Applied and Computational Mathematics
2017; 6(6): 238-242
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20170606.11
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

Solving a Nonlinear Equation Using a New Two-Step Derivative Free Iterative Methods

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To cite this article:
Alyauma Hajjah. Solving a Nonlinear Equation Using a New Two-Step Derivative Free Iterative Methods. *Applied and Computational Mathematics*. Vol. 6, No. 6, 2017, pp. 238-242. doi: 10.11648/j.acm.20170606.11

Received: August 8, 2017; **Accepted**: September 26, 2017; **Published**: November 7, 2017

Abstract: In this paper, suggest anew two step iterative method for solving a nonlinear equation, which is derivative free by approximating a derivative in the iterative method by central difference with one parameter θ. The anew derivative free iterative method has a convergence of order four and computational cost the family requires three evaluations of functions per iteration. Numerical experiments show that the proposed a method is comparable to the existing method in terms of the number of iterations.

Keywords: Nonlinear Equation, Iterative Method, Derivative Free, Central Difference, Convergence of Order

1. Introduction

Finding for a root of a nonlinear equations

$$f(x)=0 \tag{1}$$

Where f: $D \subset R \to R$ is most important problem in science and engineering. Analytical methods for solving such a equations are almost nonexistent and therefore it is only possible to obtain approximate solutions by relying on numerical techniques based on iteration procedures, so solving numerically becomes an alternative.

A numerical method used finding the solution of a nonlinear equation (1) is a one step iterative methods [1, 16], a two step iterative methods [4, 5, 7, 8, 12] and a three step iterative methods [2, 3, 9]. Some other paper have discuss about a two step iterative methods for solving a nonlinear equation, that is a methods proposed by Kingis call as King Method (KM), with convergence of order four and three evaluations of the function per iteration so it possesses 1.587 as the efficiency index, can express their formula as [12]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

$$x_{n+1} = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)} \tag{3}$$

The other a two step iterative methods are is a combination of Newton method [1] and Helley method [16] with order of convergence three, in the form:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

$$x_{n+1} = y_n - \frac{2\left(f(y_n)\right)f'(y_n)}{2f'(y_n)^2-\left(f(y_n)f''(y_n)\right)} \tag{5}$$

The other a two step iterative methods are the iterative method proposed by Weerakon and Fernando [17], can express their formula as:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{6}$$

$$x_{n+1} = y_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)} \tag{7}$$

Methods proposed by King in equation (2) and equation (3), combination of Newton and Helley methods in equation (4) and equation (5) and methods proposed by Weerakon and Fernando in equation (6) and equation (7) is methods for solving a nonlinear equation, they can fast to approximating of a roots, but they not success finding a root if $f'(x_n)$=0 or $f'(y_n)$=0, therefore must use derivative free iterative method.

Some other paper have discuss about a two step iterative methods free derivative for solving a nonlinear equation is a method proposed by Dehghan-Hajarian can be call as Dehghan Method (DM), with order of convergence three and four evaluations of the function per iteration, so it possesses 1.316 as the efficiency index, can express their formula as [5]:

$$y_n = x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \quad (8)$$

$$x_{n+1} = y_n - \frac{2f(x_n)^2\left(f(x_n)+f(y_n)\right)}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \quad (9)$$

and the one method proposed by Hajjah is call as Hajjah Method (HM), with order of convergence three and three evaluations of the function per iteration, so it possesses 1.442 as the efficiency index, can express their formula as [7]:

$$y_n = x_n - \frac{\theta f^2(x_n)}{f(x_n + \theta f(x_n)) - f(x_n)} \quad (10)$$

$$x_{n+1} = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{\theta f(x_n)f(y_n)}{f(x_n+\theta f(x_n))-f(x_n)} \quad (11)$$

Some other papers discuss the a two step type iterative methods for solving a nonlinear equation and their applications, these include in [4, 8, 10, 12]

In this paper will discuss a two step iterative method of proposed by King. This method is free from derivatives by approximating derivatives by central difference with one parameter θ. The new method is proved to be convergence of order four. Some examples are given to compare the new method with some benchmarked methods.

2. Basic Definitions

In order to establish the order of convergence of the new derivative free methods, state some of the definitions:

Definition 1 (Order of Convergence). Let $\alpha \in \Re$, n=1,2,3,... then, the sequence $\{x_n\}$ is said to converge to α if

$$\lim_{n\to\infty} |x_n - \alpha| = 0$$

If, in addition, there exists a constant $c \geq 0$, an integer $n_0 \geq 0$, and $p \geq 0$ such that for all $n \geq n_0$,

$$|x_{n+1} - \alpha| \leq c|x_n - \alpha|^p, \quad (12)$$

Then $\{x_n\}$ is said to converge to α with *q-order* at least $p = 2$ or 3, the convergence is said to be *q-quadratic* or *q-cubic*, respectively [13].

When $e_n = x_n - \alpha$ is the error in the n^{th} iterate, the relation

$$e_{n+1} = ce_n^p + O(e_n^{p+1}), \quad (13)$$

Is called the error of a equation. By substituting $e_n = x_n - \alpha$ for all n in any iterative method and after simplifying, obtain the error equation for that method. The value of p thus obtained is called the order of this method.

Definition 2 (Efficiency Index). Let r be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [6] and defined as:

$$\sqrt[r]{p},$$

where p is the order of the method.

Definition 3 (Computation Order of Convergence). Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of equation (1). Then the computational order of convergence [17] may be approximated by:

$$COC = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|} \quad (14)$$

3. Proposed Methods

In this section will discussed twostep iterative method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (15)$$

$$x_{n+1} = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)} \quad (16)$$

Use central difference with one parameter θ [1], to approximate $f'(x_n)$ in equation (15) and equation (16) by:

$$f'(x_n) \approx \frac{f\left(x_n+\theta f(x_n)\right) - f(x_n - \theta f(x_n))}{\theta f(x_n)} \quad (17)$$

Substituting equation (17) into equation (15) and equation (16), can be change as:

$$y_n = x_n - \frac{2\theta f^2(x_n)}{f(x_n+\theta f(x_n)) - f(x_n - \theta f(x_n))} \quad (18)$$

$$x_{n+1} = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{2\theta f(x_n)f(y_n)}{f(x_n+f(x_n))-f(x_n-\theta f(x_n))} \quad (19)$$

Equation (18) and equation (19) are called Solving Nonlinear Equation Using A New Two Step Derivative Free Iterative Method.

Theorem 1 Let $f: D \to \Re$ for an open interval D. Assume that f has sufficiently continuous derivative in the interval D. If x^* has a simple root at $f(x)$, and if x_0 is sufficiently close to x^*, then the new iterative method defined by equation (18) and equation (19) satisfies the following error equation:

$$e_{n+1} = \left(c_2^3 - c_2c_3 - \theta^2F_1^2c_2c_3 + 2\beta c_2^3\right)e_n^4 \quad (20)$$

Where $F_1 = f'(x^*), c_j = \frac{1}{j!}\frac{f^{(j)}(x^*)}{f'(x^*)}, j > 1,$ and $e_n = x_n - x^*$.

Proof. Let x^* is be a simple root of $f(x) = 0$, then $f(x^*) = 0$ and $f(x_n) \neq 0$. Let $e_n = x_n - x^*$. With Taylor expansion of $f(x)$ about $x_n = x^*$, can be obtain as following:

$$f(x_n) = F_1\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4\right) + O(e_n^5) \quad (21)$$

Where $c_j = \frac{1}{j!}\frac{f^{(j)}(x^*)}{f'(x^*)}, j > 1.$

Then, computing $2\theta f^2(x_n)$ using equation (21) then multiplied by 2θ, can be obtain as following:

$$2\theta f^2(x_n) = 2\theta F_1^2e_n^2 + 4\theta c_2F_1^2e_n^3 + \left(4\theta c_3 + 2\theta c_2^2\right)F_1^2e_n^4 + O(e_n^5) \quad (22)$$

Computing $f((x_n) + \theta f(x_n))$, can be get after simplifying:

$$f((x_n) + \theta f(x_n)) = (F_1 + \theta F_1^2)e_n + (c_2F_1 + 3\theta F_1^2c_2 + \theta^2F_1^3c_2)e_n^2 + (c_3F_1 + 4\theta c_3F_1^2 + 2\theta c_2^2F_1^2$$
$$+\theta^3c_3F_1^4 + 2\theta^2c_2^2F_1^3 + 3\theta^2c_3F_1^3)e_n^3 + (\theta^4c_4F_1^5 + c_4F_1 + 3\theta^3c_2c_3F_1^4$$
$$+4\theta^3c_4F_1^4 + 5\theta c_2c_3F_1^2 + 5\theta c_4F_1^2 + 8\theta^2c_2c_3F_1^3 + 6\theta^2c_4F_1^3$$
$$+\theta^2c_2^3F_1^3)e_n^4 + O(e_n^5) \quad (23)$$

Then, computing $f((x_n) - \theta f(x_n))$, can be have after simplifying:

$$f((x_n) - \theta f(x_n)) = (F_1 - \theta F_1^2)e_n + (c_2F_1 - 3\theta F_1^2c_2 + \theta^2F_1^3c_2)e_n^2 + (c_3F_1 - 4\theta c_3F_1^2 - 2\theta c_2^2F_1^2$$
$$+\theta^3c_3F_1^4 + 2\theta^2c_2^2F_1^3 + 3\theta^2c_3F_1^3)e_n^3 + (\theta^4c_4F_1^5 + c_4F_1 - 3\theta^3c_2c_3F_1^4 - 4\theta^3c_4F_1^4 - 5\theta c_2c_3F_1^2 - 5\theta c_4F_1^2 + 8\theta^2c_2c_3F_1^3 + 6\theta^2c_4F_1^3$$
$$+\theta^2c_2^3F_1^3)e_n^4 + O(e_n^5) \quad (24)$$

Using equation (23) and equation (24), compute $f((x_n) + \theta f(x_n)) - f((x_n) - \theta f(x_n))$. Then can be obtain after simplifying:

$$f((x_n) + \theta f(x_n)) - f((x_n) - \theta f(x_n))$$
$$= 2\theta F_1^2e_n + 6\theta c_2F_1^2e_n^2 + (8\theta c_3F_1^2 + 2\theta^3c_3F_1^4 + 4\theta c_2^2F_1^2)e_n^3 + (10\theta c_4F_1^2 + 6\theta^3c_2c_3F_1^4 + 8\theta^3c_4F_1^4 + 10\theta c_2c_3F_1^2)e_n^4 + O(e_n^5) \quad (25)$$

Considering a geometric series and using equation (22) and equation (25) computing $\frac{2\theta f(x_n)^2}{f((x_n) + \theta f(x_n)) - f((x_n) + \theta f(x_n))}$, can be obtain after simplifying:

$$\frac{2\theta f(x_n)^2}{f((x_n) + \theta f(x_n)) - f((x_n) + \theta f(x_n))} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3 - \theta^2c_3F_1^2)e_n^3 + (\theta^2c_2c_3F_1^2 - 4\theta^2c_4F_1^2 + 7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + O(e_n^5). \quad (26)$$

Then, substituting equation (26) into equation (18) and $x_n = x^* + e_n$, can obtain as following:

$$y_n = x^* + c_2e_n^2 + (2c_3 + \theta^2c_3F_1^2 - 2c_2^2)e_n^3 + (3c_4 + 4\theta^2c_4F_1^2 + 4c_2^3 - \theta^2c_2c_3F_1^2 - 7c_2c_3)e_n^4 + O(e_n^5) \quad (27)$$

Applying Taylor expansion of $f(y_n)$ and using equation (27), can be obtain as following:

$$f(y_n) = c_2F_1e_n^2 + (\theta^2c_3F_1^3 + 2c_3F_1 - 2c_2^2F_1)e_n^3 + (4\theta^2c_4F_1^3 - \theta^2c_2c_3F_1^3 + 3c_4F_1 + 5c_2^3F_1 - 7c_2c_3F_1)e_n^4 + O(e_n^5) \quad (28)$$

From equation (19), by doing simple calculation for equation (21) and equation (28), end up with:

$$e_{n+1} = \left(c_2^3 - c_2c_3 - \theta^2F_1^2c_2c_3 + 2\beta c_2^3\right)e_n^4 \quad (29)$$

From equation (29) can get that the new iterative method free derivative (PFDM) has four order of convergence. Thisends the proof.

4. Numerical Experiments

In this section will present five numerical examples, numerical simulation are performed in order to compare the number of iteration DM by equation (8) and equation (9), HM by equation (10) and equation (11), KM by equation (2) and equation (3) and PFDM by equation (18) and equation (19) for the solution of the nonlinear equation. In this comparison use the following test five functions:

$$f_1 = \cos(x) - x \text{ , } x^* = 0.7390851332151606 \quad [10]$$

$$f_2 = \sin(x) - 0.5 \text{ , } x^* = 0.5235987755982989 \quad [14, 15]$$

$$f_3 = \sin^2(x) - x^2 + 1 \text{ , } x^* = 1.4044916482153412 \quad [10]$$

$$f_4 = x^3 - x + 3 \text{ , } x^* = -1.6716998816571610 \quad [14, 15]$$

$$f_5 = \tan(x) - \cos(x) - 0.5 \text{ , } x^* = 0.8570567764718169 \quad [14, 15]$$

All the computation is done in Maple 13, use tolerance is $\varepsilon = 1.0 \times 10^{-15}$, and the maximum number of iteration allowed is100 iteration. Stop the iteration process by the following criteria:

1. $|x_{n+1} - x_n| < \varepsilon$,
2. $|f(x_{n+1})| < \varepsilon$.

Table 1. *Comparisons of the discussed methods.*

Functions	x_0	The number of iteration of DM	HM	KM	PFDM	x^*
f_1	-1.5	5	4	36	4	0.7390851332151606
	-1.0	5	4	4	4	
	-0.5	6	3	3	3	
	0.0	4	3	3	3	
f_2	-0.2	3	3	3	3	0.5235987755982989
	0.0	3	3	3	2	
	0.5	3	2	2	2	
	1.0	2	3	3	3	
f_3	0.3	7^*	5	9^*	4	1.4044916482153412
	0.5	5^*	5	6^*	3	
	1.3	3	3	2	2	
	1.5	3	3	2	2	
f_4	-1.7	3	3	2	3	−1.6716998816571610
	-0.9	4	4	90	2	
	-0.5	5	5	23	2	
	0.0	6	8	55	3	
f_5	0.5	3	4	3	3	0.8570567764718169
	0.7	3	3	3	3	
	1.1	7^*	7	3	2	
	2.5	3^*	9	3^*	3	

In Table 1, sign"*" on the number of iterations states that the methods converge to another root (not the same as the root). Based on numerical computations in Table 1, in general for all functions after comparison show that the PFDM is superior to the iteration methods such as DM, HM, and MK. This is evident from the number of iteration required to obtain an approximation of the root or solving a nonlinear equation.

5. Conclusion

Suggest and analyze a new two step iterative methods free derivative to solve a nonlinear equation. The new iterative method free derivative has four order of convergence and four evaluations of the function per iteration, so it possesses 1.414 as the efficiency index. Although the efficiency index of our method is worse than that of DM, KM and HM but numerical experiment show that it is comparable to the exiting method in terms of the number of iteration. So it can be state that the proposed method is superior in finding the success of roots from a nonlinear equation.

Acknowledgements

This work was supported by the Department of Informatics Technical STIKOM Pelita Indonesia. Also authors thanks to Mr. M. Imran and M. D. H. Gamal the Department of Mathematics, University of Riau, Pekanbaru, Indonesia for their constant encouragement.

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