An Analytical Solution for Queue: M/D/1 with Balking

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To cite this article:

Received: January 17, 2018; Accepted: January 31, 2018; Published: February 27, 2018

Abstract: In this paper we examine the how to of deriving analytical solution in steady-state for non-truncated single-server queueing and service time are fixed (deterministic) with addition the concept balking, using iterative method and the probability generating function. Some measures of effecting of queueing system are obtained using a smooth and logical manner also some special cases of this system. Finally, some numerical values are given showily the effect of correlation between the $(p_0, p_n, L, W)$ and the additional concepts.

Keywords: Deterministic, Queueing System, Measures of Effectiveness, Generating Function

1. Introduction


In this paper, we have proposed analytical solution of the steady-state in the non-truncated single-channel Markovian queue M/D/1 subject to balking. The probability that there are n customers in the system, the probability of empty system and some measures of effectiveness are obtained using iterative method, probability generating function. Some special cases are deduced. Finally, a simulation study has been considered to illustrate the numerical application for the model.

2. Basic Notations and Assumptions

To construct the system of this paper, we define the following parameters:

$P(z)$ - The Probability generating function.
$p_n$ - Stead-state probability that there are n customers in the system.
$\lambda$ - Mean arrival rate.
$\mu$ - Mean service rate.
$D$ - The fixed time of service between each customer and the other.
$n$ - Number of customers in the system.
$\beta$ - The probability that the customer joins the queue.
$\rho = \lambda D$ - Utilization factor.
$L$ - Expected number of customers in the system.
$L_q$ - Expected number of customers waiting to be served.
$W$ - Expected waiting time in the system.
$L_q =$ Expected waiting time in the queue.

The assumptions of this model are listed as follows:

(1) Customers arrive at the server one by one according to Poisson process with rate $\lambda$. Assume $(1-\beta)$ be the probability that a customer balks, $0 \leq \beta < 1$, and $\beta = 1, n = 0$. Thus it is clear that:

\[
\beta_n = \begin{cases} 
\lambda, & n = 0 \\
\beta \lambda, & n \geq 1
\end{cases}
\]

(2) Service times of the customers are deterministic time $D$ with rate $\mu$, where

\[
\mu = \begin{cases} 
0, & \text{no service} \\
1, & \text{timer unit}
\end{cases}
\]

(3) A single server serves entities one at a time from the front of the queue, according to a first-come, first-served discipline. When the service is complete the entity leaves the queue and the number of entities in the system reduces by one.

(4) The buffer is of infinite size, so there is no limit on the number of entities it can contain.

3. Model Formulation and Analysis

Due to the lack of a Poisson condition for the server, it is using the equation a no degenerate solution to the stationary

\[
p = \frac{(\beta \lambda)^2}{2!} p_0 + \frac{(\beta \lambda)^2}{2!} p_1 + \frac{(\beta \lambda)^2}{2!} p_2 + e^{-\lambda} p_3
\]

\[
p = \frac{(\beta \lambda)^3}{3!} p_0 + \frac{(\beta \lambda)^3}{3!} p_1 + \frac{(\beta \lambda)^3}{3!} p_2 + \frac{(\beta \lambda)^3}{3!} p_3 + e^{-\lambda} p_4
\]

\[
p = \frac{(\beta \lambda)^{n-1}}{n!} p_0 + \frac{(\beta \lambda)^{n-1}}{n!} p_1 + \frac{(\beta \lambda)^{n-1}}{n!} p_2 + \frac{(\beta \lambda)^{n-1}}{n!} p_3 + e^{-\lambda} p_{n+1}
\]

Thus

\[
p_n = \sum_{i=1}^{n} \frac{\beta \lambda}{i!} p_{n-i} + \frac{(\beta \lambda)^n}{n!} p_0 + e^{-\lambda} p_{n+1}, \quad n \geq 1
\]

To be finding explicit $p_0$ in $\lambda$, we use the probability generating function $P(z)$ whereas:

\[
P(z) = \sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad a(z) = \sum_{n=0}^{\infty} a_n z^n
\]

Multiplying each equation (4), (5), (6) and (8) by the appropriate power of $z$, we obtain:

\[
z p_0 = a_0 p_0 z + a_0 p_1 z
\]
\begin{align*}
& p_1 z^2 = a_1 p_0 z^2 + a_1 p_1 z^2 + a_0 p_2 z^2 \\
& p_2 z^3 = a_2 p_0 z^3 + a_2 p_1 z^3 + a_1 p_2 z^3 + a_0 p_3 z^3 \\
& p_n z^{n+1} = p_0 a_n z^{n+1} + p_1 a_{n-1} z^{n+1} + p_2 a_{n-2} z^{n+1} + \ldots
\end{align*}

(12)

(13)

(14)

Taking \( \sum_{n=0}^{\infty} \) into equation (14), we get:

\begin{align*}
& z \sum_{n=0}^{\infty} p_n z^n = p_0 z \sum_{n=0}^{\infty} a_n z^n + p_1 \sum_{n=0}^{\infty} a_n z^n + p_2 \sum_{n=1}^{\infty} a_{n-1} z^{n-1} + p_3 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} + \ldots
\end{align*}

(15)

From (10) and (15), obtain as:

\begin{align*}
P(z) &= \frac{p_0 (1-z) a(z)}{a(z) - z} \\
P(z) &= \frac{p_0 (1-z) \left( \sum_{n=1}^{\infty} (\beta \lambda)^n e^{-\beta \lambda} n! + e^{-\lambda} \right)}{\sum_{n=1}^{\infty} (\beta \lambda)^n e^{-\beta \lambda} n! + e^{-\lambda} - z}
\end{align*}

(16)

(17)

Thus

\begin{align*}
P(z) &= \frac{p_0 (1-z) \left( e^{-\beta \lambda (1-z)} + e^{-\lambda} - e^{-\beta \lambda} \right)}{\left( e^{-\beta \lambda (1-z)} + e^{-\lambda} - e^{-\beta \lambda} \right) - z}
\end{align*}

(18)

Using the fact that \( P(1) = 1 \), along with LHopital’s rule, we find:

\begin{align*}
p_0 &= \left( 1 + e^{-\lambda} - e^{-\beta \lambda} \right)^{-1} \left[ 1 - \beta \lambda \left( 1 + e^{-\lambda} - e^{-\beta \lambda} \right)^{-1} \right]
\end{align*}

(19)

4. Measures of Effectiveness

To calculate the expected number of units in the system, using as:

\begin{align*}
L = E(n) &= \sum_{n=0}^{\infty} n p_n
\end{align*}

(20)

Consider

\begin{align*}
A &= \sum_{n=0}^{\infty} n^2 p_n
\end{align*}

(21)

From equation (9) and (22), we find:

\begin{align*}
A &= \sum_{n=1}^{\infty} n^2 \sum_{i=0}^{n} \frac{e^{-\beta \lambda}}{i!} p_{n-i+1} + \sum_{n=0}^{\infty} n^2 \frac{(\beta \lambda)^n e^{-\beta \lambda}}{n!} - p_0 + e^{-\lambda} \sum_{n=0}^{\infty} n^2 p_{n+1}
\end{align*}

(22)

with

\begin{align*}
B &= \sum_{n=0}^{\infty} n^2 \sum_{i=0}^{n} \frac{e^{-\beta \lambda}}{i!} p_{n-i+1} \quad \text{and} \quad C = \sum_{n=0}^{\infty} n^2 \frac{(\beta \lambda)^n e^{-\beta \lambda}}{n!} p_0
\end{align*}

(23)

(24)

From equation (24) and same algebra, we get:

\begin{align*}
C &= \beta \lambda p_0 (1 + \beta \lambda)
\end{align*}

(25)

and

\begin{align*}
B &= \sum_{i=0}^{\infty} \frac{e^{-\beta \lambda}}{i!} \sum_{n=1}^{\infty} n^2 p_n + 2 \sum_{i=0}^{\infty} \frac{i e^{-\beta \lambda}}{i!} \sum_{n=1}^{\infty} n p_n \sum_{m=1}^{\infty} e^{-\beta \lambda} \frac{(\beta \lambda)^{i}}{i!} \sum_{m=1}^{\infty} m p_m
\end{align*}
From equation (21), (22), (25) and (26), we find:

\[
L = \frac{\beta \lambda (1 + \beta \lambda)(1 - p_0)}{2(1 - \beta \lambda)},
\]

Also, calculate the expected number of units in the queue, using as:

\[
L_q = L - (1 - p_0),
\]

So, calculate the expected waiting time in the system, using as:

\[
W = \frac{L}{\lambda},
\]

And, Calculate expected waiting time in the queue, using as:

\[
W_q = \frac{L_q}{\lambda},
\]

Where

\[
p_0 = \left(1 + e^{-\lambda} - e^{-\beta \lambda}\right)^{-1} \left[1 - \beta \lambda \left(1 + e^{-\lambda} - e^{-\beta \lambda}\right)^{-1}\right]
\]

5. Special Cases

Some queuing systems can be obtained as special cases of this system:

Case (1): Let \( \beta = 1 \), this is the queue: M/D/1 without any concepts. Then relations (9), (20), (27), (28), (29) and (30) are expressed as:

The steady-seat probability that there are no customers in the system is:

\[
p_0 = 1 - \rho,
\]

The expected number of customers in the system is:

\[
L = \frac{\rho}{1 - \rho},
\]

The expected number of units in the queue is:

\[
L_q = \frac{\rho^2}{1 - \rho},
\]

The expected waiting time in the system is:

\[
W = \frac{1}{\lambda} \left(\frac{\rho}{1 - \rho}\right),
\]

And the expected waiting time in the queue is:

\[
W_q = \frac{1}{\lambda} \left(\frac{\rho^2}{1 - \rho}\right)
\]

where \( \rho = \lambda / \mu \)

Relations (38-42) are the same results as Harris [9], Prasad and Usha [10].

6. An Illustrative Example

The results of \( p_0 \) and \( L \) for different values of \( \beta \) and \( \lambda \) are shown in the following table1:
Table 1. The results of $p_0$ and $L$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\lambda$</th>
<th>$p_0$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.150</td>
<td>0.050</td>
<td>0.999</td>
<td>0.004</td>
</tr>
<tr>
<td>0.300</td>
<td>0.200</td>
<td>0.923</td>
<td>0.092</td>
</tr>
<tr>
<td>0.450</td>
<td>0.350</td>
<td>0.872</td>
<td>0.160</td>
</tr>
<tr>
<td>0.600</td>
<td>0.500</td>
<td>0.755</td>
<td>0.349</td>
</tr>
<tr>
<td>0.750</td>
<td>0.650</td>
<td>0.510</td>
<td>0.720</td>
</tr>
<tr>
<td>0.880</td>
<td>0.780</td>
<td>0.295</td>
<td>1.430</td>
</tr>
<tr>
<td>0.950</td>
<td>0.900</td>
<td>0.130</td>
<td>3.340</td>
</tr>
</tbody>
</table>

Solution of the model may be determined more readily by plotting $p_0$ against $\beta$ and $\lambda$ as shown in Figure 1. Also $L$ is drawn against $\beta$ and $\lambda$ as given in Figure 2.

As we can see in figure 1, shows that the increased the both of (arrival rate and Balking) offset it decrease the probability that there are no customers in the system. It is seen in figure 2; shows that the increased the both of (arrival rate and Balking) offset it increase the expected number of customers in the system.

Figure 1. The relation between $p_0$ & $(\beta, \lambda)$.

Figure 2. The relation between $L$ & $(\beta, \lambda)$.
Also, assume the \( n = 3 \) units. The results of \( p_0 \), \( p_1, p_2, p_3, L \) and \( W_q \) for different values of \( \lambda \) are shown in the following table 2:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( L )</th>
<th>( W_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.05</td>
<td>0.01</td>
<td>0.0002</td>
<td>0.11</td>
<td>0.060</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.18</td>
<td>0.02</td>
<td>0.0020</td>
<td>0.23</td>
<td>0.125</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>0.24</td>
<td>0.05</td>
<td>0.0070</td>
<td>0.36</td>
<td>0.214</td>
</tr>
</tbody>
</table>

Solution of the model may be determined more readily by plotting \( p_0, p_1, p_2, p_3, L \) and \( W_q \) against \( \lambda \) as given in Figures 3, 4, 5, 6, 7 and 8 respectively.

![Figure 3](image3.png)  
*Figure 3. The relation between \( p_0 \) & \( \lambda \).*

![Figure 4](image4.png)  
*Figure 4. The relation between \( p_1 \) & \( \lambda \).*
Figure 5. The relation between $p_2$ & $\lambda$.

Figure 6. The relation between $p_3$ & $\lambda$. 
Figure 7. The relation between $L$ & $\lambda$.

Figure 8. The relation between $W_q$ & $\lambda$. 
As figure 3, shows that the increased the arrival rate offset it decrease of the probability that there are no customers in the system. Also, from figures 4, 5 and 6, note all increased in arrival rate Offset by an increase and then decrease in the probability that there are n of customers in the system. And in figures 7 and 8, shows that the increased the arrival rate offset it increase in the expected number of customers in the system and the queue.

7. Conclusion

This paper has explained the analytical solution in steady-state for M/D/1 with addition the concept balking a probability generating function and iterative method were devised to determine the probability that there are n customers in the system, the probability that no customers are in the service department, the expected number of customers in the system and the expected number of customers in the queue. Finally, the numerical example was confirmed to confirm the model.

References


