
Close Form Solutions of the Fractional Generalized Reaction Duffing Model and the Density Dependent Fractional Diffusion Reaction Equation

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Abstract: The two variable $(G'/G, 1/G)$ -expansion method is significant for finding the exact traveling wave solution to nonlinear evolution equations (NLEEs) in mathematical physics, applied mathematics and engineering. In this article, we exert the two variable $(G'/G, 1/G)$ -expansion method for investigating the fractional generalized reaction Duffing model and density dependent fractional diffusion reaction equation and obtain exact solutions containing parameters. When the parameters are taken particular values, traveling wave solutions are transferred into the solitary wave solutions. The two variable $(G'/G, 1/G)$ -expansion method is the generalization of the original (G'/G) -expansion method established by Wang et al [21].

Keywords: Nonlinear Evolution Equation, Fractional Generalized Reaction Duffing Model, Density Dependent Fractional Diffusion Equation, Traveling Wave Solution

1. Introduction

The significance of nonlinear evolution equations is now well established. In the last three decades nonlinear phenomena are one of the most impressive fields of research. Nonlinear phenomena occur in various branches of science, engineering and biology, such as fluid mechanics, plasma physics, solid state physics, optical fiber, gas dynamics, elasticity, biomechanics, relativity, ecology, biophysics and so on. Since the appearance of solitary wave in natural science is expanding day by day, it is important to find the solitary wave solutions to NLEEs. The exact solutions to NLEEs help us to provide information about the structure of complex phenomena. As a key problem, finding their exact solutions is of great importance and it is actually executed through various efficient and powerful method, such as, the Hirota method [1], the Backlund transform method [2, 3], the inverse scattering transform method [4], the Jacobi elliptic function expansion method [5-7], the truncated Painleve expansion method [8-11], the tanh function method [12-15],

the Exp-function method [16-22], the (G'/G) -expansion method [23-30], the improved (G'/G) -expansion method [31-32], the two variable $(G'/G, 1/G)$ -expansion method [33, 34], the first integral method [35] etc. The main concept of the (G'/G) -expansion method is the exact solution of nonlinear NLEEs are revealed by a polynomial in one variable (G'/G) in which $G = G(\xi)$ satisfies the second order ordinary differential equation (ODE) $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants. In this article, we use the two variable $(G'/G, 1/G)$ -expansion method, which is the general case of the (G'/G) -expansion method. The main concept of the two variable $(G'/G, 1/G)$ -expansion method is the exact traveling wave solutions of nonlinear NLEEs can be written as a polynomials in two variables (G'/G) and $(1/G)$, in which $G = G(\xi)$ satisfies a second order linear ODE $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. The degree of the polynomial can be evaluated by taking homogeneous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs, where the coefficient of the polynomial can be determined by solving a set of algebraic equations. Recently, Li et al [33]

and Zayed et al [34] applied the two variable $(G'/G, 1/G)$ -expansion method and determined the exact solution of nonlinear NLEEs.

The objective of this article is study the fractional generalized reaction Duffing model and density dependent fractional diffusion reaction equation by making use of the two variable $(G'/G, 1/G)$ -expansion method.

2. Description of the Two Variable $(G'/G, 1/G)$ -expansion Method

Before starting the description of the $(G'/G, 1/G)$ -expansion method [33, 34], we discuss about the following fundamental concepts. Let us consider the second order ordinary differential equation (ODE):

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{1}$$

where, $\varphi = (G'/G)$ and $\psi = 1/G$, then we obtain

$$\varphi' = -\varphi^2 + \mu\psi - \lambda\psi' = -\varphi\psi. \tag{2}$$

Remark 1: If $\lambda < 0$, the general solution of equation (1) is:

$$G(\xi) = A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda} \tag{3}$$

where A_1 and A_2 are arbitrary constants.

Consequently, we obtain

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\varphi^2 - 2\mu\psi + \lambda) \tag{4}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 2: If $\lambda > 0$, the general solution of equation (1) is:

$$G(\xi) = A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos \sqrt{\lambda}\xi + \frac{\mu}{\lambda} \tag{5}$$

where A_1 and A_2 are arbitrary constants. Consequently, we obtain

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}(\varphi^2 - 2\mu\psi + \lambda) \tag{6}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 3: If $\lambda = 0$, the general solution of equation (1) is:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2 \tag{7}$$

where A_1 and A_2 are arbitrary constants and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\varphi^2 - 2\mu\psi) \tag{8}$$

$$\frac{\partial^{2\alpha}U(x,t)}{\partial t^{2\alpha}} + p \frac{\partial^{2\alpha}U(x,t)}{\partial t^{2\alpha}}qU(x,t) + rU^2(x,t) + sU^3(x,t) = 0 \tag{12}$$

where $t > 0, 0 < \alpha \leq 1$, here p, q, r and s are constants. If we take $r = 0$, then Eq. (12) converts into

$$\frac{\partial^{2\alpha}U(x,t)}{\partial t^{2\alpha}} + p \frac{\partial^{2\alpha}U(x,t)}{\partial t^{2\alpha}}qU(x,t) + sU^3(x,t) = 0 \tag{13}$$

where $t > 0, 0 < \alpha \leq 1$.

Now we introduce the fractional wave transformation:

Assume the nonlinear partial differential equation is in the form

$$Q = (U, U_z, U_x, U_{xx}, U_{xz}, U_{zz}, \dots) \tag{9}$$

where $U = U(x, t)$ is an unknown function and Q is a polynomial of $U(x, t)$ and its partial derivative.

Step 1: Consider the traveling wave transformation

$$\xi = x - vt, U(x, t) = U(\xi),$$

where v is the speed of traveling wave.

The wave variable permits us to reduce equation (9) into an ODE for $U = U(\xi)$:

$$R = (U, U', U'', U''', \dots) = 0 \tag{10}$$

where R is a polynomial of $U(\xi)$ and its total derivative with respect to ξ .

Step 2: Assume that the solution of equation (10) can be written as a polynomial in two variables φ and ψ as follows:

$$U(\xi) = \sum_{i=0}^N a_i \varphi^i + \sum_{i=1}^N b_i \varphi^{i-1} \psi \tag{11}$$

where, a_i ($i = 0, 1, \dots, N$) and b_i ($i = 0, 1, \dots, N$) are constant to be determined later.

Step 3: Taking homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (10) to determine the positive integer N in equation (11).

Step 4: Substitute equation (11) into equation (10) along with (2) and (4), the equation (10) can be reduce into a polynomial in φ and ψ , in which the degree of ψ is no longer than one. Equating the coefficients of this polynomial of like power to zero gives a system of algebraic equations which can be solved by using the software Maple or Mathematica to get the values of $a_i, b_i, v, \mu, A_1, A_2$ and λ where $\lambda < 0$.

Step 5: Similar application to step 4, substitute equation (11) into equation (10) along with (2) and (6) for $\lambda > 0$ (or (2) and (8) for $\lambda = 0$), we attain the exact solutions of equation (10) represented by trigonometric functions (or by rational function respectively).

3. Application

3.1. The Fractional Generalized Duffing Model

In this subsection, we apply the two variables $(G'/G, 1/G)$ -expansion method to obtain exact solutions of fractional generalized Duffing model which is in the form:

$$\xi = \frac{Kx^\alpha}{\Gamma(1+\alpha)} - \frac{Ct^\alpha}{\Gamma(1+\alpha)} \tag{14}$$

where C and K are non-zero constants.

Using the transformation (14), Eq. (12) reduced into the following ODE for $U = U(\xi)$:

$$(C^2 + Pk^2)U'' + qU + sU^3 = 0 \tag{15}$$

Balancing the highest order derivative U'' and nonlinear term U^3 , yields $N = 1$.

Therefore the solution (11) is of the following form:

$$U(\xi) = a_0 + a_1 \varphi(\xi) + b_1 \psi(\xi) \tag{16}$$

Case 1.1: For $\lambda < 0$, substituting Eq. (16) into Eq. (15) along with Eq. (2) and Eq. (4) yields a polynomial equation and setting each coefficient polynomial to zero gives a set of algebraic equations for $a_0, a_1, b_1, \mu, \sigma, \lambda, p, q, s, c$ and k as follows:

$$\varphi^0: 2sb_1^3\lambda^3\mu - qa_0\lambda^4\sigma^2 - c^2b_1\lambda^2\mu^3 - sa_0^3\lambda^4\sigma^2 + 3sa_0b_1^2\lambda^2\mu^2 - sa_0^3\mu^4 - c^2b_1\lambda^4\mu\sigma - 2sa_0^3\mu^2\sigma\lambda^2 - 2qa_0\mu^2\lambda^2\sigma - pk^2b_1\lambda^2\mu^3 - qa_0\mu^4 + 3sa_0b_1^2\lambda^4\sigma - pk^2b_1\lambda^4\mu\sigma = 0$$

$$\varphi^1: -qa_1\lambda^4\sigma^2 - 2pk^2a_1\lambda\mu^4 - 6sa_0^2a_1\lambda^2\sigma\mu^2 + 3sa_1b_1^2\lambda^2\mu^2 - 4pk^2a_1\lambda^3\sigma\mu^2 - qa_1\mu^4 - 3sa_0^1a_1\mu^4\sigma^2 - 2c^2a_1\lambda^5\sigma^2 - 2c^2a_1\lambda\mu^4 - 2pk^2a_1\lambda^5\sigma^2 - 3sa_0^1a_1\mu^4 - 2qa_1\mu^2\lambda^2\sigma + 3sa_1b_1^2\lambda^4\sigma - 4c^2a_1\lambda^3\sigma\mu^2 = 0$$

$$\varphi^2: -c^2b_1\lambda^3\mu\sigma - pk^2b_1\lambda^3\mu\sigma - 3sa_0a_1^2\lambda^4\sigma^2 + 3sa_0b_1^2\lambda^3\sigma + 2sb_1^3\lambda^2\mu - 6sa_0a_1^2\lambda^2\sigma\mu^2 - pk^2b_1\lambda\mu^3 - 3sa_0a_1^2\mu^4 - c^2b_1\lambda\mu^3 + 3sa_0b_1^2\lambda\mu^2 = 0$$

$$\varphi^3: 3sa_1b_1^2\lambda\mu^2 - 4pk^2a_1\lambda^2\sigma\mu^2 - sa_1^3\mu^4 - 2c^2a_1\mu^4 - sa_1^3\lambda^4\sigma^2 - 2pk^2a_1\lambda^4\sigma^2 + 3sa_1b_1^2\lambda^3\sigma - 4c^2a_1\mu^2\lambda^2\sigma - 2sa_1^3\mu^2\lambda^2\sigma - 2pk^2a_1\mu^4 - 2c^2a_1\lambda^4\sigma^2 = 0 \tag{17}$$

$$\psi: c^2b_1\lambda\mu^4 - 6sa_0^2b_1\lambda^2\sigma\mu^2 - 3sa_0^2b_1\mu^4 - pk^2b_1\lambda^5\sigma^2 - 2b_1\mu^4 + pk^2b_1\lambda\mu^4 - qb_1\lambda^4\sigma^2 - c^2b_1\lambda^5\sigma^2 - 2qb_1\mu^2\lambda^2\sigma - 6sa_0b_1^3\lambda^3\mu\sigma - 3sb_1^3\lambda^2\mu^2 - 6sa_0b_1^2\lambda\mu^3 - 3sa_0^2b_1\lambda^4\sigma^2 + sb_1^3\lambda^4\sigma = 0$$

$$\varphi\psi: 6pk^2a_1\mu^3\lambda^2\sigma + 6c^2a_1\mu^3\lambda^2\sigma - 6sa_0a_1b_1\mu^4 + 3pk^2a_1\mu\lambda^4\sigma^2 - 12sa_0a_1b_1\mu^2\lambda^2\sigma + 3c^2a_1\mu\lambda^4\sigma^2 + 3pk^2a_1\mu^5 - 6sa_1b_1^2\lambda\mu^3 - 6sa_1b_1^2\lambda^3\mu\sigma - 6sa_0a_1b_1\lambda^4\sigma^2 + 3c^2a_1\mu^5 = 0$$

$$\varphi^2\psi: sb_1^3\lambda^3\sigma + sb_1^3\lambda\mu^2 - 3sa_0^2b_1\lambda^4\sigma^2 - 2pk^2b_1\mu^4 - 3sa_0^2b_1\mu^4 - 2c^2b_1\lambda^4\sigma^2 - 2c^2b_1\mu^4 - 4c^2b_1\lambda^2\sigma\mu^2 - 4pk^2b_1\mu^2\lambda^2\sigma - 6sa_0^2b_1\mu^2\lambda^2\sigma - 2pk^2b_1\lambda^4\sigma^2 = 0$$

Solving the system of algebraic equations in (17) by using symbolic computation software like, Maple or Mathematica, we obtain the following results:

$$a_0 = 0, a_1 = \pm\sqrt{\frac{q}{s\lambda}}, b_1 = \pm\sqrt{\frac{-q\mu^2 - q\lambda^2\sigma}{s}}, k = k, c = \sqrt{\frac{-2q - pk^2\lambda}{\lambda}}$$

$$\text{Family 1.1.1: } U(\xi) = \pm\sqrt{\frac{q}{s\lambda}} \times \left(\frac{A_1\sqrt{-\lambda} \cosh(\sqrt{-\lambda}\xi) + A_2\sqrt{-\lambda} \sinh(\sqrt{-\lambda}\xi)}{A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} \right) \pm \sqrt{\frac{-q\mu^2 - q\lambda^2\sigma}{s}} \times \frac{1}{A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} \tag{18}$$

where $\sigma = A_1^2 - A_2^2$.

Family 1.1.2: If $A_1 = 0, A_2 \neq 0$ and $\mu = 0$ in (18), we obtain the solitary wave solution

$$U(\xi) = \pm\sqrt{\frac{q}{s\lambda}} \cdot \sqrt{-\lambda} \tanh(\sqrt{-\lambda}\xi) \pm \sqrt{\frac{-q\lambda^2\sigma}{s}} \cdot \frac{1}{A_2} \text{sech}(\sqrt{-\lambda}\xi). \tag{19}$$

Family 1.1.3: If $A_1 \neq 0, A_2 = 0$ and $\mu = 0$, we obtain the solitary wave solution

$$U(\xi) = \pm\sqrt{\frac{q}{s\lambda}} \sqrt{-\lambda} \coth(\sqrt{-\lambda}\xi) \pm \sqrt{\frac{-q\lambda^2\sigma}{s}} \frac{1}{A_2} \text{cosech}(\sqrt{-\lambda}\xi) \tag{20}$$

Case 1.2: For $\lambda > 0$, Substituting Eq. (16) into Eq. (15) along with Eq. (2) and Eq. (6) and equating the coefficients to zero yields a set of algebraic equation set of algebraic equations for $a_0, a_1, b_1, \mu, \sigma, \lambda, c, p$ and k as follows:

$$\varphi^0: c^2b_1\lambda^4\mu\sigma - 3sa_0b_1^2\lambda^4\sigma + 3sa_0b_1^2\lambda^2\mu^2 + 2sb_1^3\lambda^3\mu - qa_0\lambda^4\sigma^2 - c^2b_1\lambda^2\mu^3 - sa_0^3\lambda^4\sigma^2 + 2qa_0\mu^2\lambda^2\sigma - sa_0^3\mu^4qa_0\mu^4 + pk^2b_1\lambda^4\mu\sigma - pk^2b_1\lambda^2\mu^3 + 2sq_0^3\mu^2\lambda^2\sigma = 0$$

$$\varphi^1: -qa_0\mu^4 + 4c^2a_0\lambda^3\sigma\mu^2 - qa_1\lambda^2\sigma^2 + 3sa_1b_1^2\lambda^2\mu^2 - 2pk^2a_1\lambda^5\sigma^2 + 4pk^2a_1\lambda^3\sigma\mu^2 - 3sa_0^2a_1\mu^4 + 2qa_1\mu^2\lambda^2\sigma - 3sa_0^2a_1\lambda^4\sigma^2 + 6sa_0^2a_1\mu^2\lambda^2\sigma - 2pk^2a_1\lambda\mu^4 - 2c^2a_1\lambda\mu^4 - 3sa_1b_1^2\lambda^2\sigma - 2c^2a_1\lambda^5\sigma^2 = 0$$

$$\varphi^2: pk^2b_1\lambda^3\mu\sigma - pk^2b_1\lambda\mu^3 + 6sa_0a_1^2\mu^2\lambda^2\sigma + c^2b_1\lambda^2\mu\sigma - 3sa_0b_1^2\lambda^3\sigma - 3sa_0a_1^2\mu^4 + 2sb_1^3\lambda^2\mu - 3sa_0a_1^2\lambda^4\sigma^2 + 3sa_0b_1^2\lambda\mu^2 - c^2b_1\lambda\mu^3 = 0$$

$$\begin{aligned}
 \varphi^3: & -sa_1^3\mu^4 - 2c^2a_1\mu^4 - sa_1^3\lambda^4\sigma^2 - 2pk^2a_1\mu^4 - 2c^2a_1\lambda^4\sigma^2 - 3sa_1b_1^2\lambda^2\sigma + 4pk^2a_1\mu^2\lambda^2\sigma - 2pk^2a_1\lambda^4\sigma^2 + 2sa_1^3\mu^2\lambda^2\sigma \\
 & + 4c^2a_1\mu^2\lambda^2\sigma + 3sa_1b_1^2\lambda\mu^2 = 0 \\
 \psi: & 2qb_1\mu^2\lambda^2\sigma - pk^2b_1\lambda^5\sigma^2 - qb_1\mu^4 - 3sa_0^2b_1\mu^4 - 3sa_0^2b_1\lambda^4\sigma^2 - sb_1^3\lambda^4\sigma - qb_1\lambda^4\sigma^2 - 6sa_0b_1^2\lambda\mu^3 - c^2b_1\lambda^5\sigma^2 \\
 & + 6sa_0^2b_1\mu^2\lambda^2\sigma - 3sb_1^3\lambda^2\mu^2 + c^2b_1\lambda\mu^4 + pk^2b_1\lambda\mu^4 + 6sa_0b_1^2\lambda^3\mu\sigma = 0 \\
 \varphi\psi: & -6pk^2a_1\mu^3\lambda^2\sigma - 6sa_0a_1b_1\mu^4 + 3c^2a_1\mu^5 - 6c^2a_1\mu^3\lambda^2\sigma - 6sa_1b_1^2\lambda\mu^3 - 6sa_0a_1b_1\lambda^4\sigma^2 + 6sa_1b_1^2\lambda^3\mu\sigma \\
 & + 3pk^2a_1\mu\lambda^4\sigma^2 + 3pk^2a_1\mu^5 + 12sa_0a_1b_1\mu^2\lambda^2\sigma + 3c^2a_1\mu\lambda^4\sigma^2 = 0 \\
 \psi\varphi^2: & sb_1^3\lambda\mu^2 - 2c^2b_1\lambda^4\sigma^2 - sb_1^3\lambda^3\sigma - 2pk^2b_1\mu^4 - 3sa_1^2b_1\mu^4 + 4pk^2b_1\mu^2\lambda^2\sigma - 2pk^2b_1\lambda^4\sigma^2 - 3sa_1^2b_1\lambda^4\sigma^2 + \\
 & 4c^2b_1\mu^2\lambda^2\sigma - 2c^2b_1\mu^4 + 6sa_1^2b_1\mu^2\lambda^2\sigma = 0 \tag{21}
 \end{aligned}$$

Solving the algebraic equations given in (21) with the aid of Maple and Mathematica, we obtain the following results:

$$c = \pm\sqrt{\frac{-pk^2\lambda-2q}{\lambda}}, a_0 = 0, a_1 = \pm\sqrt{\frac{q}{s\lambda}}, b_1 = \pm\sqrt{\frac{q\lambda^2\sigma-q\mu^2}{s}}, \text{ and } k = k$$

Substituting these values into Eq. (16), we get the following solutions of Eq. (12).

Family 1.2.1:

$$U(\xi) \pm \sqrt{\frac{q}{s\lambda}} \times \left(\frac{A_1\sqrt{\lambda} \cos(\sqrt{\lambda}\xi) - A_2\sqrt{\lambda} \sin(\sqrt{\lambda}\xi)}{A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}} \right) \pm \sqrt{\frac{q\lambda^2\sigma-q\mu^2}{s}} \times \left(\frac{1}{A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}} \right) \tag{22}$$

where $\sigma = A_1^2 + A_2^2$.

Family 1.2.2: When $A_1 = 0, A_2 \neq 0$ and $\mu = 0$ in Eq. (22), we obtain the solitary wave solution

$$U(\xi) = \pm\sqrt{\frac{q}{s}} \tan(\sqrt{\lambda}\xi) \pm \sqrt{\frac{q\lambda^2\sigma-q\mu^2}{s}} \frac{1}{A_2} \sec(\sqrt{\lambda}\xi). \tag{23}$$

Family 1.2.3: When $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in Eq. (22), we obtain the solitary wave solution

$$U(\xi) = \pm\sqrt{\frac{q}{s}} \cot(\sqrt{\lambda}\xi) \pm \sqrt{\frac{q\lambda^2\sigma-q\mu^2}{s}} \frac{1}{A_2} \operatorname{cosec}(\sqrt{\lambda}\xi). \tag{24}$$

Case 1.3: For $\lambda = 0$, substituting Eq. (16) into Eq. (15) along with Eq. (2) and Eq.(8) yields a set of algebraic equations for a_0, a_1, b_1, μ, C, P , and K as follows:

$$\varphi^0: 4qa_0a_1^2\mu a_2 - 4sa_0^3\mu^2a_1^2 - sa_0^3a_1^4 - 4qa_0\mu^2a_2^2 + 4sa_0^3a_1^2\mu a_2 - qa_0a_1^4 = 0$$

$$\varphi^1: 4qa_1^3\mu a_2 - 8pk^2a_1\lambda\mu^2a_2^2 - 3sa_0^2a_1^5 + 12sa_0^2a_1^3\mu a_2 - 2c^2a_1^5\lambda - 8c^2a_1\lambda a_2^2 + 8c^2a_1^3\lambda a_2 + 8pk^2a_1^3\lambda\mu a_2 - 2pk^2a_1^5\lambda \\
 - 4qa_1\mu^2a_2^2 - 12sa_0^2a_1\mu^2a_2^2 - qa_1^5 = 0$$

$$\varphi^2: 12S a_0a_1^4\mu a_2 - 2c^2b_1\mu^2a_2 + pk^2b_1\mu a_1^2 - 2pk^2b_1\mu^2a_2 + 6sa_0b_1^2\mu a_2 + 2sb_1^3\mu + c^2b_1\mu a_1^2 - 12sa_0a_1^2\mu^2a_2^2 - 3sa_0a_1^6 \\
 - 3sa_0b_1^2a_1^2 = 0$$

$$\varphi^3: 6sa_1b_1^2\mu a_2 - 2pk^2a_1^5 + 8pk^2a_1^3\mu a_2 - 8c^2a_1\mu^2a_2^2 - 3sa_1^3b_1^2 - sa_1^7 - 4sa_1^3\mu^2a_2^2 - 8pk^2a_1\mu^2a_2^2 + 4sa_1^5\mu a_2 \\
 + 8c^2a_1^3\mu a_2 - 2c^2a_1^5 = 0$$

$$\psi: -c^2b_1\lambda a_1^4 - qb_1a_1^4 - pk^2b_1a_1^4\lambda + 12sa_0^2b_1\mu a_2a_1^2 + 4c^2b_1\mu^3a_2 + 4qb_1a_1^2\mu a_2 + 4c^2b_1\lambda\mu a_2a_1^2 - 4sb_1^3\mu^2 \\
 - 12sa_0b_1^2\mu^2a_2 - 4pk^2b_1\lambda\mu^2a_2^2 - 3sa_0^2b_1a_1^4 - 2pk^2b_1\mu^2a_1^2 + 4pk^2b_1\mu^3a_2 - 12sa_0^2b_1\mu^2a_2^2 - 4qb_1\mu^2a_2^2 \\
 - 4c^2b_1\lambda\mu^2a_2^2 + 6sa_0b_1^2\mu a_1^2 + 4pk^2b_1\lambda a_1^2\mu a_2 - 2c^2b_1\mu^2a_1^2 = 0$$

$$\psi\varphi: -12c^2a_1^3\mu^2a_2 + 3pk^2a_1^5\mu + 6sa_1^3b_1^2\mu + 12pk^2a_1\mu^3a_2^2 + 3c^2a_1^5\mu - 24sa_0a_1b_1\mu^2a_2^2 - 6sa_0a_1^5b_1 - 12pk^2a_1^3\mu^2a_2 \\
 + 12c^2a_1\mu^3a_2^2 + 24sa_0a_1^3b_1\mu a_2 - 12sa_1b_1^2\mu^2a_2 = 0$$

$$\psi\varphi^2: -3sa_1^6b_1 - 2c^2b_1a_1^4 - 12sa_1^2b_1\mu^2a_2^2 - 8c^2b_1\mu^2a_2^2 + 12sa_1^4b_1\mu a_2 - 2pk^2b_1a_1^4 + 8c^1b_1\mu a_2a_1^2 + 8pk^2b_1a_1^2\mu a_2 - \\
 8pk^2b_1\mu^2a_2^2 - sb_1^3a_1^2 + 2sb_1^3\mu a_2 = 0 \tag{25}$$

Solving the algebraic equation with the aid of Maple or Mathematica, we obtain the following results:

$$a_0 = a_0, a_1 = \pm\sqrt{2\mu a_2}, b_1 = 0, k = k, c = c$$

Substituting this into Eq. (17), we get the solution of Eq. (12) as follows:

Family: 1.3.1:

$$U(\xi) = a_0 \pm \sqrt{2\mu a_2} \left(\frac{\mu\xi + A_1}{\frac{\mu}{2}\xi^2 + A_1\xi + A_2} \right) \tag{26}$$

3.2. The Density Dependent Fractional Diffusion Reaction Equation

In this subsection we apply the $(G'/G, 1/G)$ -expansion method to obtain exact solution of density dependent fractional diffusion reaction equation which can be given in the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + ku(x,t) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = D \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + au(x,t) - bu^2(x,t) \tag{27}$$

where, $t > 0, 0 < \alpha < 1$.

Introducing the following transformation:

$$\xi = \frac{px^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, u(x, t) = U(\xi) \tag{28}$$

where p and c are non-zero constants. Using traveling wave variable, equation (27) reduced into the following ODE for $u = u(\xi)$:

$$Dp^2U'' + cU' - kpUU' + aU - bU^2 = 0 \tag{29}$$

Balancing the highest order derivatives in linear and nonlinear terms, we get $N = 1$. Therefore the solution of (29) is of the form:

$$U(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi) \tag{30}$$

Case 2.1: For $\lambda < 0$, substituting equation (3.19) into equation (29) along with equation (2) and equation (4) yields a set of algebraic equation for $a_0, a_1, b_1, D, p, c, \lambda, \mu, \sigma$ as follows:

$$\phi^0: -kpa_0a_1\lambda\mu^2 + ca_1\lambda\mu^2 + ba_0^2\lambda^2\sigma - aa_0\lambda^2\sigma - kpa_0a_1\lambda^3\sigma - kpb_1\lambda^2a_1\mu + ca_1\lambda^3\sigma - Dp^2b_1\lambda^2\mu - bb_1^2\lambda^2 + ba_0^2\mu^2 - aa_0\mu^2 = 0$$

$$\phi^1: -aa_1\lambda^2\sigma - 2Dp^2a_1\lambda\mu^2 + kpb_1^2\lambda^2 - aa_1\mu^2 - 2Dp^2a_1\lambda^3\sigma + 2ba_0a_1\lambda^2\sigma + 2ba_0a_1\mu^2 - kpa_1^2\mu^2\lambda - kra_1^2\lambda^3\sigma = 0$$

$$\phi^2: -kpa_0a_1\mu^2 - kpb_1\lambda a_1\mu + ba_1^2\mu^2 - kpa_0a_1\lambda^2\sigma + ca_1\lambda^2\sigma - bb_1^2\lambda + ca_1\mu^2 - Dp^2b_1\lambda\mu + ba_1^2\lambda^2\sigma = 0$$

$$\phi^3: -kpa_1^2\lambda^2\sigma + kpb_1\lambda - kpa_1^2\mu^2 - 2Dp^2a_1\lambda^2\sigma - 2Dp^2a_1\mu^2 = 0$$

$$\psi: -ab_1\mu^2 - ca_1\mu\lambda^2\sigma + kpa_0a_1\mu^3 - Dp^2b_1\lambda^3\sigma + 2ba_0b_1\lambda^2\sigma - ca_1\mu^3 + 2ba_0b_1\mu^2 + kpb_1a_1\lambda\mu^2 + Dp^2b_1\lambda\mu^2 - ab_1\lambda^2\sigma + kpa_0a_1\mu\lambda^2\sigma + 2bb_1^2\lambda\mu - kpb_1a_1\lambda^3\sigma = 0$$

$$\psi\phi: cb_1\lambda^2\sigma + 3Dp^2a_1\mu^3 - 2kpb_1^2\lambda\mu + 3Dp^2a_1\mu\lambda^2\sigma - kpa_0b_1\lambda^2\sigma - kpa_0b_1\mu^2 + cb_1\mu^2 + 2ba_1b_1\mu^2 + kpa_1^2\mu^3 + kpa_1^2\mu\lambda^2\sigma + 2ba_1b_1\lambda^2\sigma = 0$$

$$\psi\phi^2: -2kpa_1b_1\mu^2 - 2Dp^2b_1\mu^2 - 2Dp^2b_1\lambda^2\sigma - 2kpa_1b_1\lambda^2\sigma = 0 \tag{31}$$

Solving the algebraic equation (31) by using the software Maple or Mathematica, we obtain the following results:

$$a_0 = \frac{a}{2b}, a_1 = \frac{\pm a \sqrt{-\frac{1}{4\lambda}}}{b}, b_1 = \frac{\pm a \sqrt{\mu^2 + \lambda^2 \sigma}}{2\lambda b}, c = \pm \frac{(4b^2D + k^2a)a}{8b^2D\lambda \sqrt{-\frac{1}{4\lambda}}}$$

and $p = \pm \frac{ak \sqrt{-\frac{1}{4\lambda}}}{bD}$.

$$U(\xi) = \frac{a}{2b} \pm \frac{a \sqrt{-\frac{1}{4\lambda}}}{b} \frac{A_1 \sqrt{-\lambda} \cosh(\sqrt{-\lambda}\xi) + A_2 \sqrt{-\lambda} \sinh(\sqrt{-\lambda}\xi)}{A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} \pm \frac{a \sqrt{\mu^2 + \lambda^2 \sigma}}{2\lambda b} \times \frac{1}{A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} \tag{32}$$

where, $\sigma = A_1^2 - A_2^2$.

Substituting this into equation (30) we get the solution of (29) as follows:

Family 2.1.1.1:

Family 2.1.2: If $A_1 = 0, A_2 \neq 0$ and $\mu = 0$ in (32), we get the solitary wave solution

$$U(\xi) = \frac{a}{2b} \pm \frac{a\sqrt{-\frac{1}{4\lambda}} \cdot \sqrt{\lambda} \tanh(\sqrt{\lambda}\xi)}{b} \pm \frac{a\sqrt{\mu^2 + \lambda^2\sigma}}{2\lambda b} \cdot \frac{1}{A_2} \operatorname{sech}(\sqrt{\lambda}\xi) \quad (33)$$

$$U(\xi) = \frac{a}{2b} \pm \frac{a\sqrt{-\frac{1}{4\lambda}}}{b} \cdot \sqrt{\lambda} \coth(\sqrt{\lambda}\xi) \pm \frac{a\sqrt{\mu^2 + \lambda^2\sigma}}{2\lambda b} \cdot \frac{1}{A_1} \operatorname{cosech}(\sqrt{\lambda}\xi) \quad (34)$$

Family 2.1.13: If $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in (32), we get the solitary wave solution

Case 2.2: For $\lambda > 0$, substituting equation (30) into equation (29) along with equation (2) and (6) yields a set of algebraic equation for $a_0, a_1, b_1, D, p, c, \lambda, \mu, \sigma$ as follows:

$$\phi^0 : -Dp^2 b_1 \lambda^2 \mu - ba_0^2 \lambda^2 \sigma - kpb_1 \lambda^2 a_1 \mu - bb_1^2 \lambda^2 - ca_1 \lambda^3 \sigma + ca_1 \lambda \mu^2 + aa_0 \lambda^2 \sigma + ba_0^2 \mu^2 - aa_0 \mu^2 - kpa_0 a_1 \lambda \mu^2 + kpa_0 a_1 \lambda^3 \sigma = 0$$

$$\phi^1 : kpa_1^2 \lambda^3 \sigma + 2ba_0 a_1 \mu^2 + aa_1 \lambda^2 \sigma - aa_0 \mu^2 - 2ba_0 a_1 \lambda^2 \sigma - 2Dp^2 a_1 \lambda \mu^2 + Dp^2 a_1 \lambda^3 \sigma + kpb_1^2 \lambda^2 - kpa_1^2 \lambda \mu^2 = 0$$

$$\phi^2 = ca_1 \mu^2 + ba_1^2 \mu^2 - Dp^2 b_1 \lambda \mu - bb_1^2 \lambda - kpa_0 a_1 \mu^2 - kpb_1 \lambda a_1 \mu + ba_1^2 \lambda^2 \sigma + kpa_0 a_1 \lambda^2 \sigma - ca_1 \lambda^2 \sigma = 0$$

$$\phi^3 : kpa_1^2 \lambda^2 \sigma - 2Dp^2 a_1 \mu^2 + kpb_1^2 \lambda + 2Dp^2 a_1 \lambda^2 \sigma - kpa_1^2 \mu^2 = 0$$

$$\psi : -Ca_1 \mu^3 + Ca_1 \mu \lambda^2 \sigma + KPa_0 a_1 \mu^3 + DP^2 b_1 \lambda \mu^2 + 2ba_0 b_1 \mu^2 + DP^2 b_1 \lambda^3 \sigma + Kpb_1 a_1 \lambda^3 \sigma + Kpb_1 a_1 \lambda \mu^2 - KPa_0 a_1 \mu \lambda^2 \sigma - 2ba_0 b_1 \lambda^2 \sigma + 2bb_1^2 \lambda \mu - ab_1 \mu^2 + ab_1 \mu^2 + ab_1 \lambda^2 \sigma = 0$$

$$\psi\phi : -KPa_0 b_1 \mu^2 + Cb_1 \mu^2 + KPa_1^2 \mu^3 - Cb_1 \lambda^2 \sigma + KPa_0 b_1 \lambda^2 \sigma - 2ba_1 b_1 \lambda^2 \sigma - 2Kpb_1^2 \lambda \mu + 2ba_1 b_1 \mu^2 + 3DP^2 a_1 \mu^3 - KPa_1^2 \mu \lambda^2 \sigma - 3DP^2 a_1 \mu \lambda^2 \sigma = 0$$

$$\psi\phi^2 : 2DP^2 b_1 \lambda^2 \sigma - 2KPa_1 b_1 \mu^2 - 2DP^2 b_1 \mu^2 + 2KPa_1 b_1 \lambda^2 \sigma = 0 \quad (35)$$

Solving the algebraic equation (35) by using the software Maple or Mathematica, we obtain the following results:

$$a_0 = \frac{a}{2b}, a_1 = \pm \frac{a\sqrt{-\frac{1}{4\lambda}}}{b}, b_1 = \pm \frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda b},$$

$$C = \pm \frac{(4b^2D + K^2\alpha)a}{8b^2D\lambda \cdot \sqrt{-\frac{1}{4\lambda}}}, \text{ and } P = \frac{aK\sqrt{-\frac{1}{4\lambda}}}{bD}.$$

Substituting this into Eq. (30), we get the solution of (29) as follows:

Family 2.2.1:

$$U(\xi) = \frac{a}{2b} \pm \frac{a\sqrt{-\frac{1}{4\lambda}}}{b} \times \frac{A_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\xi) + A_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\xi)}{A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}}$$

$$\pm \frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda b} \times \frac{1}{A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}} \quad (36)$$

where $\sigma = A_1^2 + A_2^2$.

Family 2.2.2: If $A_1 = 0, A_2 \neq 0$ and $\mu = 0$ in (36), we get the solitary wave solution

$$U(\xi) = \frac{a}{2b} \pm \frac{a\sqrt{-\frac{1}{4\lambda}}}{b} \sqrt{\lambda} \tan(\sqrt{\lambda}\xi) \pm \frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda b} \cdot \frac{1}{A_2} \operatorname{sec}(\sqrt{\lambda}\xi) \quad (37)$$

Family 2.2.3: If $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in (36), we get the Solitary wave solution

$$U(\xi) = \frac{a}{2b} \pm \frac{a\sqrt{-\frac{1}{4\lambda}}}{b} \sqrt{\lambda} \cot(\sqrt{\lambda}\xi) \pm \frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2\lambda b} \cdot \frac{1}{A_2} \operatorname{cosec}(\sqrt{\lambda}\xi) \quad (38)$$

Case 2.3: For $\lambda = 0$, substituting Eq. (30) into Eq. (29) along with Eq. (2) and Eq. (7) yields a set of algebraic equations for $a_0, a_1, b_1, D, P, C, \lambda, \mu, C, P$ as follows:

$$\phi^0 : -Ca_1^3 \lambda - ba_0^2 a_1^2 - 2aa_0 \mu a_2 - 2KPa_0 a_1 \lambda \mu a_2 + aa_0 a_1^2 + 2Ca_1 \lambda \mu a_2 + KPa_0 a_1^3 \lambda + 2ba_0^2 \mu a_2 = 0$$

$$\phi^1 : -2aa_1 \mu a_2 - 2KPa_1^2 \lambda \mu a_2 - 2ba_0 a_1^3 + 2DP^2 a_1^3 \lambda - 4DP^2 a_1 \lambda \mu a_2 + 4ba_0 a_1 \mu a_2 + KPa_1^4 \lambda + aa_1^3 = 0$$

$$\varphi^2: KPa_0a_1^3 - Kpb_1a_1\mu - bb_1^2 + 2ba_1^2\mu a_2 + 2Ca_1\mu a_2 - DP^2b_1\mu - 2KPa_0a_1\mu a_2 - ba_1^4 - Ca_1^3 = 0$$

$$\varphi^3: 2DP^2a_1^3 + Kpb_1^2 - 4DP^2a_1\mu a_2 + KPa_1^4 - 2KPa_1^2\mu a_2 = 0$$

$$\psi: 2Kpb_1a_1\mu^2 - KPa_0a_1^3\mu + 2KPa_0a_1\mu^2a_2 + Ca_1^3\mu + 2DP^2b_1\mu^2 + ab_1a_1^2 + 2bb_1^2\mu + 4ba_0b_1\mu a_2 - 2Ca_1\mu^2a_2 + Kpb_1a_1^3\lambda + DP^2b_1\lambda a_1^2 - 2a_1\mu a_2 - 2DP^2b_1\lambda\mu a_2 - 2ba_0b_1a_1^2 - 2Kpb_1a_1\lambda\mu a_2 = 0$$

$$\varphi\psi: 6DP^2a_1\mu^2a_2 - 3DP^2a_1^3\mu - KPa_1^4\mu + 4ba_1b_1\mu a_2 + 2KPa_1^2\mu^2a_2 - 2ba_1^3b_1 + 2Cb_1\mu a_2 + 2Kpb_1^2\mu - 2KPa_0b_1\mu a_2 + KPa_0b_1a_1^2 - Cb_1a_1^2 = 0$$

$$\psi\varphi^2: 2KPa_1^3b_1 + 2DP^2b_1a_1^2 - 4KPa_1b_1\mu a_2 - 4DP^2b_1\mu a_2 = 0 \tag{39}$$

Solving the algebraic equation (39) by using the software Maple or Mathematica, we obtain the following results:

$$a_0 = a_0, a_1 = \pm\sqrt{2\mu a_2}, b_1 = 0, C = C \text{ and } P = P.$$

Substituting this into Eq. (30), we get the solution of (29) as follows:

Family 2.3.1:

$$U(\xi) = a_0 \pm \sqrt{2\mu a_2} \times \left(\frac{\mu\xi + A_1}{\frac{\mu}{2}\xi^2 + A_1\xi + A_2} \right). \tag{40}$$

4. Conclusion

In this article, the two variable $(G'/G, 1/G)$ -expansion method is used to obtain further general and some new as well as some known solutions of the well-known fractional generalized reaction Duffing model and density dependent fractional diffusion reaction equation. If we take the special value of two parameters A_1 and A_2 , we get the solitary wave solutions. When $\mu = 0$ and $b_i = 0$ in (1) and (11) respectively, the two variable $(G'/G, 1/G)$ -expansion method transferred to the original (G'/G) -expansion method. Thus, the two variable $(G'/G, 1/G)$ -expansion is an extension of the (G'/G) -expansion method. The two variable $(G'/G, 1/G)$ -expansion method applied in this article is more efficient and more general than the original (G'/G) -expansion method.

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