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On the Solution of a Optimal Control Problem for a Hyperbolic System

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Abstract: In this study, the problem of determining the control function that is at the right hand side of a hyperbolic system from the final observation is investigated. Using the Fourier-Galerkin method, the weak solution of this hyperbolic system is obtained. The necessary conditions for the existence and uniqueness of the optimal solution are proved. We also find the approximate solutions of the test problems in numerical examples by a MAPLE® program. Finally, the numerical results are presented in the form of tables.

Keywords: Optimal Control, Partial Differential Equation, Numerical Approximation

1. Introduction

The problem of determining the control function that is at the right hand side of the hyperbolic system has been studied by different authors. Lions [3] examined the problems in detail when the control function is at the right hand side of the hyperbolic problem by using different cost function. Periago [4] has investigated the problem of optimizing the shape and position of the support of the internal exact control of minimal $L_2(0,l)-$ norm for the 1-D wave equation.

Yamamoto [5] has studied the inverse problem of determining $f(x)$ from $\frac{\partial u(f)}{\partial n}$ subject to the hyperbolic problem

$$u''(x,t) = \Delta u(x,t) + \sigma(t) f(x), x \in \Omega, t > 0$$

$$u(x,0) = 0, u'(x,0) = 0, x \in \Omega$$

$$u(x,t) = 0, x \in \partial\Omega, t > 0$$

where $\sigma \in C^1(0,T)$.

Benamou [6] has used the domain decomposition method to solve the optimal control problem in the hyperbolic system and has taken the set of admissible control as a convex subset of $L^2((0,T)\times\Omega)$.

Kim and Pavol [7] have minimized the cost functional

$$J(v) = \int_0^T \int_0^\pi \Big(q\big(u(x,t)\big) + h\big(v(x,t)\big)\Big)\, dx\, dt$$

governed by periodic nonlinear 1-D wave equation. The necessary and sufficient conditions for an admissible pair $(u^*, v^*) \in L^\infty(\mathbb{Q}) \times L^\infty(\mathbb{Q}), \mathbb{Q} = (0,\pi)\times(0,T)$ to be an optimal pair have given by authors.

Lopez *at all.* [8] have considered problem of controlling the function $f(x,t)$ related to the hyperbolic problem

$$\varepsilon u_{tt} - \Delta u + u_t = f 1_w\ (x,t) \in \Omega \times (0,T)$$

$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), x \in \Omega$$

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T).$$

Privat et al. [9] have minimized the norm of the control for given initial data in the wave equation defined on $(0,\pi)$ with Homogeneous Dirichlet boundary condition when the control is in at the right hand side of the equation.

Subaşı and Saraç [10] have obtained a minimizer function for the optimal control problem of the initial velocity in a wave equation.

Saraç and Şener [11] have determined the transverse distributed load in Euler-Bernoulli beam problem from of admissible control. The set of admissible controls has been taken as a subspace of the space $L_2[a,b]$.

Saraç [12] has obtained symbolic and numeric solutions by using the initial velocity as a control function in hyperbolic problem.

Şener et al. [14] have explained applications of the Galerkin method to wave equation.

The problem of determining of unknown spatial load distributions in a vibrating Euler–Bernoulli beam from limited measured data has been solved in [16].

The space $L_2(0,l)$ consist of the functional which are square integrable, inner product and norm in $L_2(0,l)$ are defined respectively as;

$$(u,v)_{L_2} = \int_0^l u(x)v(x)dx \text{ and } \|u\|_{L_2} = \sqrt{(u,u)_{L_2}}.$$

Let U_{ad} be closed, convex subset of $L_2(0,l)$.

In this study, we consider an optimal control problem for a wave equation with homogeneous Dirichlet boundary conditions, the control being the one from the functions that are at the right hand side of the equation. We determining the unknown function $v(x)$ in the closed and convex subset $U_{ad} \subset L_2(0,l)$ from the target $u(x,T;v)$, which correspond to final position using L_2- norm. We are interested in generating Maple® procedure easy to used for obtaining approximate optimal control. The useful approximate optimal control function is easily obtained in some numeric examples.

We consider the following final optimal control problem: Choose a control $v(x) \in L_2(0,l)$ and a corresponding u such that the pair (v,u) minimizes the functional

$$J_\alpha(v) = \|u(x,T;v) - y(x)\|^2_{L_2(0,l)} + \alpha\|v(x)\|^2_{L_2(0,l)} \quad (1)$$

subject to the hyperbolic problem;

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = f(t)v(x), && (x,t) \in \Omega := (0,l)\times(0,T] \\ &u(x,0) = \varphi(x), u_t(x,0) = \psi(x), && x \in (0,l) \\ &u(0,t) = 0, u(l,t) = 0, && t \in (0,T]. \end{aligned} \quad (2)$$

where y is given target function and φ, ψ and f are known functions.

With the choice of the functional in (1), we mentioned the observation of $u(x,T;v)$ in $L_2(0,l)$ for the control $v \in L_2(0,l)$. Our aim is to obtain suitable function v^* which approaches the solution of the problem (2) to desired target $y(x) \in L_2(0,l)$ at the final time $t = T$. Another word, we want to find the function $v^* \in U_{ad}$ such that

$$\text{Inf}\, J_\alpha(v) = J_\alpha(v^*).$$

Here $\alpha > 0$ is a regularization parameter which ensures both the uniqueness of the solution and a balance between the norms $\|u(x,T;v) - y(x)\|^2_{L_2(0,l)}$ and $\|v(x)\|^2_{L_2(0,l)}$. Detailed information as regards the regularization parameter can be found in [2]. The term $\|v\|^2{}_{L_2}$ is called penalization term; its role is to avoid using too large controls in the minimization of $J_\alpha(v)$.

In system (2), the term $f(t)v(x)$ is considered to be an external force. External forces in this form of separation of variables are important in modelling vibrations. In [5] Yamamoto point out that the system (2) is regarded as approximation to a model for elastic waves from a point dislocation source.

This paper is organized as follows. In section 2, we state the definition for solution of the wave equation considered and give the necessary conditions for the existence and uniqueness of the optimal solution. In section 3, we give Frechet derivative of the cost functional and construct a minimizing sequence that converge to the optimal solution. In the last section, we obtain the approximate solutions on numeric examples.

2. Existence of Unique Optimal Solution

In this section, we give the solvability of the optimal control problem (1)-(2). First we state the generalized solution of the hyperbolic problem (2) in view of [1].

Definition 2.1. The generalized (weak) solution of the problem (2) will be defined as the function $u \in H_0^1(\Omega)$, with $u(x,0) = \varphi(x), x \in (0,l)$ which satisfies the following integral identity:

$$\int_0^T \int_0^l (-u_t\eta_t + a^2 u_x \eta_x) dxdt = \int_0^T \int_0^l f v \eta dxdt + \int_0^l \psi \eta(x,0)\, dx \quad (3)$$

for all $\eta \in H_0^1(\Omega)$ with $\eta(x,T) = 0$. To have this solution the followings are needed;

$$f \in L_2(0,T), v \in L_2(0,l), \varphi \in H_0^1(0,l), \psi \in L_2(0,l) \quad (4)$$

Theorem 2.2. Suppose that the condition (4) holds, then the problem (2) has a unique generalized solution and the following estimate is valid for this solution;

$$\|u\|^2_{H_0^1(\Omega)} \le c_0 \left(\|\varphi\|^2_{H_0^1(0,l)} + \|\psi\|^2_{L_2(0,l)} + \|f\|^2_{L_2(0,T)}\|v\|^2_{L_2(0,l)}\right) \quad (5)$$

Proof of this theorem can easily be obtained by Galerkin method used in [1].

Let's give the increment Δv to v such that $v + \Delta v \in U_{ad}$ and show the solution of (2) corresponding $v + \Delta v$ by $u_\Delta = u(x,t;v + \Delta v)$. Then the function $\Delta u = u_\Delta - u$ will be the solution of the following difference problem:

$$\Delta u_{tt} = a^2 \Delta u_{xx} + f(t)\Delta v(x)$$
$$\Delta u(x,0) = 0, \Delta u_t\,(x,0) = 0 \qquad (6)$$
$$\Delta u(0,t) = 0, \Delta u(l,t) = 0$$

Lemma 2.3: Let Δu be the solution of the problem (6). Then the following estimate is valid:

$$\|\Delta u(x,T)\|_{L_2(0,l)} \le c_1 \,\|\Delta v\|_{L_2(0,l)} \qquad (7)$$

Proof: We can proof this lemma in view of [15]. We multiply both sides of the hyperbolic equation (6) by Δu_t, then integrate it on $[0,l]$. After some transformations, we have

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^l [(\Delta u_t)^2 + a^2(\Delta u_x)^2]dx\right\} = \int_0^l f(t)\Delta v(x)\Delta u_t(x,t;v)dx + a^2(\Delta u_x \Delta u_t)|_{x=0}^{x=l}.$$

Using here the homogeneous boundary conditions of the system (6), we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^l [(\Delta u_t)^2 + a^2(\Delta u_x)^2]dx\right\} = \int_0^l f(t)\Delta v(x)\Delta u_t(x,t;v)dx.$$

Integrating both sides on $[0,t], t \in [0,T]$, we get

$$I^2(t) = \int_0^t \int_0^l f(\tau)\Delta v(x)\Delta u_t(x,\tau;v)dx\,d\tau, \forall t \in [0,T]$$

where

$$I^2(t) = \frac{1}{2}\int_0^l [(\Delta u_t)^2 + a^2(\Delta u_x)^2]dx, t \in [0,T]$$

We differentiate both the sides

$$2I(t)I'(t) = \int_0^l f(t)\Delta v(x)\Delta u_t(x,t;v)dx, \forall t \in [0,T].$$

Appling to the right-hand side the Cauchy inequality, we obtain

$$2I(t)I'(t) \le f(t)\|\Delta v\|_{L_2(0,l)}\|\Delta u_t\|_{L_2(0,l)}, \forall t \in [0,T].$$

Since we have

$$\|\Delta u_t\|^2_{L_2(0,l)} \le \int_0^l [(\Delta u_t)^2 + a^2(\Delta u_x)^2]dx = 2I^2(t), \forall t \in [0,T]$$

we get

$$I'(t) \le \frac{1}{\sqrt{2}} f(t)\|\Delta v\|_{L_2(0,l)}, \forall t \in [0,T].$$

Integrating both the sides on $[0,t], t \in [0,T]$ and taking into account $I(0) = 0$, we obtain

$$I(t) \le \frac{1}{\sqrt{2}}\|\Delta v\|_{L_2(0,l)} \int_0^t f(\tau)d\tau, \forall t \in [0,T].$$

Substituting in last inequality $t = T$, we write

$$I(t) \le \frac{C}{\sqrt{2}}\|\Delta v\|_{L_2(0,l)}$$

where $\int_0^T f(t)dt \le C$ (C is a constant).

We have

$$\int_0^l [\Delta u(x,T)]^2 dx = \int_0^l \left(\int_0^T \Delta u_t(x,t)dt\right)^2 dx$$
$$\leq T\int_0^l \int_0^T [\Delta u_t(x,t)]^2 dtdx$$
$$\leq 2T\int_0^T I^2(t)dt.$$

Combining last inequalities, we get

$$\int_0^l [\Delta u(x,T)]^2 dx \leq 2T\int_0^T \frac{C^2}{2}\|\Delta v\|^2_{L_2(0,l)}dt$$
$$\leq (TC)^2\|\Delta v\|^2_{L_2(0,l)}$$

which implies the required estimate (7).

We can write the cost functional (1) in the following way;

$$J_\alpha(v) = \int_0^l [u(x,T;v) - u(x,T;0) + u(x,T;0) - y(x)]^2\, dx + \alpha\int_0^l v^2\, dx$$

So we rewrite $J_\alpha(v)$ as

$$J_\alpha(v) = \pi(v,v) - 2Lv + b \tag{8}$$

for

$$\pi(v,v) = \int_0^l [u(x,T;v) - u(x,T;0)]^2\, dx + \alpha\int_0^l v^2\, dx \tag{9}$$

$$Lv = \int_0^l [u(x,T;v) - u(x,T;0)][y(x) - u(x,T;0)]dx \tag{10}$$

and

$$b = \int_0^l [y(x) - u(x,T;0)]^2\, dx \tag{11}$$

Due to the linearity of the transform $v \to u[v] - u[0]$, it can easily be seen that the functional $\pi(v,v)$ is bilinear and symmetric. Further, we write the following;

$$|\,\pi(v,v)| \geq \alpha\|v\|^2_{L_2(0,l)} \tag{12}$$

and this implies the coercivity of $\pi(v,v)$. Since

$$\pi(v,\eta) = \int_0^l [u(x,T;v) - u(x,T;0)]\,[u(x,T;\eta) - u(x,T;0)]dx + \alpha\int_0^l v\eta dx$$

applying Cauchy-Schwartz inequality and using (7), we get

$$|\,\pi(v,\eta)| \leq c_2\|v\|_{L_2(0,l)}\|\eta\|_{L_2(0,l)} \tag{13}$$

for $c_2 = max\{c_1,\alpha\}$. Then $\pi(v,\eta)$ is continuous.

The functional Lv is linear. We can easily write that

$$Lv \leq c_3\|v\|_{L_2(0,l)} \tag{14}$$

using (7). Hence we see that the functional Lv is continuous.

Theorem 2.4. Let $\pi(v,v)$ be a continuous symmetric bilinear coercive form and Lv be a continuous linear form. Then there exists a unique element $v^* \in U_{ad}$ such that

$$J_\alpha(v^*) = \operatorname*{Inf}_{v\in U_{ad}} J_\alpha(v).$$

Proof of this theorem can easily be obtained by showing the weak lower semi-continuity of J_α same as in [3].

3. Frechet Differential of the Cost Functional and Minimizing Sequence

Let us introduce the Lagrangian $L(u,v,z)$ given by

$$L(u,v,z) = \int_0^l [u(x,T;v) - y(x)]^2\, dx + \alpha\int_0^l v^2 dx + \int_0^T\int_0^l [u_{tt} - a^2u_{xx} - f(t)v(x)]zdxdt \tag{15}$$

Using the $\delta L = 0$ stationarity condition, we get the following adjoint problem:

$$z_{tt} - a^2 z_{xx} = 0$$
$$z(x,T) = 0, z_t(x,T) = 2[u(x,T;v) - y(x)] \tag{16}$$
$$z(0,t) = 0, z(l,t) = 0$$

Now, we investigate the variation of the functional $J_\alpha(v)$. The difference functional $\Delta J_\alpha(v) = J_\alpha(v + \Delta v) - J_\alpha(v)$ is such as

$$\Delta J_\alpha(v) = \int_0^l [2u(x,T;v) - 2y(x)]\Delta u(x,T)dx + \int_0^l [\Delta u(x,T)]^2 dx + \alpha \int_0^l (2v + \Delta v)\Delta v dx \tag{17}$$

Here, the term

$$2\int_0^l [u(x,T;v) - y(x)]\Delta u(x,T)dx$$

must be evaluated. Using the problems (6) and (16) we have

$$2\int_0^l [u(x,T;v) - y(x)]\Delta u(x,T)dx = -\int_0^T \int_0^l f(t)z(x,t)\Delta v(x)dxdt$$

So the relation (17) can be written as

$$\Delta J_\alpha(v) = \int_0^l \left\{-\int_0^T f(t)z(x,t) + 2\alpha v\right\}\Delta v dx + \int_0^l [\Delta u(x,T)]^2 dx + \alpha \int_0^l (\Delta v)^2 dx. \tag{18}$$

Using Lemma 2.3 in the (18), we can write the following equality:

$$\Delta J_\alpha(v) = \langle -\int_0^T f(t)z(x,t)dt + 2\alpha v, \Delta v\rangle_{L_2(0,l)} + o(\|v\|^2_{L_2(0,l)})$$

By the definition of Frechet differential at $v \in U_{ad}$ we get the gradient

$$J'_\alpha(v) = -\int_0^T f(t)z(x,t)dt + 2\alpha v.$$

So, we can state the following theorem in view of [2].

Theorem 3.1. The control v^* and the state $u^* = u(v^*)$ are optimal if there exists a multiplier $z^* \in U_{ad}$ such that z^* and v^* satisfy the following optimality conditions:

$$\langle -\int_0^T f(t)z^*(x,t)dt + 2\alpha v^*, v - v^*\rangle_{L_2(0,l)} \geq 0 \tag{19}$$

for $\forall v \in U_{ad}$.

Now, we can apply standard steepest descent iteration. We write an iterative procedure to compute a sequence of controls $\{v_k\}$ convergent to the optimal one.

Select an initial control v_0. If v_k is known ($k \geq 0$) then v_{k+1} is computed according to the following scheme.

1. Solve the state problem (2) in the sense (3) and get corresponding u_k.
2. Knowing u_k solve the adjoint problem (16).
3. Using z_k get the gradient $(J'_\alpha)_k$
4. Set

$$v_{k+1} = v_k - \beta_k J'_\alpha(v_k) \tag{20}$$

and select the relaxation parameter β_k in order to assure that

$$J_\alpha(v_{k+1}) - J_\alpha(v_k) = \beta_k\left[-\|J'_\alpha(v_k)\|^2 + \frac{o(\beta_k)}{\beta_k}\right] < 0 \tag{21}$$

for sufficiently small $\beta_k > 0$. The term $o(\beta_k)$ is infinite decreasing term with high order respect to β_k. Computations of the β_k can be carried out by one of the methods shown in [13].

One of the following can be taken as a stopping criterion to the iteration process;

$$\|v_{k+1}-v_k\|<\varepsilon_1, |J_\alpha(v_{k+1})-J_\alpha(v_k)|<\varepsilon_2, \|J'_\alpha(v_k)\|<\varepsilon_3.$$

Lemma 3.2. The cost functional (1) is strongly convex with the strong convexity constant α.

From the following strongly convex functional definition for $\lambda\in[0,1]$:

$$J_\alpha(\lambda v_1+(1-\lambda)v_2)\le \lambda J_\alpha(v_1)+(1-\lambda)J_\alpha(v_2)-\chi\lambda(1-\lambda)\|v_1-v_2\|^2_{L_2(0,l)}$$

we can see that cost functional (1) is strongly convex the constant $\chi=\alpha$.

So, we can give the following theorem which states the convergence of the minimizer to optimal solution.

Theorem 3.3. Let v^* be optimum solution of the problem (1)-(2). Then the minimizer given in (20) satisfies the following inequality;

$$\|v_k-v^*\|^2\le\frac{2}{\alpha}\left(J_\alpha(v_k)-J_\alpha(v^*)\right), k=0,1,2,\dots \tag{22}$$

Proof of this theorem is obtained by taking $\lambda=\frac{1}{2}$ in the definition of the above strongly convex functional.

4. Numerical Example

In this section we test the method in a numerical example. The used trigonometric basis functions are chosen such as;

$$\left\{\sqrt{\frac{2}{l}}\sin\left(\frac{\pi}{l}x\right),\sqrt{\frac{2}{l}}\sin\left(\frac{2\pi}{l}x\right),\sqrt{\frac{2}{l}}\sin\left(\frac{3\pi}{l}x\right),\dots,\sqrt{\frac{2}{l}}\sin\left(\frac{N\pi}{l}x\right)\right\}$$

for the generalized solution of the hyperbolic problem (2).

Example 4. 1: Let us consider the following problem of minimizing the cost functional:

$$J_\alpha(v)=\int_0^2[u(x,1;v)]^2dx+\alpha\int_0^2 v^2dx$$

under the following condition:

$$u_{tt}-u_{xx}=(t-1)v(x),(x,t)\in(0,2)\times(0,1]$$

$$u(x,0)=-\begin{cases}x^3+x^2 & 0<x\le 1\\ -7x^2+19x-10 & 1\le x\le 2\end{cases}$$

$$u_t(x,0)=\begin{cases}x^3+x^2 & 0<x\le 1\\ -7x^2+19x-10 & 1\le x\le 2\end{cases}$$

$$u(0,t)=0, u(2,t)=0, t\in(0,1].$$

The weak solution of this problem is

$$u(x,t)=(t-1)\begin{cases}x^3+x^2 & 0<x\le 1\\ -7x^2+19x-10 & 1\le x\le 2\end{cases}.$$

The function u and its partial derivatives u_x,u_t belong to $C(\Omega)$. The function $u(x,t)$ is not a classical solution since $u_{xx}\notin C(\Omega)$. Here the force function is discontinuous.

Rewrite the functional as

$$J_\alpha(v)=J^1_\alpha(v)+\alpha J^2_\alpha(v)$$

where

$$J^1_\alpha(v)=\int_0^2[u(x,1;v)]^2dx$$

$$J^2_\alpha(v)=\int_0^2 v^2dx$$

Choosing $\alpha=0.1$, starting the initial element $v_0=sin\pi x$ and the relaxation parameter

$\beta_k=0.1$ assures the inequality $J_{0.1}(v_{k+1})<J_{0.1}(v_k)$.

We get the following approximate minimizing function and the values of the $J^1_{0.1}(v_{88})$ and $J^2_{0.1}(v_{88})$, respectively;

$$\begin{aligned}v_{88}=&-0.088086149\sin(3.14159265x)-1.66789652\sin(1.57079632x)\\&-0.011854776\sin(4.71238898x)+0.00127332\sin(6.28318530x)\\&-0.000781035\sin(7.85398163x)-0.00106163\sin(9.42477796x)\\&-0.000162151\sin(10.9955742x)+0.00003977\sin(12.5663706x)\\&-0.000042586\sin(14.1371669x)-0.00008255\sin(15.7079632x)\end{aligned},$$

$$J^1_{0.1}(v_{88})=5.303067954,$$

$$J^2_{0.1}(v_{88})=2.789781915$$

when the stopping criteria $J_{0.1}(v_{k+1})-J_{0.1}(v_k)>-0.2\times10^{-8}$ are chosen.

Table 1. *The values of functions and iteration numbers for different α values of Example.*

α	$J_\alpha^1(v^*)$	$J_\alpha^2(v^*)$	k
0.9	5.880206210	0.0502480698	15
0.7	5.855965285	0.0801608045	18
0.5	5.812154139	0.1484081594	23
0.2	5.606363377	0.7782944222	47
0.08	5.148875247	4.2750794700	112
0.06	4.927442646	7.1645327120	147
0.05	4.764295988	9.8422178970	174

Table 2. *The values of $J_\alpha^1(v^*), J_\alpha^2(v^*)$ and the optimal controls v^* for different α values of Example.*

α	$J_\alpha^1(v^*)$	$J_\alpha^2(v^*)$	v^*
0.9	5.8802062	0.0502480	$0.0183201\sin(3.141592x) - 0.224057\sin(1.570796x)$ $-0.0015055\sin(4.712388x) + 0.000161\sin(6.283185x)$ $-0.0000991\sin(7.853981x) - 0.000134\sin(9.424777x)$ $-0.000020\sin(10.99557x) + 0.0000087\sin(12.56637x)$ $-0.000005\sin(14.137166x) - 0.000011\sin(15.70796x)$
0.5	5.8121541	0.1484081	$0.0322759\sin(3.141592x) - 0.223405\sin(1.570796x)$ $-0.002602\sin(4.712388x) + 0.0002792\sin(6.283185x)$ $-0.000171\sin(7.853981x) - 0.0002328\sin(9.424777x)$ $-0.000035\sin(10.99557x) + 0.0000087\sin(12.56637x)$ $-0.000009\sin(14.137166x) - 0.000018\sin(15.70796x)$
0.2	5.6063633	0.7782944	$0.0150037\sin(3.141592x) - 0.882060\sin(1.570796x)$ $-0.006088\sin(4.712388x) + 0.0006535\sin(6.283185x)$ $-0.0004001\sin(7.853981x) - 0.000544\sin(9.424777x)$ $-0.000083\sin(10.99557x) + 0.0000020\sin(12.56637x)$ $-0.000002\sin(14.137166x) - 0.000042\sin(15.70796x)$
0.08	5.1488752	4.2750794	$0.1593668\sin(3.141592x) - 2.061421\sin(1.570796x)$ $-0.014898\sin(4.712388x) + 0.0016001\sin(6.283185x)$ $-0.000981\sin(7.853981x) - 0.0013346\sin(9.424777x)$ $-0.000203\sin(10.99557x) + 0.0000501\sin(12.56637x)$ $-0.000053\sin(14.137166x) - 0.000103\sin(15.70796x)$
0.05	4.7642959	9.8422178	$-0.339056\sin(3.141592x) - 3.118764\sin(1.570796x)$ $-0.023551\sin(4.712388x) + 0.0025322\sin(6.283185x)$ $-0.001552\sin(7.853981x) - 0.0021105\sin(9.424777x)$ $-0.000322\sin(10.99557x) + 0.0000791\sin(12.56637x)$ $-0.000084\sin(14.137166x) - 0.000164\sin(15.70796x)$

5. Conclusion

In this paper, we show that the external force in the wave equation be controlled by minimizing the distance between final situation distance and the desired target function. By using the adjoint approach in the mathematical analysis of the optimal control problem for wave equation, the gradient of the cost functional can be obtained. The minimizing sequence is constructed via this gradient.

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