}, \beta_2^{(i+1)} = \frac{\beta_1^{(i)} \int_{0}^{\infty} g_i(s)e^{-\gamma s}ds - b_i - c_i (\alpha_i^{(i)} - \varepsilon)}{r_i}
\]
\[
\beta_i^{(0)} = \frac{\alpha \int_0^\infty g_i(s)e^{-\gamma_i s}ds - b_i}{r_i}, \quad \beta_j^{(0)} = \frac{\beta \int_0^\infty g_j(s)e^{-\gamma_j s}ds - b_j}{r_j}
\]

(17)

And satisfying

\[
0 < \alpha^{(k)}_i \leq \liminf_{h \to +\infty} \min_{u \in [0, x]} u_i(t, x) \leq \limsup_{h \to +\infty} \max_{u \in [0, x]} u_i(t, x) \leq \beta^{(k)}_i \\
0 < \alpha^{(k)}_j \leq \liminf_{h \to +\infty} \min_{u \in [0, x]} u_j(t, x) \leq \limsup_{h \to +\infty} \max_{u \in [0, x]} u_j(t, x) \leq \beta^{(k)}_j
\]

\[
\begin{bmatrix} \alpha^{(k+1)}_i, \beta^{(k+1)}_i \end{bmatrix} \subseteq \begin{bmatrix} \alpha^{(k)}_i, \beta^{(k)}_i \end{bmatrix}, \quad \begin{bmatrix} \alpha^{(k+1)}_j, \beta^{(k+1)}_j \end{bmatrix} \subseteq \begin{bmatrix} \alpha^{(k)}_j, \beta^{(k)}_j \end{bmatrix}
\]

We need to testify the following \( \alpha_i = \beta_i = u_i^*, \quad \alpha_j = \beta_j = u_j^* \)

Let \( k \to +\infty \) in (17), we derive that

\[
\begin{align*}
\alpha r_i + c_i \beta_j &= \alpha \int_0^\infty g_i(s)e^{-\gamma_i s}ds - b_i \\
\alpha r_j + c_j \beta_i &= \beta \int_0^\infty g_j(s)e^{-\gamma_j s}ds - b_j \\
r_i \beta_j + c_i \alpha_j &= \alpha \int_0^\infty g_i(s)e^{-\gamma_i s}ds - b_i \\
r_j \beta_i + c_j \alpha_i &= \beta \int_0^\infty g_j(s)e^{-\gamma_j s}ds - b_j
\end{align*}
\]

(18)

Which yields

\[
\begin{cases}
(\alpha_i - \beta_i)r_i - c_i(\alpha_j - \beta_j) = 0 \\
(\alpha_j - \beta_j)r_j - c_j(\alpha_i - \beta_i) = 0
\end{cases}
\]

Since

\[
\begin{bmatrix} r_i \\ -c_i \\ r_j \\ -c_j \end{bmatrix} = \begin{bmatrix} r_i \\ -c_i \\ r_j \\ -c_j \end{bmatrix} > 0,
\]

system (4) has only zero solution with respect to \( \alpha_i = \beta_i, \quad \alpha_j = \beta_j \). Therefore, from (18), we can get \( \alpha_i = \beta_i = u_i^*, \quad \alpha_j = \beta_j = u_j^* \).

This completes the proof.

The methods are also appropriate for a class of cooperation model and Epidemic model with stage structure, and so on. for example:

\[
\begin{align*}
\frac{\partial u_i(x, t)}{\partial t} - d_1 \frac{\partial^2 u_i(x, t)}{\partial x^2} &= \alpha ((g_i \ast u_i)(x, t)) - r_i u_i^2(x, t) + c_i u_i(x, t)u_j(x, t) \\
\frac{\partial u_j(x, t)}{\partial t} - d_2 \frac{\partial^2 u_j(x, t)}{\partial x^2} &= \beta ((g_j \ast u_j)(x, t)) - r_j u_j^2(x, t) + c_j u_i(x, t)u_j(x, t)
\end{align*}
\]

Where, the parameters have the same the meanings with consistent.

4. Conclusion

Using of constant linear methods, we considered the local asymptotic stability; Employing the upper-lower solutions and monotone iterative methods, we considered the global stability of the positive equilibrium point about the competition model with diffusion terms and stage structure. The conclusions are also appropriate for the corresponding parabolic-ordinary differential system \((d_i = 0 \text{ for some or all } i)\). Besides, The conclusions are also appropriate for the predator-prey model and epidemic model and so on. So, The traditional results are improved and this result adds to the previous results and applies to broader frameworks. But, with the increase of the invasive species, we can study the multi-group reaction diffusion model in the next few years.

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References


