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Some Metric Properties of Semi-Regular Equilateral Nonagons

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Abstract: A simple polygon that either has equal all sides or all interior angles is called a semi-regular nonagon. In terms of this definition, we can distinguish between two types of semi-regular polygons: equilateral polygons (that have equal all sides and different interior angles) and equiangular polygons (that have equal interior angles and different sides). Unlike regular polygons, one characteristic element is not enough to analyze the metric properties of semi-regular polygons, and an additional one is needed. To select this additional characteristic element, note that the following regular triangles can be inscribed to a semi-regular equilateral nonagon by joining vertices: $\Delta A_1 A_4 A_7$, $\Delta A_2 A_5 A_8$, $\Delta A_3 A_6 A_9$. Now have a look at triangle $\Delta A_1 A_4 A_7$. Let us use the mark $\varphi=\measuredangle(a,b_1)$ to mark the angle between side a of the semi-regular nonagon and side b_1 of the inscribed regular triangle. In interpreting the metric properties of a semi-regular equilateral nonagon, in addition to its side, we also use the angle that such side creates with the side of one of the three regular triangles that can be inscribed to such semi-regular nonagon. We consider the way in which convexity, possibility of construction, surface area, and other properties depend on a side of the semi-regular nonagon and angle $\varphi=\measuredangle(a,b_1)$.

Keywords: Polygons, Semi-Regular Nonagon. Surface Area, Convexity Poligons

1. Introduction

A simple polygon $\mathcal{A}_9 \equiv A_1A_2A_3\ldots A_9$ that has equal either all sides or all interior angles is called a semi-regular nonagon. In terms of this definition, we can distinguish between two types of semi-regular polygons: equilateral polygons (that have equal all sides and different interior angles) and equiangular polygons (that have equal interior angles and different sides), [5, 8-13]. We consider vertex $A_i, i = 1,\ldots,9$ of a semi-regular equilateral nonagon to be in an even, or in an odd position, if index i is an even, or an odd number, respectively. (Figure 1). In this paper, we consider the metric properties of convex equilateral semi-regular nonagons. Unlike with regular polygons, the length of one of the sides is not sufficient for analyzing the metric properties of equilateral semi-regular polygons, and another characteristic element is needed, [2, 3, 7]. For the selection of this element, note that the following regular triangles can be inscribed to a semi-regular equilateral nonagon: $\Delta A_1A_4A_7,$ $\Delta A_2A_5A_8,$ $\Delta A_3A_6A_9,$ (Figure 1). Let us look at the inscribed equilateral triangle, i.e. $\Delta A_1A_4A_7,$. Let $\varphi = \measuredangle(a, b_1)$ be an angle between side a of the semi-regular nonagon and side b_1 of the inscribed regular triangle. Note that $\varphi = 2\delta$, g where $\delta = \measuredangle(a, d_1) = \measuredangle(d_1, b_1)$, marks the angle between diagonal d_1 of isosecles quadrilateral $A_1A_2A_3\,A_4$ drawn from vertex A_1 and side a, or side b. In this way, in addition to side $A_iA_{i+1} = a, A_{i+1} = A_1, i = 1,2,\ldots,9$ of the semi-regular nonagon, we also get the second characteristic element, angle $\varphi = \measuredangle(a, b)$, which we need in order to analyze the metric properties [3-5]. In this paper, the following marks are used:

1. the side of a semi-regular polygon $\mathcal{P}_9$ is marked with a
2. the number of sides of the “inscribed” regular triangle is marked with n
3. the side of a regular triangle $\mathcal{P}_3^i$ ”inscribed” to a semi-regular nonagon, $\mathcal{A}_9 \equiv A_1A_2\ldots A_9$ constructed by joining its vertices in even (or odd) positions is marked with -$b_i, i \in \mathbb{N}$, i=1,2,3, [1, 4].
4. the interior angles of a semi-regular nonagon at the vertices of the "inscribed" regular triangle $\Delta A_1A_4A_7,$ are marked with α, and at the other vertices of the semi-regular nonagon, they are marked with β

1. Isocescles quadrilaterals $A_1A_2A_3A_4, A_4A_5A_6A_7, A_7A_8A_9A_1$ are "edge" quadrilaterals constructed above each side of the "inscribed" regular triangle, $\triangle A_1A_4A_7,$ (Figure 1), [8-13].
2. d_1 is a diagonal drawn from vertex A_1 of an isosceles quadrilateral $A_1A_2A_3A_4$ constructed above side $A_1A_4 = b_1$ of the "inscribed" regular triangle $\triangle A_1A_4A_7,$ (Figure 1),[1.3.4].

In addition to these marks, all other marks will be interpreted when mentioned in a given definition. To a semi-regular equilateral nonagon $\mathcal{A}_9 \equiv A_1A_2\ldots A_9$ with equal sides there can be "inscribed" three regular triangles, i.e.: $\mathcal{P}_3^1 \equiv A_1A_4A_7, \mathcal{P}_3^2 \equiv A_2A_5A_8, \mathcal{P}_3^3 \equiv A_3A_6A_9$. In interpreting the metric properties of the semi regular nonagon in this paper, in addition to side a we also use angle φ which this side a closes with side b_1 of a regular triangle $\triangle A_1A_4A_7,$ that is inscribed to it, and for which the following applies: $\varphi = \angle(a, b_1)$((Figure 1). To show that a semi-regular equilateral nonagon is given by side a and angle φ we write $\mathcal{A}_9^{a,\varphi}$.

2. My Result

2.1. Convexity of a Semi-regular Equilateral Nonagon

First, let us analyze the dependence of the internal angles of a semi-regular nonagon $\mathcal{P}_9^{a,\varphi}$ (dependence on parameter φ), and then the manner in which convexity depends on the value of this parameter. For this purpose, let us use mark α to mark its interior angles at the vertices of regular triangle $\triangle A_1A_4A_7$ which is "inscribed" to it, and constructed by connecting vertices A_1, A_4, A_7, and let us use mark β to mark the interior angles at other vertices of the semi-regular equilateral nonagon (Figure 1).

Note that the following is valid for the interior angles of the semi-regular equilateral nonagon at vertices A_1, A_4, A_7

$$\alpha = \gamma + 2\varphi = \frac{\pi}{3} + 2\varphi, \tag{1}$$

where $\gamma = \frac{\pi}{3}$ is the interior angle of the regular triangle, while for the interior angles equal to angle

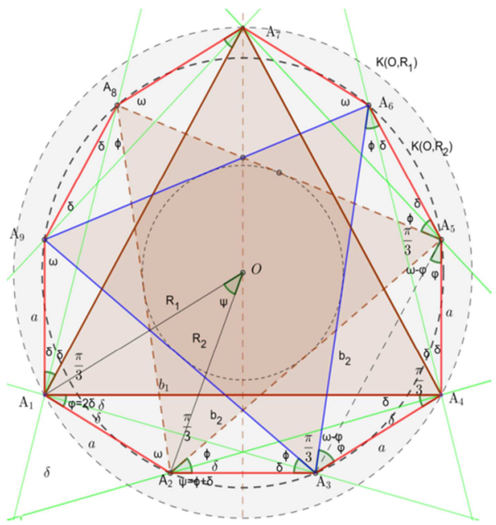

Figure 1. *A semi-regular nonagon – marks and elements.*

β at vertices $A_2, A_3, A_5, A_6, A_8, A_9$, the following applies

$$\beta = \pi - \varphi. \tag{2}$$

The relation between the angles which are created by the side of the inscribed regular triangle and the side of the semi-regular nonagon (Figure 1) is given in a Theorem.

Theorem 1. Let $\mathcal{P}_3^1 \equiv A_1A_4A_7, \mathcal{P}_3^2 \equiv A_2A_5A_8, \mathcal{P}_3^3 \equiv A_3A_6A_9$ be the regular triangles inscribed to semi-regular equilateral nonagon $\mathcal{A}_9 \equiv A_1, A_2, \ldots, A_9$. Then the following equality applies for angles $\varphi = \measuredangle(\mathrm{a}, \mathrm{b}_1)$, $\psi = \measuredangle(\mathrm{a}, \mathrm{b}_2)$, $\omega = \measuredangle(\mathrm{a}, \mathrm{b}_3)$ between the sides of the inscribed triangles and side a of the semi-regular equilateral nonagon

$$\varphi + \psi + \omega = \frac{2\pi}{3} \tag{3}$$

Proof: Let regular equilateral triangles, as marked in Figure 1, be inscribed to a semi-regular equilateral nonagon. Then, note that the following condition is valid for the interior angles of isosceles quadrilateral $A_1A_2A_3A_4$, (and quadrilaterals $A_4A_5A_6A_7, A_7A_8A_9A_1$) constructed above the sides of inscribed regular triangle$\mathcal{P}_3^1 \equiv A_1A_4A_7$:

$$\measuredangle A_1 + \measuredangle A_2 + \measuredangle A_3 + \measuredangle A_4 = 2\pi \tag{4}$$

which then, according to markings in Figure 1, takes the form of:

$$\varphi + \omega + \frac{\pi}{3} + \psi + \psi + \frac{\pi}{3} + \omega + \varphi = 2\pi \tag{5}$$

where:

$$\measuredangle A_2 = \measuredangle A_3 = \beta = \pi - \varphi = \omega + \frac{\pi}{3} + \psi.$$

From equality (5) we get that

$$\varphi + \psi + \omega = \frac{2\pi}{3}.$$

This is similar for the interior angles of isosceles quadrilateral $A_2A_3A_4A_5, A_5A_6A_7A_8, A_8A_9A_1A_2$ constructed above the sides of the inscribed regular triangle, and all angles in inscribed regular triangle $\mathcal{P}_3^2 \equiv A_2A_5A_8$. For example, for the interior angles of a quadrilateral $A_2A_3A_4A_5, A_5$ the following applies

$$\measuredangle A_2 + \measuredangle A_3 + \measuredangle A_4 + \measuredangle A_5 = 2\pi \tag{6}$$

According to markings (Figure 1), the equality (6) has the form of

$\psi + \psi + \frac{\pi}{3} + \omega + 2\varphi + \frac{\pi}{3} + \omega = 2\pi$ from which we find that equality $\varphi + \psi + \omega = \frac{2\pi}{3}$ is valid.

As for the other quadrilaterals, it also turns out that the equality is valid. In case of the interior angles of isosceles quadrilaterals $A_3A_4A_5A_6, A_6A_7A_8A_9, A_9A_1A_2A_3$ it also shows in a similar procedure that the equality is valid, i.e. $\varphi + \psi + \omega = \frac{2\pi}{3}$.

Consequence 2. If all angles between the sides of the inscribed regular triangles and side a of the semi-regular equilateral nonagon are equal, then the nonagon is regular.

Proof. From equality $\varphi+\psi+\omega=\frac{2\pi}{3}$ for $\varphi=\psi=\omega$, it follows that $\varphi=\frac{2\pi}{9}$. For this value of angle φ the interior angles of the semi-regular nonagon are equal, i.e. $=\beta$, in other words, it is regular.

The manner in which the convexity of the semi-regular nonagon depends on the value of angle φ is shown in a Theorem.

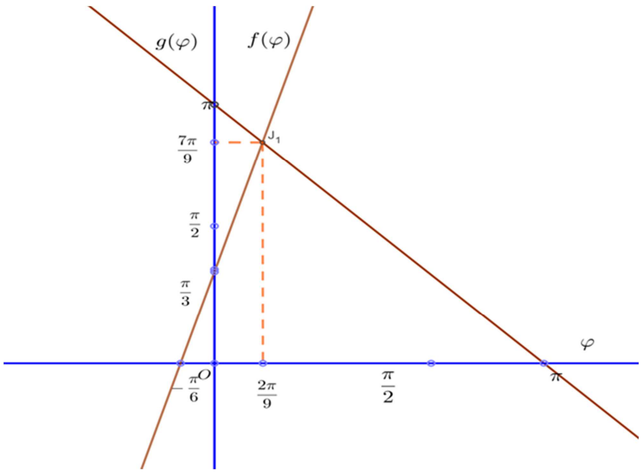

Figure 2. *Convexity of a semi-regular nonagon in relation to value of angle φ.*

Theorem 3. *A semi-regular equilateral polygon $\mathcal{A}_9\equiv A_1A_2A_3\ldots A_9$, with interior angles defined by relations (1) and (2), side* a, *and angle $\varphi=\measuredangle(a,b_1)$,* where b_1 *is the side of inscribed regular triangle $\mathcal{P}_3^1$ is:* convex for all values of angle

$\varphi\in\langle 0,\frac{\pi}{3}\rangle$ (for $\varphi=\frac{2\pi}{9}$ the nonagon is convex and regular, and that is not the subject of this paper), non-convex for all values of angle $\varphi\in[\frac{\pi}{3},\pi\rangle$ and not defined for $\varphi>\pi$ (it degenerates).

Proof: Let us present the interior angles of a semi-regular equilateral nonagon as linear functions of parameter φ. Let us say that

$$f(\varphi)=\alpha=\frac{\pi}{3}+2\varphi, g(\varphi)=\beta=\pi-\varphi.$$

A graph of function f – let us mark it as Γ_f – cuts across graph Γ_g of function g at point $(\frac{2\pi}{9},\frac{7\pi}{9})$, with abscissa being $\varphi=\frac{2\pi}{9}$ (Figure 2). For this value of angle φ, the nonagon is regular. Further, using a definition of a polygon convexity, $(\alpha,\beta<\pi)$, we get the value of angle $0<\varphi<\frac{\pi}{3}$. Given the intersection point of straight lines (Figure 2) and the value of angle φ, we can distinguish between the following:

when $\varphi\in\langle 0,\frac{2\pi}{9}\rangle$ and $\alpha<\beta$

when $\varphi\in\langle\frac{2\pi}{9},\frac{\pi}{3}\rangle$ and $\alpha>\beta$

when $\varphi\in\langle\frac{\pi}{3},\pi\rangle$

when $\varphi\geq\pi$.

Let us consider these in order:

For $\varphi\in\langle 0,\frac{2\pi}{9}\rangle$ the values of the interior angles of a semi-regular equilateral nonagon defined by relations (1) and (2) are at intervals; $\alpha\in\langle\frac{\pi}{3},\frac{7\pi}{9}\rangle,\beta\in\langle\frac{7\pi}{9},\pi\rangle$ and $\alpha<\beta$. Since $\alpha,\beta<\pi$ for all values $\varphi\in\langle 0,\frac{2\pi}{9}\rangle$ the semi-regular equilateral nonagon is convex. If $\delta=\frac{2\pi}{9}$ the equilateral semi-regular nonagon is convex and regular, with internal angles $\alpha=\beta=\frac{7\pi}{9}$.

If $\varphi\in\langle\frac{2\pi}{9},\frac{\pi}{3}\rangle$ then, starting from inequality $\frac{2\pi}{9}<\varphi<\frac{\pi}{3}$, by elemental transformations we get that the values of the interior angles belong to intervals $\alpha\in\langle\frac{7\pi}{9},\pi\rangle$, and $\beta\in\langle\frac{2\pi}{3},\frac{7\pi}{9}\rangle$ and $\alpha>\beta$, respectively, and the semi-regular nonagon is convex.

For $\varphi\in\left[\frac{\pi}{3},\pi\right\rangle$ the value of the internal angles of a semi-regular nonagon equal to angle α are at interval $[\pi,3\pi\rangle$, and the angles equal to angle β are at interval $\left[0,\frac{2\pi}{3}\right\rangle$ Consequently, it folows that a semi-regular nonagon is not convex, given these values of angle φ.

Let $\varphi\geq\pi$. Then, for $\varphi=\pi$ the interior angles equal to angle β disappear, and for $\varphi>\pi$ it is $\beta<0$, in other words, a semi-regular nonagon disappears, i.e. it transforms into other polygons

2.2. Construction of a Semi-Regular Nonagon

Let us analyze the possibility of constructing a semi-regular equilateral nonagon if the length of its side a and angle $\varphi=\angle(a,b_1)$ are given, where b_1 is the side of an "inscribed" regular triangle $\mathcal{P}_3^1\equiv A_1A_4A_7$.

Let us suppose that the problem has been solved and that the semi-regular nonagon is the required one as shown in Figure 3. Let $\mathcal{P}_3^1\equiv A_1A_4A_7$ be the inscribed regular triangle and $|A_1A_3|=b_1$ be the side of this traingle. A quadrilateral $A_1A_2A_3A_4$ constructed above side $A_1A_3=b_1$ of regular triangle $\mathcal{P}_3^1$ is an isosceles one.

The rays of that quadrilateral are equal to the given side a. An isosceles triangle $A_1A_2A_3$ is an auxiliary triangle which can be constructed as it is (rule ASA),: $|A_1A_2|=|A_2A_3|=a$ and $\angle A_1=\delta=\frac{\varphi}{2}$ and$\angle A_2=\pi-\varphi$. On a straight line p we choose point $A_1\in p$ and construct a given angle $\varphi=\measuredangle(p,q)$ and its perpendicular bisector s, and we get angle $\delta=\frac{\varphi}{2}$. Then $\{A_2\}=K(A_1,a)\cap q$ and $\{A_3\}=K(A_2,a)\cap s$.

Vertex A_4 of quadrilateral $A_1A_2A_3A_4$ is the intersection of circle $K(A_3,a)$ and side A_1p of angle $\mathrm{p}A_1q=\varphi$, and its base $A_1A_4=b_1$ is the side of inscribed regular triangle $\mathcal{P}_3^1\equiv A_1A_4A_7$.

If we construct an equilateral triangle over side A_1A_4 we get vertex A_7. Further, above ray A_4A_7 of the "inscribed" regular triangle we construct an isosceles quadrilateral such that at vertices A_4 and A_7 we construct the given angle φ and its bisector, so that one ray of that angle is base A_4A_7. The following applies to vertices $A_5\,A_6$: $\{A_5\}=K(A_4,a)\cap s_{A_7}$ and $\{A_6\}=K(A_7,a)\cap s_{A_4}$ where s_{A_7} and s_{A_4} are bisectors of angle φ constructed at vertex $A_7\,A_4$. If we repeat the process by constructing an isosceles quadrilateral over side A_1A_7, we get vertices A_8,A_9.

Construction and its description:

We complete the geometric construction of a semi-regular nonagon with a given length of side a and angle φ as follows:

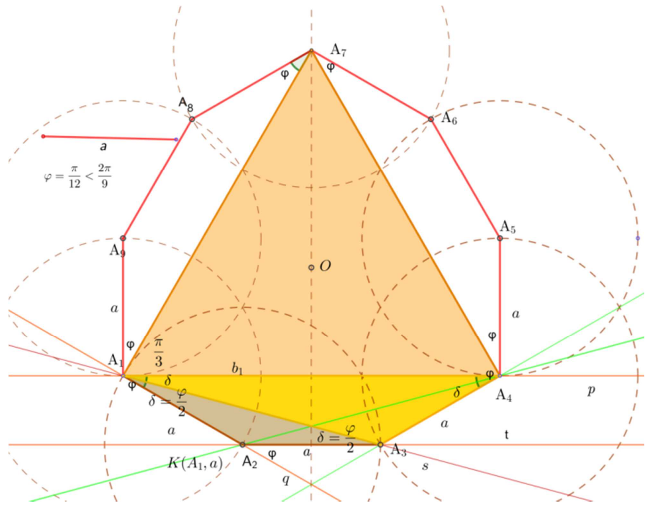

Figure 3. *Construction of a semi-regular nonagon with the given side* a *and angle* $\delta = \frac{\pi}{12}$.

1. Construct the given angle at vertex A_1, where A_1q is its ray, and half-line A_1p is its other ray.
2. Construct an auxiliary isosceles triangle $A_1A_2A_3$ on the grounds of the known elements: $|A_1A_2| = a, \angle A_1 = \varphi, \angle A_2 = \pi - \varphi$ using the elementary rule of ASA.
3. Construct vertex A_4 as follows;

$$K(A_3, a) \cap A_1p = \{A_4\}.$$

4. Construct an equilateral triangle of side $A_1A_4 = b_1$. Thus, we also get vertex A_7 because $\Delta\, A_1A_4A_7$ is the triangle that is "inscribed" to a semi-regular nonagon.
5. Vertex A_5 is the intersection of a bisector s_{A_7} of angle φ constructed at vertex A_7 and circle $K(A_4, a)$ i.e.

$$K(A_4, a) \cap s_{A_7} = \{A_5\}$$

6. Construct vertex A_6 as the intersection of a bisector s_{A_4} of angle φ that is constructed at vertex A_4 and circle $K(A_7, a)$ i.e.

$$\{A_6\} = K(A_7, a) \cap s_{A_4}$$

7. Construct vertices A_8 and A_9 similarly as vertices A_5 and A_6, with a note that the construction is to be compelted on side A_7A_1. By a similar procedure, construct vertices A_8, A_9. The construction of a semi-regular nonagon with the given length of side a and angle φ is shown in Figure 3.

2.3. Surface Area of a Semi-Regular Nonagon

The manner in which the surface area of a semi-regular nonagon is related to the given side a and angle φ is shown in the Theorem.

Theorem 4. *The surface area of equilateral semi-regular nonagon* $\mathcal{P}_9^{a,\varphi} \equiv A_1A_2 \dots A_9$ *with the given length of side* a *and angle* $\varphi = \angle(a, b_1)$ *between the sides of inscribed regular triangle* $\mathcal{P}_3^1 \equiv A_1A_4A_7$ *and side* a *of the semi-regular nonagon is calculated in the following formula*

$$\mathcal{A}(\mathcal{P}_9^{a,\varphi}) = \frac{3a^2}{4}\left[(1+2\cos\varphi)^2\cot\frac{\pi}{3} + \frac{3\sin\varphi - \sin 3\varphi}{2\sin^2\frac{\varphi}{2}}\right] \quad (7)$$

Proof: Let semi-regular equilateral nonagon $\mathcal{P}_9^{a,\varphi} = A_1A_2 \dots A_9$ be given (Figure 1). The following applies for surface area $\mathcal{A}(\mathcal{P}_9^{a,\varphi})$

$$\mathcal{A}(\mathcal{P}_9^{a,\varphi}) = \mathcal{A}(P_3^{b_1}) + 3 \cdot \mathcal{A}(P_4^{a}),$$

where $\mathcal{A}(P_3^{b_1})$ is the surface area of the inscribed equilateral triangle of side b_1, and $\mathcal{A}(P_4^a)$ is the surface area of the isosceles quadrilateral constructed above side b_1 of the inscribed triangle, the side of which is equal to the length of side a of semi-regular nonagon $\mathcal{P}_9^{a,\varphi}$.

Let $A_1A_2A_3A_4$ be an isosceles quadrilateral (Figure 4) constructed over side $A_1A_4 = b_1$.

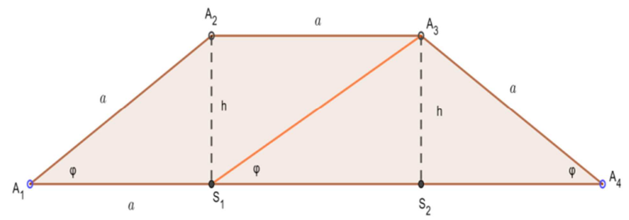

Figure 4. *Isosceles polygon constructed above side* b_1.

Note that triangle $S_1A_3A_4$ is an isosceles one. Therefore, we get height $h = a \cdot \sin\varphi$, and side $b = a(1 + 2\cos\varphi)$ from a special right triangle $S_1A_3S_2$. Since the surface area of parallelogram $A_1A_2A_3S_1$ is $a^2\sin\varphi$ and the surface area of isosceles triangle $S_1A_3A_4$ is $\frac{a^2}{2}\sin 2\varphi$, based on that, we find that the surface area of isosceles quadrilateral $A_1A_2A_3A_4$ constructed above side b_1 is equal to $\mathcal{A}\left(P_4^{a,\delta}\right) = a^2\sin\varphi(1 + \cos\varphi)$.

The surface area of equilateral triangle $\Delta\, A_1A_2A_3$ (Figure 1) is

$$\mathcal{A}(P_3^{b_1}) = \frac{\sqrt{3}}{4} \cdot a^2(1 + 2\cos\varphi)^2.$$

If we include the obtained values in the starting equality, we get

$$\mathcal{A}(P_9^{a,\delta}) = \mathcal{A}(P_3^{b}) + 3 \cdot \mathcal{A}(P_4^{a})$$

$$= \frac{\sqrt{3}}{4} \cdot a^2(1 + 2\cos\varphi)^2 + 3a^2 \cdot \sin\varphi(1 + \cos\varphi)$$

$$= \cdots$$

$$= \frac{3a^2}{4}\left[(1 + 2\cos\varphi)^2\cot\frac{\pi}{3} + 16\sin\frac{\varphi}{2}\cos^3\frac{\varphi}{2}\right]$$

$$= \cdots$$

$$= \frac{3a^2}{4}[(1+2\cos\varphi)^2\cot\frac{\pi}{3} + \frac{3\sin\varphi - \sin 3\varphi}{2\sin^2\frac{\varphi}{2}}].$$

The last equality is obtained if we notice that

$$16\sin\frac{\varphi}{2}\cos^3\frac{\varphi}{2} = \frac{32}{2}\frac{\sin^3\frac{\varphi}{2}\cos^3\frac{\varphi}{2}}{\sin^2\frac{\varphi}{2}} = \frac{4(2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2})^3}{2\sin^2\frac{\varphi}{2}}$$

$$= \frac{4\sin^3\varphi}{2\sin^2\frac{\varphi}{2}}$$

$$= \frac{3\sin\varphi - 3\sin\varphi + 4\sin^3\varphi}{2\sin^2\frac{\varphi}{2}} = \frac{3\sin\varphi - (3\sin\varphi - 4\sin^3\varphi)}{2\sin^2\frac{\varphi}{2}}$$

$$= \frac{3\sin\varphi - \sin 3\varphi}{2\sin^2\frac{\varphi}{2}}.\lozenge$$

Consequence 5. The surface area of a regular nonagon with the given side a is

$$\mathcal{A}\left(\mathcal{P}_9^{a,\frac{2\pi}{9}}\right) = 6.181a^2 \quad (8)$$

Proof. Since the following is valid for the regular nonagon: $= \frac{2\pi}{9}$. If in the formula for the surface area

$$\mathcal{A}(\mathcal{P}_9^{a,\varphi}) = \frac{3a^2}{4}[(1+2\cos\varphi)^2\cot\frac{\pi}{3} + \frac{3\sin\varphi - \sin 3\varphi}{2\sin^2\frac{\varphi}{2}}]$$

we include the value of angle $\varphi = \frac{2\pi}{9}$, after calculation, we get $\mathcal{A}\left(\mathcal{P}_9^{a,\frac{2\pi}{9}}\right) = 6.181a^2$, The resulting value is equal to the value obtained when using formula $P = \frac{9a^2}{4}cot\frac{\pi}{9} \approx 6.18182\, a^2$ to calculate the surface area of the regular nonagon of side a.

2.4. Circle Inscribed to a Regular Nonagon

A problem related to the construction of the inscribed circle to a convex equilateral semi-regular nonagon is considered in a Theorem.

Theorem 6. *A circle cannot be inscribed to semi-regular equilateral convex nonagon* $\mathcal{P}_9^{a,\varphi}$

Proof. Let us suppose, on the contrary, that a circle can be inscribed. Let there be given semi-regular equilateral nonagon $\mathcal{P}_9^{a,\varphi}$ (Figure 5) and let point O be the center of the inscribed circle, and let $A_1A_2A_3A_4$ be an isosceles quadrilateral constructed above side $b_1 = A_1A_4$ of the inscribed regular triangle$\triangle A_1A_4A_7$. The interior angles of a semi-regular nonagon at vertices A_1, A_4 are equal to angle $\alpha = \frac{\pi}{3} + 2\varphi$, and they are equal to angle $\beta = \pi - \varphi$ at vertices A_2 and A_3. Let us construct triangles: $\triangle OA_1A_2$,$\triangle OA_2A_3$,$\triangle OA_3A_4$.

Note that $\triangle OA_1A_2 \cong \triangle OA_3A_4$ because: $OA_1 = OA_4 = \frac{\sqrt{3}}{3}\cdot b$, $A_1A_2 = A_3A_4 = a$ and $\angle OA_1A_2 = \angle OA_4A_3 = \frac{\alpha}{2} = \frac{\pi}{6} + \varphi$. Based on the congruence of these triangles, it follows that: $OA_1 = OA_4$ and $OA_2 = OA_3$ and $\angle OA_2A_1 \cong \angle OA_3A_4$. As the angles at vertices A_2 and A_3 are equal, we conclude that $\triangle OA_2A_3$ is an isosceles triangle. Let L be the point of contact between side $A_1A_2 = a$ and the inscribed circles with radius r, then $OL = r$ and let K be the contact point of side $A_2A_3 = a$ and the inscribed circle, and let $OK = r_1$. If there is an inscribed circle, it is unique and its radii are equal i.e. $r = r_1$. Let us show that equality is not true and that $r \neq r_1$.

Note the special right angle triangle $\triangle OA_1L$. The following is true for its elements

$$OA_1 = \frac{\sqrt{3}}{3}\cdot a(1+2\cos\varphi), \angle OA_1L = \frac{\pi}{6} + \varphi, \angle A_1OL$$

$$= \frac{\pi}{3} - \varphi.$$

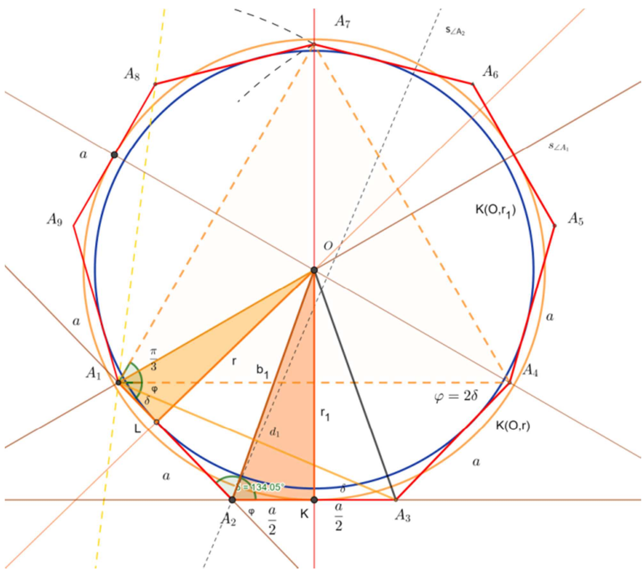

Figure 5. *Semi-regular nonagon and inscribed circle.*

Since $\cos(\frac{\pi}{3} - \varphi) = \frac{r}{OA_1}$ we find that $r = \frac{\sqrt{3}}{3}\cdot a(1 + 2\cos\varphi)\cdot\cos(\frac{\pi}{3} - \varphi)$.

Let us write $|A_1L| = t \neq \frac{a}{2}$, because triangle OA_1A_2 is not an isosceles one. Then $|LA_2| = a - t$. From special right angle triangle A_1OL it follows that $t = r\tan(\frac{\pi}{3} - \varphi)$.

Similarly, from special right angle triangle OLA_2 it follows that $|OA_2|^2 = r^2 + (a-t)^2$.

For isosceles triangle OA_2A_3 it follows that $OA_2 = OA_3$ and $\angle A_2 = \angle A_3$, and for special right angle triangle OA_2K it is $OK = r_1$ and $A_2K = \frac{a}{2}$ and the following is valid $|OA_2|^2 = r_1^2 + (\frac{a}{2})^2$

If we equate the last equalities, we get $r_1^2 + (\frac{a}{2})^2 = r^2 + (a-t)^2$.

As by assumption $r_1 = r$ (there is an inscribed circle), from $r_1^2 + (\frac{a}{2})^2 = r^2 + (a-t)^2$

we have the quadratic equality

$$a^2 - 2at + t^2 - \frac{a^2}{4} = 0$$

$$\Leftrightarrow 4t^2 - 8at + 3a^2 = 0.$$

The solution of the equality by t are $t = \pm\frac{a}{2}$.

The solution that satisfies the condition ($t > 0$) is $t = \frac{a}{2}$. If $t = \frac{a}{2}$ then triangle OA_1A_2 is an isosceles one, which is contrary to the assumption, that the angles along the base are different, i.e. $\angle OA_2A_1 \neq \angle OA_1A_2 = \frac{\pi}{6} + 2\delta$, (the equality applies only for $\delta = \frac{\pi}{9}$, in case of a regular nonagon). We obtained a contradiction.

Therefore, on the grounds of the above stated, we conclude that the radii are not equal and that $r \neq r_1$. In other words, there is no inscribed circle to a semi-regular equilateral nonagon (Figure 5).

3. Conclusion

Based on the contents presented in this paper, it follows that the basic metric properties, such as convexity, area of a semi-regular equilateral nonagons, radius of an inscribed circle and ratios of semi-regular equilateral nonagons can be expressed by side a and angle Δ.

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