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Oscillations of Solutions of Neutral Nonlinear Differential Equations

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Abstract: This paper aims to establish a new class of differential equations and study the oscillatory behavior of a kind of first-order neutral nonlinear differential equation with time delay arguments. The oscillatory properties of the solutions of the type of first order neutral functional differential equations applied in chemomedical problems are studied. Sufficient conditions for the oscillations of solutions of the above equations are obtained. Also, some results which demonstrate in literature [1-4] will be extended, and the paper focuses on expanding the main finding of literature [2, 3]. Moreover, a new kind of method to be used to discuss the properties of oscillation of the first-order neutral nonlinear differential equations and some theorems are obtained in the paper.

Keywords: Oscillation, Differential Equations, Neutral, Piecewise Constant Arguments

1. Introduction

The oscillation theory of differential equations with deviating arguments is a relatively new and rapidly developing branch of the theory of ordinary differential equation, numerous research papers have been devoted to this study. Recently, attempts have been made by many mathematicians to develop the oscillation theory of differential equations with deviating arguments. The mathematical modeling of several real-world problems leads to differential equations rely more on the past history rather than on the present. The models may have discrete time delays as well as distributed lags or delays originated in [5]. Bernoulli (1728) studied the problem of sound vibrating in a tube of finite length and investigated the properties of first order Ordinary Differential Equation With Deviating Arguments (ODEWDA). Miskis investigated several oscillation problems of first order ODEWDA, which are recorded in his book. Since 1950 oscillation theory of ODEWDA has received the attention of several applied mathematicians as well as other scientists around the world. There are two main reasons to pursue this research, firstly, theoretically it is well-known that the ODE $y'(t) + p(t)y(t) = 0 \quad p(t) \in C(R^+)$ has no the oscillatory solution, but the equation $y'(t) + y(t - \frac{\pi}{2}) = 0$ has the oscillatory solution $y = \sin t$. Therefore, the oscillation of this type of first order equation is caused by deviating arguments. Secondly, this problem arises in many industrial and scientific problems, for example the literature of [6]. The first systematic research of the oscillation of first order equations with deviating arguments was given in the book “Linear differential equations with retarded argument” by Miskis, A. D.

The new development of the oscillation theory of differential equation with deviating arguments has two cases, those are the research of oscillatory solution of partial differential equation with deviating arguments and the research of oscillatory solution of difference equation with deviating arguments. The difference equation has been considered in its own right as a method of biology model of single species with nonoverlapping generation, on the other hand, some methods for the latter in the special case when the deviation of argument vanishes at individual points have been used to investigate differential equations with piecewise constant delays, this note continues the investigation of differential equations with piecewise constant arguments (EPCA) originated in literature [7]. They are closely related to impulse and loaded equation, especially, to difference equation of a discrete argument, these equations have the structure of continuous dynamical systems within intervals of certain length. Continuity of a solution at a point joining any two consecutive intervals implies recursion for the solution at such points, and the equations are thus similar in structure to those found in certain "sequential-continuous" models of disease dynamics. The cited works show that all types of EPCA share similar characteristics. First of all, it is natural pose the initial value problem for such equations not on interval but at a number of individual points, second, two-sided solutions exist for all types of EPCA, finally, since EPCA combine the features of both differential and difference equations, their asymptotic-behavior as tending to infinity resembles in some cases the solutions growth of differential equation, while in other it inherits the properties of difference equation.

In recent years there has been a growing interesting in oscillation theory of the retarded and advanced functional differential equations with piecewise constant arguments, see, for example in literature [6]-[16]. And there were some papers on oscillatory properties of neutral functional differential equations with piecewise constant arguments [17]-[32], especially, there has been a lot of activities concerning the oscillatory behavior of a kind of first-order neutral linear differential equations. But there is hardly any work at the time concerning the oscillatory behavior of a kind of first-order neutral nonlinear differential equations with time delay arguments.

2. Preliminaries

In this paper, the first-order neutral nonlinear differential inequalities with time delay arguments and the neutral nonlinear functional differential equation with time delay arguments are considered

$$\left[x(t)-\sum_{j=1}^{n}c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)),x(t-\sigma_2(t)),\ldots,x(t-\sigma_k(t))]$$
$$+\,p(t)f[x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_k(t))]\le 0, \tag{1}$$

$$\left[x(t)-\sum_{j=1}^{n}c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)),x(t-\sigma_2(t)),\ldots,x(t-\sigma_k(t))]$$
$$+\,p(t)f[x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_k(t))]\ge 0 \tag{2}$$

and

$$\left[x(t)-\sum_{j=1}^{n}c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)),x(t-\sigma_2(t)),\ldots,x(t-\sigma_k(t))]$$
$$+\,p(t)f[x(t-\tau_1(t)),x(t-\tau_2(t)),\ldots,x(t-\tau_k(t))]=0, \tag{3}$$

where $a(t), p(t), \sigma_i(t)$ and $\tau_i(t)$ are continuous functions for $t\in R^+$, $i=1,2,\ldots,k.$ such that $a(t)\ge 0$, $p(t)>0$, $0\le\sigma_i(t)\le M$ and $0<m\le\tau_i(t)\le M$ (m, M are positive constants). c_j, θ_j are positive constants too, $0\le\sum_{j=1}^{n}c_j<1, 0<\theta_1<\theta_2<\cdots<\theta_n$. Also suppose that the functions f and g satisfy the following condition:

(A) f (or g) is continuous on R^k and such that

$$y_j>0,\ \text{for } j=1,2,\ldots,k. \Rightarrow f(y_1,y_2,\cdots,y_k)>0, (\text{ or } g(y_1,y_2,\cdots,y_k)>0)$$

and

$$y_j<0,\ \text{for } j=1,2,\ldots,k. \Rightarrow f(y_1,y_2,\cdots,y_k)<0, (\text{or } g(y_1,y_2,\cdots,y_k)<0).$$

Some sufficient conditions are given as follows:

1. Differential inequality (1) has no eventually positive solutions;
2. Differential inequality (2) has no eventually negative solutions;
3. Differential equation (3) has only oscillatory solutions.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros.

3. Two Lemmas

Lemma 1 Suppose that $x(t)$ is an eventually positive solution of (1). Let

$$z(t) = x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j) \tag{4}$$

Then there is a $T \ge t_0$, such that $z'(t) < 0, z(t) > 0$ for $t \ge T$.

Proof Suppose that $x(t) > 0$, $x(t-\theta_n - M) > 0$, for $t \ge T \ge t_0$. From (1), we get that $z'(t) < 0$, so $z(t)$ is strictly decreasing for $t > T$.

Let $\lim_{t\to\infty} z(t) = L$, then L is a finite constant or $-\infty$.

It can be proved that $L = -\infty$ is impossible. Assume that $L = -\infty$, then we get the fact $x(t)$ is unbounded from (4). So there is a sequence $\{t_k\}$, $t_k \to \infty (k \to \infty)$ such that

$$x(t_k) = \max_{s \le t_k} \{x(s)\}, \lim_{t\to\infty} x(t_k) = +\infty$$

$$z(t_k) = x(t_k) - \sum_{j=1}^{n} c_j x(t_k - \theta_j) \ge x(t_k) - \sum_{j=1}^{n} c_j x(t_k)$$

$$= (1 - \sum_{j=1}^{n} c_j)\, x(t_k) \to +\infty, (k \to \infty)\ (0 \le \sum_{j=1}^{n} c_j < 1) \tag{5}$$

This is a contradiction to $L = -\infty$. Hence, L is a finite constant. Thus, $z(t)$ is bounded and $x(t)$ is bounded too. Then $\overline{\lim_{t\to\infty}} x(t) < \infty$.

It is easy to prove that $L < 0$ is impossible from (5). Now we shall prove that $L > 0$.

Take $\{t_k\}$ such that $t_k \to \infty$ ($k \to \infty$) and $\lim_{k\to\infty} x(t_k) = \overline{\lim_{t\to\infty}} x(t)$,

Then

$$L = \lim_{k\to\infty} z(t_k) = \lim_{k\to\infty}[x(t_k) - \sum_{j=1}^{n} c_j x(t_k - \theta_j)]$$

$$= \lim_{k\to\infty} x(t_k) - \lim_{k\to\infty} \sum_{j=1}^{n} c_j x(t_k - \theta_j).$$

Since $\{x(t_k - \theta_j)\}$ ($j = 1,2,\ldots,n.$) is bounded, we have a subsequence $\{t_{k_i}\} \subset \{t_k\}$ such that $\lim_{i\to\infty} x(t_{k_i} - \theta_j)$ ($j = 1,2,\ldots,n.$) exist.

So

$$L = \lim_{i\to\infty} z(t_{k_i}) = \lim_{k\to\infty}[x(t_{k_i}) - \sum_{j=1}^{n} c_j x(t_{k_i} - \theta_j)]$$

$$= \lim_{i\to\infty} x(t_{k_i}) - \lim_{i\to\infty} \sum_{j=1}^{n} c_j x(t_{k_i} - \theta_j)$$

$$\ge \overline{\lim_{t\to\infty}} x(t) - \sum_{j=1}^{n} c_j \overline{\lim_{t\to\infty}} x(t)$$

$$= (1 - \sum_{j=1}^{n} c_j) \overline{\lim_{t\to\infty}} x(t) \ge 0.$$

Since $z(t)$ is strictly decreasing, so $z(t) > 0$ for $t \ge T$. we obtain that

$$z'(t) < 0, z(t) > 0 \quad \text{for } t \ge T.$$

Lemma 2 Suppose that $x(t)$ is an eventually positive solution of (1). Let

$$z(t) = x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j)$$

and

$$\omega(t) = \frac{z(t-m)}{z(t)},$$

If $\liminf_{t\to\infty} \omega(t) = X \ge 1$, and X is finite then $\lim_{t\to\infty} x(t) = 0$.

Proof By Lemma 1, there is a $T \ge t_0$, such that $z'(t) < 0, z(t) > 0$ for $t \ge T$, then

$$\lim_{t\to\infty} z(t) = L, (0 \le L < +\infty).$$

Integrating both sides of (1) from $t - m$ to t, for $t > t_0 + 3M$

$$z(t) - z(t-m) + \int_{t-m}^{t} a(s) g[x(s-\sigma_1(s)), \cdots, x(s-\sigma_k(s))] ds$$

$$+\int_{t-m}^{t} p(s)f[x(s-\tau_1(s)),\cdots,x(s-\tau_k(s))]ds \le 0,\ t > t_0 + 3M.$$

If $0 < L < +\infty$, which contradicts the above inequality. Therefore $\lim_{t\to\infty} z(t) = 0$, and $\lim_{t\to\infty} x(t) = 0$.

Therefore $z(t)$ is bounded and so is $x(t)$.

If $\lim_{t\to\infty} x(t) \neq 0$, Assume that $0 < \overline{\lim_{t\to\infty}} x(t) < \infty$. Take $\{t_k\}$ such that $t_k \to \infty\,(k\to\infty)$, the proof is similar to the proof of (5) in Lemma 1.

It can get

$$\lim_{k\to\infty} x(t_k) = \overline{\lim_{t\to\infty}} x(t)$$

$$L \ge \overline{\lim_{t\to\infty}} x(t) - \sum_{j=1}^{n} c_j \overline{\lim_{t\to\infty}} x(t)$$

$$= (1-\sum_{j=1}^{n} c_j)\overline{\lim_{t\to\infty}} x(t) \neq 0,$$

which leads to a contradiction. Hence, $\overline{\lim_{t\to\infty}} x(t) = 0$. So $\lim_{t\to\infty} x(t) = 0$.

4. Oscillations of the Neutral Nonlinear

Theorem 1 Consider the delay differential inequality

$$\left[x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)), x(t-\sigma_2(t)),\ldots, x(t-\sigma_k(t))]$$

$$+ p(t)f[x(t-\tau_1(t)), x(t-\tau_2(t)),\ldots, x(t-\tau_k(t))] \le 0,$$

where $a(t), p(t), \sigma_i(t)$ and $\tau_i(t)$ are continuous functions for $t \in R^+$, $i = 1,2,\ldots,k.$ such that $a(t) \ge 0$, $p(t) > 0$, $0 \le \sigma_i(t) \le M$ and $0 < m \le \tau_i(t) \le M$ (m, M are positive constants). c_j, θ_j are positive constants too, $0 \le \sum_{j=1}^{n} c_j < 1$, $0 < \theta_1 < \theta_2 < \cdots < \theta_n \le M$. Also suppose that the functions f and g satisfy the condition (A). Assume that there exist nonnegative numbers $\alpha_i, i = 1,2,\ldots,k.$, with $\sum_{i=1}^{k}\alpha_i = 1$ and $\sum_{i=1}^{k}\beta_i = 1$ such that

$$G = \lim_{\substack{y_i\to 0\\ 1\le i\le k}} \sup \frac{|y_1|^{\partial_1}|y_2|^{\partial_2}\cdots|y_k|^{\partial_k}}{|g(y_1,y_2,\cdots,y_k)|}$$

and

$$H = \lim_{\substack{y_i\to 0\\ 1\le i\le k}} \sup \frac{|y_1|^{\beta_1}|y_2|^{\beta_2}\cdots|y_k|^{\beta_k}}{|f(y_1,y_2,\cdots,y_k)|}$$

where $G > 0, H > 0$. Assume further that

$$\frac{1}{H}\lim_{t\to\infty}\inf\int_{t-m}^{t} p(s)ds > \frac{1}{e}\exp\left(-\frac{1}{G}\lim_{t\to\infty}\inf\int_{t-m}^{t} a(s)ds\right) \tag{6}$$

and

$$\lim_{t\to\infty}\inf\int_{t-\frac{m}{2}}^{t}\frac{1}{H}p(s)ds > 0 \tag{7}$$

Then (1) has no eventually positive solutions.

Proof The existence of an eventually positive solution will be proved that it leads to a contradiction. To this end suppose that $x(t)$ is solution of (1) such that for t_0 sufficiently large

$$x(t) > 0,\ \forall t > t_0$$

They can choose a $t_1 > t_0 + M$ such that $x(t-\sigma_i(t)) > 0, x(t-\tau_i(t)) > 0$, $i = 1,2,\ldots,k.$ for $t > t_1$. let $z(t) = x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j)$, and observing that the functions f and g satisfy the condition (A), and thus from (1) and Lemma1, you obtain $z'(t) < 0, z(t) > 0$ for $t > t_0 + M$, i.e. $z(t)$ is strictly decreasing for $t > t_0 + M$, Hence, you have $z(t) < z(t-m)$ and $z(t) < x(t)$ for $t > t_0 + 2M$

Set

$$\omega(t) = \frac{z(t-m)}{z(t)}, \ \forall\ t > t_0 + 2M \tag{8}$$

Then $\omega(t) > 1$ and dividing both sides of (1) by $z(t)$, they obtain

$$\frac{z'(t)}{z(t)} + a(t)\frac{g[x(t-\sigma_1(t)),\cdots,x(t-\sigma_k(t))]}{z(t)}$$
$$+ p(t)\frac{g[x(t-\tau_1(t)),\cdots,x(t-\tau_k(t))]}{z(t)} \le 0,\ \forall\ t > t_0 + 2M \tag{9}$$

Integrating both sides of (9) from $t-m$ to t, for $t > t_0 + 3M$, we have

$$\ln z(t) - \ln z(t-m) + \int_{t-m}^{t} a(s)\frac{g[x(s-\sigma_1(s)),\cdots,x(s-\sigma_k(s))]}{z(s)}ds$$
$$+ \int_{t-m}^{t} p(s)\frac{g[x(s-\tau_1(s)),\cdots,x(s-\tau_k(s))]}{z(s)}ds \le 0,\ \forall\ t > t_0 + 3M$$

From which and (8), we obtain

$$\ln \omega(t) \ge \int_{t-m}^{t} a(s)\frac{g[x(s-\sigma_1(s)),\cdots,x(s-\sigma_k(s))]}{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))}\frac{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))}{z(s)}ds$$
$$+ \int_{t-m}^{t} p(s)\frac{f[x(s-\tau_1(s)),\cdots,x(s-\tau_k(s))]}{\prod_{i=1}^{k} x^{\beta_i}(s-\sigma_i(s))}\frac{\prod_{i=1}^{k} x^{\beta_i}(s-\tau_i(s))}{z(s)}ds$$
$$\forall\ t > t_0 + 3M$$

Notice that $z(t) < x(t)$ for $t > t_0 + 2M$.

And $$\ln \omega(t) \ge \inf_{s>t-m}\frac{g[x(s-\sigma_1(s)),\cdots,x(s-\sigma_k(s))]}{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))}\frac{\prod_{i=1}^{k} z^{\alpha_i}(s-\sigma_i(s))}{z(s)}\int_{t-m}^{t} a(s)ds$$
$$+ \int_{t-m}^{t} p(s)\inf_{s>t-m}\frac{f[x(s-\tau_1(s)),\cdots,x(s-\tau_k(s))]}{\prod_{i=1}^{k} x^{\beta_i}(s-\sigma_i(s))}\frac{\prod_{i=1}^{k} z^{\beta_i}(s-\tau_i(s))}{z(s)}ds$$
$$\forall\ t > t_0 + 3M$$

$$\frac{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))}{z(s)} \ge \frac{\prod_{i=1}^{k} z^{\alpha_i}(s-\sigma_i(s))}{z(s)} \ge 1$$

$$\frac{\prod_{i=1}^{k} z^{\beta_i}(s-\tau_i(s))}{z(s)} \ge \frac{\prod_{i=1}^{k} z^{\beta_i}(s-m)}{z(s)} \ge \frac{z(s-m)}{z(s)} = \omega(s)$$

Thus

$$\ln \omega(t) \ge \inf_{0<y_i<x(t-m-M)}\frac{g(y_1,y_2,\cdots,y_k)}{\prod_{i=1}^{k} y_i^{\alpha_i}}\int_{t-m}^{t} a(s)ds$$

$$+ \inf_{0<y_i<x(t-m-M)} \frac{f(y_1, y_2, \cdots, y_k)}{\prod_{i=1}^{k} y_i^{\beta_i}} \int_{t-m}^{t} p(s)\omega(s)ds \quad \forall\ t > t_0 + 3M \tag{10}$$

Integrating both sides of (1) from $t - \frac{m}{2}$ to t, for $t > t_0 + \frac{5}{2}M$ and observe that $z(t)$ is strictly decreasing, it can be achieved that

$$\frac{z(t)}{z(s)} < 1, \text{ and } \frac{z(t-m)}{z(s-\tau_i(s))} < 1 \text{ for } t - \frac{m}{2} < s < t, s - \tau_i(s) < s - m < t - m$$

$$z(t) - z(t - \frac{m}{2}) + z(t) \inf_{s>t-\frac{m}{2}} \frac{g[x(s-\sigma_1(s)), \cdots, x(s-\sigma_k(s))]}{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} a(s)ds$$

$$+ z(t-m) \inf_{s>t-\frac{m}{2}} \frac{f[x(s-\tau_1(s)), \cdots, x(s-\tau_k(s))]}{\prod_{i=1}^{k} x^{\beta_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} p(s)ds \le 0$$

Dividing the last inequality first by $z(t)$ and then by $z(t - \frac{m}{2})$, we obtain, respectively:

$$1 - \frac{z(t-\frac{m}{2})}{z(t)} + \inf_{s>t-\frac{m}{2}} \frac{g[x(s-\sigma_1(s)), \cdots, x(s-\sigma_k(s))]}{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} a(s)ds$$

$$+ \frac{z(t-m)}{z(t)} \inf_{s>t-\frac{m}{2}} \frac{f[x(s-\tau_1(s)), \cdots, x(s-\tau_k(s))]}{\prod_{i=1}^{k} x^{\beta_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} p(s)ds \le 0 \quad \forall\ t > t_0 + \frac{5}{2}M \tag{11}$$

and

$$\frac{z(t)}{z(t-\frac{m}{2})} - 1 + \frac{z(t)}{z(t-\frac{m}{2})} \inf_{s>t-\frac{m}{2}} \frac{g[x(s-\sigma_1(s)), \cdots, x(s-\sigma_k(s))]}{\prod_{i=1}^{k} x^{\alpha_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} a(s)ds$$

$$+ \frac{z(t-m)}{z(t-\frac{m}{2})} \inf_{s>t-\frac{m}{2}} \frac{f[x(s-\tau_1(s)), \cdots, x(s-\tau_k(s))]}{\prod_{i=1}^{k} x^{\beta_i}(s-\sigma_i(s))} \int_{t-\frac{m}{2}}^{t} p(s)ds \le 0 \quad \forall\ t > t_0 + \frac{5}{2}M \tag{12}$$

Let $\lim_{t\to\infty} \inf \omega(t) = X$

Then we have $X \ge 1$. Now, the following two possible cases are considered:

Case 1 X is finite. We have $z'(t) < 0$, for $t > t_0 + M$ and therefore from (1),

We find $\lim_{t\to\infty} z(t) = 0$

Notice that $z(t) = x(t) - \sum_{j=1}^{n} c_j x(t - \theta_j)$

By Lemma 2, so $\lim_{t\to\infty} x(t) = 0$

Taking limit inferiors on both sides of (8), we obtain

$$\ln X \geq \frac{1}{G}\liminf_{t\to\infty}\int_{t-m}^{t} a(s)ds + \frac{X}{H}\liminf_{t\to\infty}\int_{t-m}^{t} p(s)ds$$

Using the fact that $\ln X - aX \leq -\ln a - 1$ for $X \geq 1$, so we get

$$\ln X - \frac{X}{H}\liminf_{t\to\infty}\int_{t-m}^{t} a(s)ds \leq -\ln\left(\frac{1}{H}\liminf_{t\to\infty}\int_{t-m}^{t} p(s)ds\right) - 1,$$

and

$$\frac{1}{H}\liminf_{t\to\infty}\int_{t-m}^{t} p(s)ds \leq \frac{1}{e}\exp\left(-\frac{1}{G}\liminf_{t\to\infty}\int_{t-m}^{t} a(s)ds\right).$$

The last inequality contradicts hypothesis (4).

Case 2 X is infinite. That is

$$\lim_{t\to\infty}\frac{z(t-m)}{z(t)} = +\infty$$

In view of (5) and the fact that $a(t) \geq 0$, from (9), we get

$$\lim_{t\to\infty}\frac{z(t-\frac{m}{2})}{z(t)} = +\infty.$$

$$\lim_{t\to\infty}\frac{z(t-m)}{z(t-\frac{m}{2})} = +\infty,$$

which contradicts (12). So (1) has no eventually positive solutions.

Theorem 2 Consider the delay differential inequality.

And, therefore

$$\left[x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)), x(t-\sigma_2(t)), \ldots, x(t-\sigma_k(t))]$$
$$+ p(t)f[x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_k(t))] \geq 0,$$

where $a(t), p(t), \sigma_i(t)$ $\tau_i(t)$ c_j, θ_j, f and g satisfy the condition of Theorem 1, $i = 1,2,\ldots,k$. $j = 1,2,\ldots,n$. Then (2) has no eventually negative solutions.

Theorem 3 Consider the delay differential equation

$$\left[x(t) - \sum_{j=1}^{n} c_j x(t-\theta_j)\right]' + a(t)g[x(t-\sigma_1(t)), x(t-\sigma_2(t)), \ldots, x(t-\sigma_k(t))]$$
$$+ p(t)f[x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_k(t))] = 0,$$

Where $a(t), p(t), \sigma_i(t)$ $\tau_i(t)$ c_j, θ_j, f and g satisfy the condition of Theorem 1, $i = 1,2,\ldots,k$. $j = 1,2,\ldots,n$. Then (3) has only oscillatory solutions.

Proof From the result of Theorem 1 it follows that (3) has no eventually positive solutions. Also, from the result of Theorem 1 it follows that (3) has no eventually negative solutions. Therefore (3) has only oscillatory solutions.

5. Applications

Example 1 Consider the first order nonlinear functional differential equation

$$\left[x(t) - \frac{1}{3}x(t-1)\right]' + x(t-\cos^2 t) + 3[x(t-1)]^{1/3}[x(t-(\cos t+2))]^{2/3} = 0 \tag{13}$$

Where $c = \frac{1}{3}, \theta = 1, p(t) \equiv 3, a(t) \equiv 1, m = 1$ and $M = 3$, Clearly, conditions of Theorem 3 are satisfied, then (13) has only oscillatory solutions.

Example 2 Consider the first order nonlinear functional differential equation

$$\left[x(t) - \frac{1}{2}x(t-\frac{1}{3})\right]' + x(t-\sin^2 t) + 2[x(t-1)]^{2/5}[x(t-(2\cos t+3))]^{3/5} = 0, \tag{14}$$

where $c = \frac{1}{2}, \theta = \frac{1}{3}, p(t) \equiv 2, a(t) \equiv 1, m = 1$ and $M = 5$, Clearly, conditions of Theorem 3 are satisfied, Then (14) has only oscillatory solutions.

6. Conclusion

In this paper, we study the oscillatory behavior of solutions of a kind of first-order neutral nonlinear functional differential equations with time delay arguments. We look for the sufficient conditions of the existence of no eventually positive solutions and no eventually negative solutions for the delay functional differential inequalities. We obtain some result by the auxiliary function method which is important tools in oscillation theory. We also get some corollaries about the linear case: $f(t, x(t-\sigma)) = x(t-\sigma)$.

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