Variants of Chebyshev's Method with Eighth-Order Convergence for Solving Nonlinear Equations

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Abstract: This paper develops the variants of Chebyshev's method by applying Lagrange interpolation and finite difference to eliminate the second derivative appearing in the Chebyshev's method. The results of this research show that the modified eight-order method has the efficiency index 1.5157. Numerical simulations show that the effectiveness and performance of the new method in solving nonlinear equations are encouraging.

Keywords: Chebyshev's Method, Finite Differences, Lagrange Interpolation, Nonlinear Equations, Order of Convergence

1. Introduction

The solution of nonlinear equations is very important problem in numerical analysis. One of the problems in nonlinear equations is to determine the roots of the equation $f(x) = 0$ so that the sophisticated iteration method to solve it is in need. Many iteration methods can be used to solve the problem. Newton's method is the oldest and widely used method written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

To accelerate the convergence of (1) many authors have modified it as we can see in [6, 8, 10, 11]. If we expand $f(x)$ about $x = x_n$ using Taylor expansion and ignoring the term of containing the third order, we obtain Chebyshev's method in the form of

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{1}{2}\frac{f(x_n)f''(x_n)}{(f'(x_n))^2}\right), \tag{2}$$

which has third-order convergence [1, 3]. Kou *et al.* [7] proposed a method which is free from second-derivative by approximating $f''(x_n)$ in (2) by a finite difference as follows

$$y_n = x_n - \theta\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

$$x_{n+1} = x_n - \left(1 + \frac{1}{2\theta}\frac{f(y_n) - f'(x_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{4}$$

This method has a third-order convergence [8]. The other methods modified from (2) having a four-order convergence can be seen in [2, 4, 5].

In this paper, we present the combination of Newton's method and Chebyshev's method into a three-step iteration method. We also incorporate finite difference to approximate the second derivative in second step and Lagrange interpolation to approximate the first derivative in the third step. The discussion of the new method and their convergence and analysis are carried out in Section 2. Then, in Section 3 we perform numerical simulations using some test functions, and compare the new method with some other methods, such as Newton's method, Halley's method, and Chebyshev's method.

2. Proposed Methods

In this section, for construction of the new iterative method, we use the iterative methods given by equations (1) and (2). We consider the following three-step method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{5}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 + \frac{1}{2}\frac{f(y_n)f''(y_n)}{(f'(y_n))^2}\right), \tag{6}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{7}$$

which requires one evaluation of the second derivative of the function. To remove this derivative, firstly we replace $f''(y_n)$ in (6) with a finite difference [10], that is

$$f''(y_n) \approx \frac{f'(y_n) - f'(x_n)}{y_n - x_n}, \tag{8}$$

where y_n is equation (1).

Secondly, we approximate $f'(z_n)$ by a derivative of Lagrange interpolation polynomial $L_2(x)$ passing the points $(x_n, f(x_n))$, $(y_n, f(y_n))$, and $(z_n, f(z_n))$, that is

$$L_2'(z_n) = \frac{2z_n - (y_n + z_n)}{(x_n - y_n)(x_n - z_n)} f(x_n)$$
$$+ \frac{2z_n - (x_n + z_n)}{(y_n - x_n)(y_n - z_n)} f(y_n)$$
$$+ \frac{2z_n - (x_n + y_n)}{(z_n - x_n)(z_n - y_n)} f(z_n). \tag{9}$$

Simplifying equation (9) yields

$$L_2'(z_n) = f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n], \tag{10}$$

where

$$f[x_n, z_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \tag{11}$$

$$f[y_n, z_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}, \tag{12}$$

$$f[x_n, y_n] = \frac{f(y_n) - f(x_n)}{y_n - x_n}. \tag{13}$$

Let $f'(z_n) \approx L_2'(z_n)$, and substituting equation (8) into (6) and (10) into (7), we get

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{14}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f(y_n)(f'(x_n))^2}{2f(x_n)(f'(y_n))^2}\right), \tag{15}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}. \tag{16}$$

We can see that the new scheme (14), (15), and (16) are free from second derivative. For the method defined by scheme (14), (15), and (16), we have the following analysis of convergence.

Theorem 1 Assume that function f is sufficiently differentiable and f has a simple root $\alpha \in I$. If the initial point x_0 is sufficiently close to α, then the method of iteration in equations (14)-(16) has eighth-order convergence and satisfying the following error equation:

$$e_{n+1} = \left(\frac{3}{2}c_2^3 c_3^2\right)e_n^8 + O(e_n^9),$$

where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k! f'(\alpha)$, $k = 2, 3, \dots, 8$.

Proof. Let α be a simple root of the equation $f(x) = 0$, then $f'(\alpha) \neq 0$. Furthermore, using Taylor expansion of the $f(x_n)$ about $x_n = \alpha$, we obtain

$$f(x_n) = f(\alpha) + (x - \alpha)f'(\alpha) + \frac{(x - \alpha)^2}{2!}f''(\alpha)$$
$$+ \cdots + O(x_n - \alpha)^9. \tag{17}$$

Because $f(\alpha) = 0$ equation (17) can be rewritten in the form of

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots + O(e_n^9)\right), \tag{18}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2, 3, \dots, 8$.

Similarly, carrying out the Taylor expansion again for $f'(x_n)$ about $x_n = \alpha$, after simplification, we obtain

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + \cdots + O(e_n^8)\right). \tag{19}$$

Dividing (18) by (19) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + \cdots + O(e_n^9). \tag{20}$$

Substituting (20) into (14), we get

$$y_n = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + \cdots + O(e_n^9). \tag{21}$$

Moreover, the Taylor expansion of $f(y_n)$ about $y_n = \alpha$ respectively are given by

$$f(y_n) = f'(\alpha)(c_2 e_n^2 + 2(c_3 + c_2^2)e_n^3 + \cdots + O(e_n^9)), \tag{22}$$

Repeating the above process, we can find approximation to $f'(y_n)$ as follows:

$$f'(y_n) = f'(\alpha)(1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_2^3)e_n^3$$
$$+ \cdots + O(e_n^8)). \tag{23}$$

Now dividing (22) by (23), we obtain

$$\frac{f(y_n)}{f'(y_n)} = c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + \cdots + O(e_n^9). \tag{24}$$

From (18), (19), (22) and (23), we get

$$f'(y_n)f'(x_n)f(y_n) - f(y_n)(f'(x_n))^2 = (f'(\alpha))^3(-2c_2^2 e_n^3$$
$$+ (2c_2^3 - 7c_2 c_3)e_n^4 + \cdots + O(e_n^9)). \tag{25}$$

Using (18) and (22), we obtain

$$2f(x_n)(f'(y_n))^2 = (f'(\alpha))^3(2e_n + 2c_2^2 e_n^2 + 2(4c_2^2 + c_3)e_n^3$$
$$+ \cdots + O(e_n^9)). \tag{26}$$

Furthermore, dividing (25) by (26), we have

$$\frac{f'(y_n)f'(x_n)f(y_n) - f(y_n)(f'(x_n))^2}{2f(x_n)(f'(y_n))^2} = -c_2^2 e_n^2 + (2c_2^3$$
$$- \frac{7}{2}c_2 c_3)e_n^3 + \cdots + O(e_n^9). \tag{27}$$

Substituting (21), (24) and (27) into (15), we obtain

$$z_n = \alpha - \frac{3}{2}c_2^2 c_3 e_n^5 + \left(2c_2^5 - 6c_2 c_3^2 - 2c_2^2 c_4 + \frac{13}{2}c_2^6\right)e_n^4$$

$$+ \cdots + O(e_n^9). \tag{28}$$

Applying Taylor expansion of $f(z_n)$ about $z_n = \alpha$, we get

$$f(z_n) = f'(\alpha)\left(-\tfrac{3}{2}c_2^2 c_3 e_n^5 + (2c_2^5 - 6c_2 c_3^2 - 2c_2^2 c_4\right.$$
$$\left. + \tfrac{13}{2}c_2^6)e_n^4 + \cdots + O(e_n^9)\right) \tag{29}$$

To obtain the $f[x_n, z_n]$, we substitute (18), (28) and (29) into (11), that is

$$f[x_n, z_n] = f'(\alpha)\left(1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3 + \cdots + O(e_n^9)\right). \tag{30}$$

Using the same strategy, $f[y_n, z_n]$ can be obtained by substituting (21), (22), (28) and (29) into (12), that is

$$f[y_n, z_n] = f'(\alpha)\left(1 + c_2^2 e_n^2 + (2c_2 c_3 - 2c_2^3)e_n^3 + \cdots + O(e_n^9)\right). \tag{31}$$

To obtain $f[x_n, y_n]$, we substitute (18), (21) and (22) into (13), that is

$$f[x_n, y_n] = f'(\alpha)\left(1 + c_2 e_n + (c_2^2 + c_3)e_n^2 + \cdots + O(e_n^9)\right). \tag{32}$$

Using the equations (30), (31) and (32), we get

$$f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n] = f'(\alpha)(1 - c_2 c_3 e_n^2$$
$$+ \cdots + O(e_n^9)) \tag{33}$$

Dividing (29) by (33), we have

$$\frac{f(z_n)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]} = -\tfrac{3}{2}c_2^2 c_3 e_n^5 + (2c_2^5$$
$$- 6c_2 c_3^2 - 2c_2^2 c_4 + \tfrac{13}{2}c_2^3 c_3)e_n^6 + \cdots + O(e_n^9) \tag{34}$$

Substituting equations (28) and (34) into (16), we obtain

$$x_{n+1} = \alpha + \left(\tfrac{3}{2}c_2^3 c_3^2\right)e_n^8 + O(e_n^9). \tag{35}$$

Putting $e_{n+1} = x_{n+1} - \alpha$, then from equation (35) we get

$$e_{n+1} = \left(\tfrac{3}{2}c_2^3 c_3^2\right)e_n^8 + O(e_n^9). \tag{36}$$

This means that the method defined by scheme (14), (15), and (16) has eighth-order convergence. The proof is completed.

Schemes (14), (15), and (16) require three evaluations of the functions and two of their first derivative per iteration. So that, if we consider the definition of eficiency index as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evalations per iteration required by the method. We have that the method obtained by schemes (14), (15), and (16) has the efficiency index equal to $8^{\frac{1}{5}} \approx 1.5157$, which is

better than the Newton's method having the efficiency index $2^{\frac{1}{2}} \approx 1.4142$, Halley's method $3^{\frac{1}{3}} \approx 1.4422$ and Chebyshev's method $3^{\frac{1}{3}} \approx 1.4422$.

3. Numerical Expriments

In this section some numerical simulations are performed to compare Chebyshev-Lagrange method to some other methods, such as Newton's method, Halley's method, and Chebyshev's method. The functions used are as follows:

$$f_1(x) = 10xe^{-x^2} - 1, \alpha \in (1.5, 2.0),$$
$$f_2(x) = x^5 + x^4 + 4x^2 - 15, \alpha \in (1.0, 1.5),$$
$$f_3(x) = e^x - 4x^2, \alpha \in (0.5, 1.0),$$
$$f_4(x) = \sin^2(x) - x^2 + 1, \alpha \in (1.0, 1.5),$$
$$f_5(x) = \cos(x) - x, \alpha \in (0.5, 1.0),$$
$$f_6(x) = (x - 1)^6 - 1, \alpha \in (1.5, 2.5).$$

We also calculate the computational order of convergence (COC) of the method using the following equation:

$$COC = \frac{\ln\left|(x_{n+1} - \alpha)/(x_n - \alpha)\right|}{\ln\left|(x_n - \alpha)/(x_{n-1} - \alpha)\right|} \tag{37}$$

The calculation is carried out using software with 800 digits accuracy and tolerance $\epsilon = 1.0 \times 10^{-100}$. The stoping criteria of the iteration are $|x_{n+1} - x_n|$ and $|f(x_{n+1})|$. The value x_{n+1} is taken as the exact root α.

In Table 1, we give initial value (x_0), number of iterations (N), and the computational order of convergence (COC). An asterisk (*) on the number of iterations indicates that the method converges to different roots. Table 1 shows a comparison of the number of iterations and COC several methods to resolve the above functions including Newton's method (NM), Halley's method (HM), Chebyshev's method (CM), and Chebyshev-Lagrange method (CLM) for some given initial values.

Based on Table 1 it is generally known that the CLM has the number of iterations less when compared to the other methods. This means that CLM has a better efficiency in computing process than other methods. From Table 1 we observe that the COC perfectly coincides with the theoretical results at Theorem 1. The results presented in Table 1 show that the CLM has higher convergence order compared to the other methods.

Tabel 1. Comparison of the number of iterations and COC.

$f(x)$	x_0	N				COC			
		NM	HM	CM	CLM	NM	HM	CM	CLM
	0.9	9	6	5	3	2.0000	3.0000	3.0000	8.0029
	1.3	8	5	5	3	2.0000	3.0000	3.0000	8.0000
$f_1(x)$	1.5	7	5	5	3	2.0000	3.0000	3.0000	8.0000
	2.0	8	5	6	3	2.0000	3.0000	3.0000	8.0000
$f_2(x)$	0.5	12	6	32	4	2.0000	3.0000	3.0000	8.0008

$f(x)$	x_0	N				COC			
		NM	HM	CM	CLM	NM	HM	CM	CLM
$f_3(x)$	1.0	9	5	6	3	2.0000	3.0000	3.0000	7.9999
	1.5	7	5	5	3	2.0000	3.0000	3.0000	8.0000
	2.0	9	6	6	3	2.0000	3.0000	3.0000	8.0001
	0.4	8	5	7	3	2.0000	3.0000	3.0000	8.0000
	1.6	7	5	5	3	2.0000	3.0000	3.0000	8.0000
	1.0	8	5	5	3	2.0000	3.0000	3.0000	8.0000
	1.4	8	5	6	3	2.0000	3.0000	3.0000	8.0000
$f_4(x)$	0.4	12	7	12*	4	2.0000	3.0000	3.0000	7.9997
	0.8	9	6	5*	3	2.0000	3.0000	3.0000	7.9973
	1.6	7	5	5	3	2.0000	3.0000	3.0000	8.0000
	2.0	8	5	6	3	2.0000	3.0000	3.0000	8.0000
$f_5(x)$	-1.0	11	7	6	4	2.0000	3.0000	3.0000	8.0000
	0.0	8	5	6	3	2.0000	3.0000	3.0000	8.0000
	1.0	7	5	5	3	2.0000	3.0000	3.0000	8.0000
	2.0	7	5	6	3	2.0000	3.0000	3.0000	8.0000
$f_6(x)$	1.7	11	6	6*	4	2.0000	3.0000	3.0000	8.0000
	1.9	8	5	5	3	2.0000	3.0000	3.0000	8.0000
	2.4	9	6	6	3	2.0000	3.0000	3.0000	8.0029
	2.6	10	6	7	4	2.0000	3.0000	3.0000	7.9996

Table 2 shows a comparison of the absolute difference $|x_{n+1} - x_n|$ and absolute value of the functions $|f(x_{n+1}))|$ of several methods to resolve the above functions including Newton's method, Halley's method, Chebyshev's method, and Chebyshev-Lagrange method for some given initial values.

Tabel 2. Comparison $|x_{n+1} - x_n|$ and $|f(x_{n+1}))|$.

$f(x)$	x_0	$|x_{n+1} - x_n|$				$|f(x_{n+1})|$			
		NM	HM	CM	CLM	NM	HM	CM	CLM
$f_1(x)$	0.9	6.85e-85	4.85e-53	1.73e-82	4.22e-40	1.24e-168	2.65e-157	2.51e-245	1.82e-317
	1.3	3.41e-81	6.03e-40	9.81e-36	2.19e-44	3.07e-161	5.12e-118	4.59e-105	9.59e-352
	1.5	1.46e-54	2.46e-65	3.09e-57	2.24e-67	5.61e-108	3.46e-194	1.43e-169	1.13e-533
	2.0	2.71e-55	1.76e-42	2.43e-45	2.12e-36	1.94e-109	1.27e-125	7.00e-134	7.45e-288
$f_2(x)$	0.5	2.08e-67	6.37e-58	8.17e-36	1.50e-18	1.71e-132	4.72e-171	3.27e-104	6.90e-142
	1.0	5.50e-97	7.04e-50	7.37e-56	6.82e-26	1.19e-191	6.38e-147	2.40e-164	1.25e-200
	1.5	2.30e-53	5.69e-80	5.14e-62	5.09e-57	2.08e-104	3.37e-237	8.17e-183	1.21e-449
	2.0	1.53e-77	2.77e-99	2.17e-63	5.33e-26	9.19e-153	3.88e-295	6.12e-187	1.76e-201
$f_3(x)$	0.4	5.76e-58	6.70e-35	2.20e-44	1.52e-43	9.86e-115	8.30e-103	5.47e-131	7.07e-345
	1.6	1.52e-63	4.03e-78	7.81e-65	1.99e-86	6.88e-126	1.81e-232	2.46e-192	6.11e-688
	1.0	2.55e-95	1.15e-57	7.42e-50	2.06e-56	1.94e-189	4.19e-171	2.11e-147	8.33e-448
	1.4	5.52e-62	4.21e-35	5.77e-88	5.06e-36	9.08e-123	2.05e-103	9.95e-262	1.08e-284
$f_4(x)$	0.4	1.07e-88	1.16e-45	8.26e-44	1.08e-26	2.23e-176	2.02e-135	1.60e-129	2.48e-210
	0.8	5.17e-54	4.14e-68	2.22e-43	3.10e-20	5.20e-107	9.27e-203	3.09e-128	1.17e-158
	1.6	2.00e-56	3.19e-72	1.11e-61	1.65e-72	7.82e-112	4.25e-215	3.84e-183	7.71e-577
	2.0	6.51e-65	3.47e-39	4.47e-96	7.10e-44	8.24e-129	5.47e-116	2.52e-286	8.91e-348
$f_5(x)$	-1.0	1.76e-71	2.68e-71	2.54e-78	5.11e-58	1.15e-142	3.75e-213	4.51e-234	5.61e-463
	0.0	1.80e-83	1.62e-43	2.54e-78	1.68e-40	1.19e-166	8.23e-130	4.51e-234	7.65e-323
	1.0	1.80e-83	4.42e-87	5.05e-83	1.05e-77	1.19e-166	1.68e-260	3.56e-248	1.86e-620
	2.0	5.63e-96	1.48e-35	6.67e-97	3.88e-75	1.17e-191	6.30e-106	8.18e-290	6.30e-600
$f_6(x)$	1.7	3.21e-66	2.08e-66	8.52e-36	1.32e-38	1.54e-130	1.58e-196	3.40e-104	1.42e-300
	1.9	3.63e-72	9.81e-62	4.00e-36	9.39e-39	1.97e-142	1.65e-182	3.52e-105	9.46e-302
	2.4	1.64e-53	2.80e-71	1.31e-42	1.41e-15	4.06e-105	3.82e-211	1.24e-124	2.41e-116
	2.6	2.74e-62	3.37e-45	5.85e-71	1.28e-21	1.13e-122	6.72e-133	1.10e-209	1.13e-164

The computational results presented in Table 2 show that in almost all of cases, the CLM has the absolute values of the function smaller when compared to Newton's method, Halley's method, and Chebyshev's method.

efficiency shows that this method is better than Newton's method, Halley's method, and Chebyshev's method.

4. Conclusions

In this paper we present the variants of Chebyshev's method by removing the second derivative using finite difference. This method requires three functions and two first derivative evaluations per iteration. We have that the order convergence of this method is eight. Analysis of the

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