Applied and Computational Mathematics
2018; 7(1-1): 1-11
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.s.2018070101.11
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

Convergence Analysis of Piecewise Polynomial Collocation Methods for System of Weakly Singular Volterra Integral Equations of The First Kind

Gholamreza Karamali[1], Babak Shiri[1, 2, *], Mahnaz Kashfi[2]

[1]Shahid Sattari Aeronautical University of Science and Technology, South Mehrabad, Tehran, Iran
[2]Department of Applied Mathematics, University of Tabriz, Bahman Boulevard, Tabriz, Iran

Email address:
g_karamali@iust.ac.ir (G. Karamali), shiri@tabrizu.ac.ir (B. Shiri), mahnaz.kash@gmail.com (M. Kashfi)
[*]Corresponding author

To cite this article:
Gholamreza Karamali, Babak Shiri, Mahnaz Kashfi. Convergence Analysis of Piecewise Polynomial Collocation Methods for System of Weakly Singular Volterra Integral Equations of The First Kind. *Applied and Computational Mathematics.* Special Issue: Singular Integral Equations and Fractional Differential Equations. Vol. 7, No. 1-1, 2018, pp. 1-11. doi: 10.11648/j.acm.s.2018070101.11

Received: March 21, 2017; **Accepted**: March 22, 2017; **Published**: April 11, 2017

Abstract: We study regularity of solutions of weakly singular Volterra integral equations of the first kind. We then study the numerical analysis of discontinuous piecewise polynomial collocation methods for solving such systems. The main purpose of this paper is the derivation of global convergent and super-convergent properties of introduced methods on the graded meshes. We apply relevant methods to a system of fractional differential equations and analyze them. The numerical experiments confirm the theoretical results.

Keywords: Discontinuous Piecewise Polynomial Spaces, Collocation Methods, Graded Meshes, Weakly Singular Volterra Integral Equations

1. Introduction

In this paper, we consider a system of weakly singular Volterra integral equations of first kind (SWSVIEFK) of the form

$$\int_0^t \frac{k(t,s)y(s)}{(t-s)^{\alpha}}ds = f(t), \qquad t \in \mathcal{I} := [0,T], \tag{1}$$

where, $0<\alpha<1$, $T \in \mathcal{R}$, $\nu \in \mathcal{N}$, $f:\mathcal{I} \to \mathcal{R}^{\nu}$. The domain of the matrix function $k(t,s):\mathcal{D} \to \mathcal{R}^{\nu\times\nu}$, is

$$\mathcal{D} = \{(t,s) \mid (t,s) \in \mathcal{I}^2, 0 \le s \le t \le T\}.$$

Also, we suppose that $k(t,t)$ is a nonsingular matrix for all $t \in \mathcal{I}$. The $y:\mathcal{R} \to \mathcal{R}^{\nu}$ is the unknown vector function. The system (1) is an Abel's integral equation if $k(t,s)=1$.

The numerical solution of weakly singular Volterra integral equations of first kind has extensively been studied (see for example [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16], but it does not mean that this subject has completely been studied. There are some unsolved problems which are important and need more challenge. One of them is convergence analysis of collocation methods on the piecewise polynomial spaces for solving system (1), [3]. The aim of this paper is to provide a complete convergence analysis of these methods for this system.

The piecewise polynomial collocation methods (PPCM) are easily programmable and they have rapid convergent order for many equations including integral or differential operator. They have extensively been examined by many authors. We refer here to [1, 3, 7, 12] and literature given therein. Therefore, it is important to analyze PPCMs for the system (1).

Suppose $\nu \in \mathcal{N}$. Let $q:\mathcal{R} \mapsto \mathcal{R}$ and $f:\Omega \mapsto \mathcal{R}^{\nu\times\nu}$ be scaler and matrix functions, respectively, where Ω is a set. In this paper, by qf we mean

$$(qf)_{ij} = qf_{ij}, \quad i,j \in \{1,\cdots,\nu\},$$

and the norm we mean the max norm

$$\| f \| = \max_{i,j\in\{1,\cdots,\nu\}} \sup_{t\in\Omega} | f_{i,j}(t) | .$$

The paper is organized as follow:

In section 2, we review existence, uniqueness and smoothness of the solutions of system (1). In section 3, we recall application of the collocation method on the continuous piecewise polynomial spaces. In section 4, we generalize Granwall's inequalities for matrix function equations. In section 5, we give the global convergence of the collocation method on the continuous piecewise polynomial spaces. In section 6, we investigate the stability function introduced in the previous section. Finally, in section 7, we present numerical experiments which support theoretical results.

2. Regularity Properties

The arguments of this section can be obtained by arguments similar to [3] (section 2.1.1). Here, we should concern that the systems we investigated are of dimension greater than 1 while the system in [3] is of dimension greater 1. Therefore, instead of dividing by a function we should multiply by an inverse of corresponding matrix function.

Suppose $G_\alpha \in \left(\mathcal{C}(\mathcal{I})\right)^\nu$ and

$$G_\alpha(z) = \frac{sin(\pi\alpha)}{\pi} k^{-1}(z,z) \frac{d}{dz}\int_0^z f(t)(z-t)^{\alpha-1}dt. \tag{2}$$

Define resolvent kernel associated with the given kernel $K_\alpha(z,s)$ as

$$R(z,s) := \lim_{n\to\infty}\sum_{i=1}^{n} H_n(z,s)$$

where

$$H_i(z,s) = \int_s^z H_1(z,v)H_{i-1}(v,s)ds, \, i\geq 2,$$

and

$$H_1(z,s)= -\frac{sin(\pi\alpha)}{\pi} k^{-1}(z,z)\frac{\partial}{\partial z}\int_0^1 \frac{k(s+\nu(z-s),s)}{\nu^\alpha(1-\nu)^{1-\alpha}}d\nu.$$

Then, we obtain corresponding Volterra's fundamental results that

$$y(z) = G_\alpha(z) + \int_0^z R(z,s)G_\alpha(s)ds. \tag{3}$$

is a unique solution of system (1). Now, we can argument about the regularity of solutions. Supposing $k\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$ and $\frac{\partial k}{\partial t}\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, we obtain $H_1(z,s)\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$. It is straightforward then to show that $R(z,s)\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$. Therefore, the regularity of y depends on the regularity of G_α.

Theorem 2.1 Let $k\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $\frac{\partial k}{\partial t}\in\left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $m\in\mathcal{N}$, $f(t)=t^\beta g(t)$ where $g\in\left(\mathcal{C}^d([0,T])\right)^\nu$, $1\leq d\leq m+1$, $d\in\mathcal{N}$, and $\beta\geq 1-\alpha$. Then, the system (1) has a unique solution and there exists $q\in\left(\mathcal{C}^{d-1}([0,T])\right)^\nu$ such that $y(z)=z^{\alpha+\beta-1}q(z)$.

Proof. Integrating by substitution $t=vz$, we have

$$\int_0^z t^\beta g(t)(z-t)^{\alpha-1}dt = z^{\alpha+\beta}q_0(z)$$

Where

$$q_0(z) =: \int_0^1 v^\beta g(zv)(1-v)^{\alpha-1}dv,$$

and hence we obtain

$$\frac{d}{dz}\int_0^z t^\beta g(t)(z-t)^{\alpha-1}dt = z^{\alpha+\beta-1}\left(zq_0^{(1)}(z) + (\alpha+\beta)q_0(z)\right).$$

Since $g\in\left(\mathcal{C}^d([0,T])\right)^\nu$, we conclude that

$$q_0\in\left(\mathcal{C}^d([0,T])\right)^\nu,$$

and consequently

$$zq_0^{(1)}(z)+(\alpha+\beta)q_0(z)\in\left(\mathcal{C}^{d-1}([0,T])\right)^\nu.$$

Thus, using equation (2), there exists

$$q_1\in\left(\mathcal{C}^{d-1}([0,T])\right)^\nu$$

such that $G_\alpha(z)=z^{\alpha+\beta-1}q_1(z)$. Taking into account that

$$\begin{aligned} q_2(z) &:= \frac{1}{z^{\alpha+\beta}}\int_0^z R(z,s)G_\alpha(s)ds \\ &= \int_0^z \frac{R(z,s)}{z^{\alpha+\beta}}s^{\alpha+\beta-1}q_1(s)ds \\ &= \int_0^1 R(z,vz)v^{\alpha+\beta-1}q_1(vz)dv \in \left(\mathcal{C}^{d-1}([0,T])\right)^\nu, \end{aligned} \tag{4}$$

and using the equation (4), we have

$$y(z) = z^{\alpha+\beta-1}(q_1(z)+zq_2(z)),$$

which completes the proof.

3. Collocation Method on the Continuous Piecewise Polynomial Spaces

For given $N\in\mathcal{N}$, Let

$$I_h = \{t_n = T(\frac{n}{N})^r : 0\leq n\leq N\},$$

be a graded mesh, with grading exponent $r \geq 1$. Assume that $\sigma_n = (t_n, t_{n+1}]$, $\bar{\sigma}_n = [t_n, t_{n+1}]$, $h_n = t_{n+1} - t_n$ and $h = \max\{h_j : j = 0, \cdots, N-1\}$. Then, it is straightforward to see that the sequence $\{h_n\}_{n=0}^{N}$ is strictly increasing and

$$h_j \leq h = h_{N-1} < rTN^{-1}, \quad j = 0, \cdots, N-1. \qquad (5)$$

We use collocation method to solve system (1) directly (without transforming to the second kind integral equation), on the discontinuous piecewise polynomial spaces

$$\mathcal{S}_{m-1}^{(-1)}(I_h) := \{v : v|_{\bar{\sigma}_n} \in \pi_{m-1} (n = 0, 1, ..., N-1)\}, \ m \in \mathcal{N}.$$

For discontinuous piecewise polynomial spaces, let $0 < c_1 < \cdots < c_m \leq 1$, $m \in \mathcal{N}$ be the collocation parameters. Therefore, the approximate solution $u_h \in \left(\mathcal{S}_{m-1}^{(-1)}(I_h)\right)^{\nu}$, has the form

$$u_l(t_l + vh_l) = \sum_{j=1}^{m} L_j(v) U_{l,j}, \ v \in (0,1], \ l = 1, \cdots, N-1,$$

in the interval σ_l ($u_l = u_h|_{\sigma_l}$, for $l = 1, \cdots, N-1$). Here, $L_j(v)$ for $j = 1, \cdots, m$, are Lagrange fundamental polynomials with respect to distinct collocation parameters and $U_{l,j} := u_h(t_{l,j})$ are approximation solutions at the collocation points $t_{l,j} := t_l + c_j h_l$ for $l = 1, \cdots, N-1$ and $j = 1, \cdots, m$. We are seeking for a collocation solution u_h such that satisfies the collocation conditions

$$\int_0^{t_{n,i}} \frac{k(t_{n,i}, s) u_h(s)}{(t_{n,i} - s)^{\alpha}} ds = f(t_{n,i}), \pm \qquad (6)$$

for $n = 0, \cdots, N$ and $i = 1, \cdots, m$. Therefore, it is straightforward to show that the solution of the system (6) can be obtained by solving recursively the systems

$$F(t_{n,i}) + h_n \sum_{j=1}^{m} \int_0^{c_i} \frac{k(t_{n,i}, t_n + vh_n) L_j(v)}{(t_{n,i} - t_n - vh_n)^{\alpha}} dv U_{n,j} = f(t_{n,i}) \qquad (7)$$

for $i = 1, \cdots, m$, and $n = 0, \cdots, N-1$, where

$$F(t_{n,i}) = \sum_{l=0}^{n-1} h_l \sum_{j=1}^{m} \int_0^{1} \frac{k(t_{n,i}, t_l + vh_l) L_j(v)}{(t_{n,i} - t_l - vh_l)^{\alpha}} dv U_{l,j}$$

is the lag term. In the system (7), the integrals can be approximated by the quadrature approximations

$$\int_0^{c_i} \frac{k(t_{n,i}, t_n + vh_n) L_j(v)}{(t_{n,i} - t_n - vh_n)^{\alpha}} dv \simeq \frac{1}{h_n^{\alpha}} a_{i,j}(\alpha) k(t_{n,i}, t_{n,j})$$

and

$$\int_0^{1} \frac{k(t_{n,i}, t_l + vh_l) L_j(v)}{(t_{n,i} - t_l - vh_l)^{\alpha}} dv \simeq \frac{1}{h_l^{\alpha}} b_{ij}(n, l, \alpha) k(t_{n,i}, t_{l,j}),$$

For $0 \leq l \leq n-1$, where

$$a_{i,j}(\alpha) = \int_0^{c_i} \frac{L_j(v)}{(c_i - v)^{\alpha}} dv$$

and

$$b_{ij}(n, l, \alpha) = \int_0^{1} \frac{L_j(v)}{\left(\frac{t_{n,i} - t_l}{h_l} - v\right)^{\alpha}} dv, \ 0 \leq l \leq n-1,$$

for $i, j \in \{1, \cdots, m\}$. Therefore, the fully discretised version can be obtained by solving the systems

$$\hat{F}(t_{n,i}) + h_n^{1-\alpha} \sum_{j=1}^{m} a_{i,j}(\alpha) k(t_{n,i}, t_{n,j}) \hat{U}_{n,j} = f(t_{n,i}), \qquad (8)$$

recursively, for $i = 1, \cdots, m$ and $n = 0, \cdots, N-1$, where

$$\hat{F}(t_{n,i}) = \sum_{l=0}^{n-1} h_l^{1-\alpha} \sum_{j=1}^{m} b_{ij}(n, l, \alpha) k(t_{n,i}, t_{l,j}) \hat{U}_{l,j},$$

for $i = 1, \cdots, m$, and $\hat{U}_{l,j}$ are the collocation solution at $t_{l,j}$ for $l = 0, \cdots, N-1$ and $j = 1, \cdots, m$. Now, the dense output approximate solution can be approximated by

$$\hat{u}_l(t_l + vh_l) = \sum_{j=1}^{m} L_j(v) \hat{U}_{l,j}, \ v \in (0,1], \ l = 1, \cdots, N-1.$$

We note that this discretised version is slightly different from discretised version introduced in [3]. However, it reduce the computation complexity by a factor of $\frac{1}{m}$, and as we will see it does not reduce the order of the collocation method. Setting $(A_3)_{ij} := a_{ij}(\alpha)$, $\hat{U}_n := [\hat{U}_{n,1}, \cdots, \hat{U}_{n,m}]^T$, $\mathbf{F} = [f(t_{n,1}) - \hat{F}(t_{n,1}), \cdots, f(t_{n,m}) - \hat{F}(t_{n,m})]^T$, and taking into account that

$$k(t_{n,i}, t_{n,j}) = k(t_n, t_n) + \mathcal{O}(h),$$

we can write (8) in the matrix form

$$h_n^{1-\alpha} A_3 \otimes (k(t_n, t_n) + \mathcal{O}(h_n)) \hat{U}_n = \mathbf{F}_n.$$

Now, since A_3 is invertible by Theorem 6.1, and $k(t_n, t_n)$ is invertible by hypotheses of the introduction, the matrix $A_3 \otimes (k(t_n, t_n) + \mathcal{O}(h_n))$ is invertible and there exists a unique solution for system (8), for sufficiently small h. Consequently, the fully discretised collocation method is well-defined. In a similar manner the system (7) has a unique solution for sufficiently small h, and the introduced collocation method is well-defined.

4. Granwall's Inequalities

First, we recall the Granwall's inequalities [3]. Note that, we write

$$v = O(h^m) \quad \text{whenever} \quad \| v \| = O(h^m).$$

Lemma 4.1. (Gronwall's inequality) Assume that $\{k_j\}$, $(j \geq 0)$ is a given non-negative sequence and the sequence $\{\epsilon_n\}$ satisfies $\epsilon_0 \leq \rho_0$ and

$$\epsilon_n \leq \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \epsilon_j$$

with $\rho_0 \geq 0$ and $q_j \geq 0$ for $(j \geq 0)$. Then

$$\epsilon_n \leq \left(\rho_0 + \sum_{j=0}^{n-1} q_j \right) \exp\left(\sum_{j=0}^{n-1} k_j \right).$$

In this paper, we need a generalization of Gronwall's inequality for the matrix functions. Thus, we consider

$$\epsilon_n = R\epsilon_{n-1} + \sum_{j=0}^{n-1} K_j \epsilon_j + \rho_0 \tag{9}$$

where R, ρ_0, K_j and ϵ_j for $j = 0, 1, \cdots,$ are matrix functions. We suppose that R is a diagonalizable matrix i.e., $R = P^{-1}DP$, where $D = diag(\lambda_1, \cdots, \lambda_r)$ is a diagonal matrix and P is a nonsingular matrix. Also, we suppose that $\lambda_i \in [-1, 1)$. Then, It is straightforward to show that there exists a positive constant C such that

$$\| \epsilon_n \| \leq (\| \epsilon_0 \| + C \| \rho_0 \|) \exp\left(C \sum_{j=0}^{n-1} \| K_j \| \right).$$

5. Global Convergence

5.1. Discontinuous Collocation Method

Lemma 5.1. For $r \geq 1$, we have

$$\frac{h_{n-1}}{h_n} = \frac{n^r - (n-1)^r}{(n+1)^r - (n)^r} = 1 - \mathcal{O}(\frac{1}{n}) = 1 + O(h_n).$$

Hence

$$\frac{h_{n-1}^{1-\alpha}}{h_n^{1-\alpha}} = 1 - O(h_n^{1-\alpha}),$$

and similarly

$$\mathcal{O}(1 - \frac{h_n}{h_{n-1}}) = O(h_n).$$

Proof. One can easily observe the results by expanding the polynomials $(n+1)^r$ and $(n-1)^r$ for case $r \in \mathcal{N}$. The other cases can be obtained by applying the Hopital's rule.

Theorem 5.2. Let $k \in \left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $\frac{\partial k}{\partial t} \in \left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $m \in \mathcal{N}$, $f(t) = t^\beta g(t)$ where $g \in \left(\mathcal{C}^d([0,T])\right)^\nu$, $1 \leq d \leq m+1$, $d \in \mathcal{N}$, and $\beta \geq 1-\alpha$. Then the approximate solution u_h of the discontinuous collocation method for system (1) with collocation parameters $0 < c_1 < c_2 < \cdots < c_m \leq 1$ and the grading exponent $r \geq 1$ converges to the solution if the eigenvalues of the stability matrix $\Re$ be in the interval $[-1, 1)$. Furthermore, the collocation error satisfies

$$\| y(t) - u_h(t) \| \leq c \begin{cases} N^{-r(\alpha+\beta-1)}, & 1 \leq r \leq \dfrac{d-1}{\alpha+\beta-1}, \\ N^{-d+1}, & r \geq \dfrac{d-1}{\alpha+\beta-1}, \end{cases}$$

for a constant $c > 0$ and sufficiently large N.

Proof. Suppose that the assumptions of Theorem \ref{th1} hold. Let $e_h(t) = y(t) - u_h(t)$. An application of Theorem 2.1 implies that $y \in \left(\mathcal{C}^{d-1}([\epsilon, T])\right)^\nu$ for all $\epsilon > 0$. Therefore, by Peano's theorem ([3], Section 1.8)

$$e_h(t_l + vh_l) = \sum_{j=1}^{m} L_j(v) E_{l,j} + h_l^{\lambda_l} R_l(v) \tag{10}$$

where $E_{l,j} = e_h(t_l + c_j h_l)$ for $j = 1, \cdots, m$, $l = 0, \cdots, N$, the remainder $R_l(v)$ is a bounded function and

$$\lambda_l = \begin{cases} \beta + \alpha - 1, & l = 0, \\ d - 1, & \text{Otherwise.} \end{cases} \tag{11}$$

By subtracting equation (1) (at $t = t_{n,i}$) from equation (6) we obtain

$$\int_0^{t_{n,i}} \frac{k(t_{n,i}, s) e_h(s)}{(t_{n,i} - s)^\alpha} ds = 0, \tag{12}$$

for $n = 1 \cdots, N-1$, $i = 1, \cdots, m$, and hence,

$$\sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \frac{k(t_{n,i}, s) e_h(s)}{(t_{n,i} - s)^\alpha} ds + \int_{t_n}^{t_{n,i}} \frac{k(t_{n,i}, s) e_h(s)}{(t_{n,i} - s)^\alpha} ds = 0. \tag{13}$$

Letting $n = 0$, substituting $s = vh_0$, and using (10) we have

$$\sum_{j=1}^{m} \int_0^{c_i} \frac{k(c_i h_0, vh_0)}{(c_i - v)^\alpha} L_j(v) dv E_{0,j} = -h_0^{\lambda_0} \int_0^{c_i} R_l(v) dv, \tag{14}$$

for $i = 1, \cdots, m$. By Taylor's theorem for multivariable matrix functions, $k(c_i h_0, vh_0) = k(0,0) + \mathcal{O}(h_0)$ and the system (14) can be written in the matrix form

$$(A_3 \otimes k(0,0) + \mathcal{O}(h_0))\mathcal{E}_0 = \mathcal{O}(h_0^{\lambda_0})$$

where $\mathcal{E}_0 = [E_{0,1}, \ldots, E_{0,m}]^T$ and

$$(A_3)_{ij} = \int_0^{c_i} \frac{L_j(v)}{(c_i - v)^\alpha} dv, \quad i, j \in \{1, \cdots, m\}. \tag{15}$$

Since, A_3 and $k(0,0)$ are invertible matrices, we can conclude that $\mathcal{E}_{0,j} = \mathcal{O}(h_0^{\lambda_0})$, for sufficiently small h. We can now proceed to obtain an estimate for $e_n = e_h|_{\bar{\sigma}_n}$, $n \in \mathcal{N}$. Substituting $s = t_l + vh_l$, $l = 0, \cdots, n$, into corresponding integrals in (13) and using (10) we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-2} h_l \sum_{j=1}^{m} \int_0^1 \left(\frac{k(t_{n,i}, t_l + sh_l)}{(t_{n,i} - t_l - sh_l)^\alpha} - \frac{k(t_{n-1,m}, t_l + sh_l)}{(t_{n-1,m} - t_l - sh_l)^\alpha} \right) L_j(s) ds E_{l,j} \\
&+ h_{n-1} \sum_{j=1}^{m} \int_0^1 \frac{k(t_{n,i}, t_{n-1} + sh_{n-1}) L_j(s)}{(t_{n,i} - t_{n-1} - sh_{n-1})^\alpha} ds E_{n-1,j} \\
&+ h_n \sum_{j=1}^{m} \int_0^{c_i} \frac{k(t_{n,i}, t_n + sh_n) L_j(s)}{(t_{n,i} - t_n - sh_n)^\alpha} ds E_{n,j} \\
&- h_{n-1} \sum_{j=1}^{m} \int_0^{c_m} \frac{k(t_{n-1,m}, t_{n-1} + sh_{n-1}) L_j(s)}{(t_{n-1,m} - t_{n-1} - sh_{n-1})^\alpha} ds E_{n-1,j} \\
&= \hat{R}_n
\end{aligned} \tag{16}$$

Rewriting (16) with replaced n by $n-1$ and $j = m$ and subtracting it from (16), we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-2} h_l \sum_{j=1}^{m} \int_0^1 \left(\frac{k(t_{n,i}, t_l + sh_l)}{(t_{n,i} - t_l - sh_l)^\alpha} - \frac{k(t_{n-1,m}, t_l + sh_l)}{(t_{n-1,m} - t_l - sh_l)^\alpha} \right) L_j(s) ds E_{l,j} \\
&+ h_{n-1} \sum_{j=1}^{m} \int_0^1 \frac{k(t_{n,i}, t_{n-1} + sh_{n-1}) L_j(s)}{(t_{n,i} - t_{n-1} - sh_{n-1})^\alpha} ds E_{n-1,j} \\
&+ h_n \sum_{j=1}^{m} \int_0^{c_i} \frac{k(t_{n,i}, t_n + sh_n) L_j(s)}{(t_{n,i} - t_n - sh_n)^\alpha} ds E_{n,j} \\
&- h_{n-1} \sum_{j=1}^{m} \int_0^{c_m} \frac{k(t_{n-1,m}, t_{n-1} + sh_{n-1}) L_j(s)}{(t_{n-1,m} - t_{n-1} - sh_{n-1})^\alpha} ds E_{n-1,j} \\
&= \hat{R}_n
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
-\hat{R}_n &= \\
&\sum_{l=0}^{n-2} h_l \int_0^1 \left(\frac{k(t_{n,i}, t_l + sh_l)}{(t_{n,i} - t_l - sh_l)^\alpha} - \frac{k(t_{n-1,m}, t_l + sh_l)}{(t_{n-1,m} - t_l - sh_l)^\alpha} \right) h_l^{\lambda_l} R_l(s) ds \\
&+ h_{n-1} \int_0^1 \frac{k(t_{n,i}, t_{n-1} + sh_{n-1}) h_{n-1}^{\lambda_{n-1}} R_{n-1}(s)}{h_{n-1}^\alpha (c_i \frac{h_n}{h_{n-1}} + 1 - s)^\alpha} ds \\
&+ h_n \int_0^{c_i} \frac{k(t_{n,i}, t_n + sh_n) h_n^{\lambda_n} R_n(s)}{h_n^\alpha (c_i - s)^\alpha} ds \\
&- h_{n-1} \int_0^{c_m} \frac{k(t_{n-1,m}, t_{n-1} + sh_{n-1}) h_{n-1}^{\lambda_{n-1}} R_{n-1}(s)}{h_{n-1}^\alpha (c_m - s)^\alpha} ds \\
&= \mathcal{O}\left(\frac{h_n}{h_{n-1}^\alpha} (h_0^{\lambda_0} + h_n^{d-1}) \right),
\end{aligned} \tag{18}$$

by Lemma 5.1 and following Remark.

Remark 5.3. We note that,

$(x, s) \mapsto \dfrac{k(x, t_l + sh_l)}{(x - t_l - sh_l)^\alpha} \in \left(C^m([t_{n-1,m}, t_{n,m}] \times [0,1]) \right)^{\nu \times \nu}$, for $l = 0, \cdots, n-2$. Therefore, if we apply mean value theorem for each components of the above matrix function, then we have

$$\begin{aligned}
&\frac{k(t_{n,i}, t_l + sh_l)}{(t_{n,i} - t_l - sh_l)^\alpha} - \frac{k(t_{n-1,m}, t_l + sh_l)}{(t_{n-1,m} - t_l - sh_l)^\alpha} \\
&\qquad = h_n \bar{k}_i(t_l + sh_l).
\end{aligned} \tag{19}$$

where there exist $t_{n-1,m} \le {}_{pq}\theta_{i,l} \le t_{n,m}$ such that

$$\begin{aligned}
&(\bar{k}_i(t_l + sh_l))_{pq} \\
&\quad = (c_i + (1 - c_m)\frac{h_{n-1}}{h_n}) \left(\frac{\frac{\partial k_{pq}}{\partial t}({}_{pq}\theta_{i,l}, t_l + sh_l)}{({}_{pq}\theta_{i,l} - t_l - sh_l)^\alpha} \right)
\end{aligned}$$

for $p, q \in \{1, \cdots, \nu\}$, $i = 1, \cdots, m$ and $l = 0, \cdots, n-2$.

By our assumptions $\dfrac{\partial k}{\partial t}$ is bounded and there exists $M > 0$ such that $\| (c_i + (1 - c_m)\dfrac{h_{n-1}}{h_n}) \dfrac{\partial k}{\partial t} \| < M$. Therefore,

$$\begin{aligned}
&\int_0^1 |(\bar{k}_i(t_l + sh_l))_{pq} L_j(s)| ds \\
&\qquad \le \int_0^1 \frac{M |L_j(s)|}{({}_{pq}\theta_{i,l} - t_l - sh_l)^\alpha} ds \\
&\qquad \le \int_0^1 \frac{M |L_j(s)|}{(t_{n-1,m} - t_l - sh_l)^\alpha} ds \\
&\qquad \le \frac{M 2^\alpha}{(1 - \alpha) h_l^\alpha} (n - 1 - l)^{-\alpha},
\end{aligned} \tag{20}$$

for $l = 0, \cdots, n-2$. The last inequality is obtained by Lemma (6.2.10) [3]. Also, setting

$$\gamma(\alpha) = \frac{M 2^\alpha}{(1 - \alpha)},$$

we conclude that

$$\| \int_0^1 (\overline{k}_i(t_l + sh_l))L_j(s)ds \| \le \gamma(\alpha)h_l^{-\alpha}(n-1-l)^{-\alpha}. \tag{21}$$

Furthermore, to obtain equation (18), we use the fact that the sum $\sum_{l=0}^{n-2} h_l^{1-\alpha}(n-1-l)^{-\alpha}$ is bounded.

Substituting (19) into (17), and using Taylor's series for other terms and components we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-2} h_l h_n \sum_{j=1}^{m} \int_0^1 \overline{k}_i(t_l + sh_l)L_j(s)dsE_{l,j} \\
&+h_{n-1}\sum_{j=1}^{m}\int_0^1 \frac{(k(t_n,t_n)+c_1(s)h_n + c_2(s)h_{n-1})L_j(s)}{h_{n-1}^{\alpha}(c_i \frac{h_n}{h_{n-1}} + 1 - s)^{\alpha}} dsE_{n-1,j} \\
&+h_n \sum_{j=1}^{m}\int_0^{c_i} \frac{(k(t_n,t_n)+c_3(s)h_n)L_j(s)}{h_n^{\alpha}(c_i - s)^{\alpha}} dsE_{n,j} \\
&-h_{n-1}\sum_{j=1}^{m}\int_0^{c_m} \frac{(k(t_n,t_n)+c_4(s)h_{n-1})L_j(s)}{h_{n-1}^{\alpha}(c_m - s)^{\alpha}} dsE_{n-1,j} \\
&= \hat{R}_n
\end{aligned} \tag{22}$$

where $c_i(s)$ for $i = 1,\cdots,4$ are bounded matrix functions. Multiplying (21) by $\mathbf{k}^{-1} = k(t_n, t_n)$ and dividing by h_n, we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-2} h_l \sum_{j=1}^{m} \int_0^1 \overline{k}_i(t_l + sh_l)L_j(s)dsE_{l,j} \\
&+\frac{h_{n-1}}{h_n h_{n-1}^{\alpha}}\sum_{j=1}^{m}\int_0^1 \frac{(1+\mathbf{k}^{-1}c_1(s)h_n + \mathbf{k}^{-1}c_2(s)h_{n-1})L_j(s)}{(c_i \frac{h_n}{h_{n-1}} + 1 - s)^{\alpha}} dsE_{n-1,j} \\
&+\frac{1}{h_n^{\alpha}}\sum_{j=1}^{m}\int_0^{c_i} \frac{(1+\mathbf{k}^{-1}c_3(s)h_n)L_j(s)}{(c_i - s)^{\alpha}} dsE_{n,j} \\
&-\frac{h_{n-1}}{h_n h_{n-1}^{\alpha}}\sum_{j=1}^{m}\int_0^{c_m} \frac{(1+\mathbf{k}^{-1}c_4(s)h_{n-1})L_j(s)}{(c_m - s)^{\alpha}} dsE_{n-1,j} \\
&= \frac{\hat{R}_n}{h_n}.
\end{aligned} \tag{23}$$

Denoting,

$$(\mathbf{B}_n^{(l)})_{i,j} := \sum_{j=1}^{m}\int_0^1 \overline{k}_i(t_l + sh_l)L_j(s)ds, \; l = 0,\cdots,n-2,$$

$$(\mathbf{A}_1)_{i,j} := \int_0^1 \frac{L_j(s)}{(c_i + 1 - s)^{\alpha}}ds,$$

$$(\mathbf{A}_2)_{i,j} := \int_0^{c_m} \frac{L_j(s)}{(c_m - s)^{\alpha}}ds,$$

$$(\mathbf{A}_3)_{i,j} := \int_0^{c_i} \frac{L_j(s)}{(c_i - s)^{\alpha}}ds,$$

$$(\mathbf{K}_n^{(1)})_{i,j} := \int_0^1 \frac{(\mathbf{k}^{-1}c_1(s)\frac{h_n}{h_{n-1}} + \mathbf{k}^{-1}c_2(s))L_j(s)}{(c_i \frac{h_n}{h_{n-1}} + 1 - s)^{\alpha}}ds$$

$$-\int_0^{c_m} \frac{\mathbf{k}^{-1}c_4(s)L_j(s)}{(c_m - s)^{\alpha}}ds,$$

$$(\mathbf{K}_n^{(2)})_{i,j} := \int_0^{c_i} \frac{\mathbf{k}^{-1}c_3(s)L_j(s)}{(c_i - s)^{\alpha}}ds,$$

and we can rewrite system (22) in the matrix form

$$\begin{aligned}
&\sum_{l=0}^{n-2} h_l \mathbf{B}_n^{(l)}\mathcal{E}_l \\
&+\frac{h_{n-1}}{h_n h_{n-1}^{\alpha}}\left(\mathbf{A}_1 + \mathcal{O}(h_n) - \mathbf{A}_2 + h_{n-1}\mathbf{K}_n^{(1)}\right) \otimes \mathbf{I}_\nu \mathcal{E}_{n-1} \\
&+\frac{1}{h_n^{\alpha}}(\mathbf{A}_3 + h_n\mathbf{K}_n^{(2)}) \otimes \mathbf{I}_\nu \mathcal{E}_n = \frac{\hat{R}_n}{h_n},
\end{aligned} \tag{24}$$

where $\mathbf{I}_\nu$ is identity matrix of dimension ν. Multiplying equation (23) by

$$\left((\mathbf{A}_3 + h_n\mathbf{K}_n^{(2)}) \otimes \mathbf{I}_\nu\right)^{-1} = (\mathbf{A}_3 + h_n\mathbf{K}_n^{(2)})^{-1} \otimes \mathbf{I}_\nu$$

(which exists for sufficiently small *h*, by Theorem 6.1), we obtain

$$\begin{aligned}
\mathcal{E}_n &= \frac{h_{n-1}^{1-\alpha}}{h_n^{1-\alpha}}(\Re + \mathcal{O}(h_n))\mathcal{E}_{n-1} \\
&+\sum_{l=0}^{n-2} h_l h_n^{\alpha}\mathbf{C}_n^{(l)}\mathcal{E}_l + \mathcal{O}(h_0^{\lambda_0} + h_n^{d-1}).
\end{aligned} \tag{25}$$

where $\mathbf{C}_n^{(l)} = -(\mathbf{A}_3 + h_n\mathbf{K}_n^{(2)})^{-1} \otimes \mathbf{I}_\nu\mathbf{B}_n^{(l)}$ and

$$\Re = (\mathbf{A}_3)^{-1}\left(\mathbf{A}_2 - \mathbf{A}_1\right) \otimes \mathbf{I}_\nu.$$

Using Lemma 5.1 we can conclude that

$$\begin{aligned}
\mathcal{E}_n &= (\Re + \mathcal{O}(h_n^{1-\alpha}))\mathcal{E}_{n-1} \\
&+ h_n^{\alpha}\sum_{l=0}^{n-2} h_l\mathbf{C}_n^{(l)}\mathcal{E}_l + \mathcal{O}(h_0^{\lambda_0} + h_n^{d-1})
\end{aligned}$$

and by (5) we have

$$\begin{aligned}
\mathcal{E}_n &= \Re\mathcal{E}_{n-1} + \mathcal{O}(h^{1-\alpha})\mathcal{E}_{n-1} \\
&+\sum_{l=0}^{n-2} h_n^{\alpha} h_l\mathbf{C}_n^{(l)}\mathcal{E}_l + \mathcal{O}(h_0^{\lambda_0} + h^{d-1}).
\end{aligned} \tag{26}$$

If eigenvalues of the stability matrix $\Re$ be in the interval $[-1,1)$, we can invoke generalized Granwall's inequality to conclude that

$$\begin{aligned}
\| \mathcal{E}_n \| &= \left(\| \mathcal{E}_0 \| + C\mathcal{O}(h_0^{\lambda_0} + h^{d-1})\right) \\
&\times \exp\left(C\left(\sum_{l=0}^{n-2} h_n^{\alpha}h_l \| \mathbf{C}_n^{(l)} \| + \mathcal{O}(h^{1-\alpha})\right)\right)
\end{aligned} \tag{27}$$

where $C > 0$ is a constant.

Remark 5.4. By continuity and non-singularity assumptions, for sufficiently small h_n (large N), we can find

$M > 0$ such that

$$\begin{aligned}
&\| (\mathbf{A}_3 + h_n \mathbf{K}_n^{(2)})^{-1} \otimes \mathbf{I}_\nu \| \\
&\le \frac{M}{\gamma(\alpha)} \| (\mathbf{A}_3 + h_n \mathbf{K}_n^{(2)})^{-1} \otimes \mathbf{I}_\nu \| \\
&\le \frac{M}{\gamma(\alpha)}
\end{aligned}$$

by (21), $\| (\mathbf{B}_n^{(l)})_{i,j} \| \le \gamma(\alpha) h_l^{-\alpha} (n-1-l)$ and therefore

$$\| (\mathbf{C}_n^{(l)})_{i,j} \| \le M(n-1-l)^{-\alpha} h_l^{-\alpha}. \tag{28}$$

Consequently, the equation (28) yields

$$\begin{aligned}
\sum_{l=0}^{n-2} h_n^\alpha h_l \| \mathbf{C}_n^{(l)} \| &\le hM \sum_{l=0}^{n-2} (n-1-l)^{-\alpha} \\
&\le hM \sum_{l=0}^{n-2} \int_l^{l+1} (n-1-s)^{-\alpha} ds \\
&= hM \int_0^{n-1} (n-1-s)^{-\alpha} ds \\
&= \frac{(n-1)^{-\alpha+1} hM}{1-\alpha} \\
&\le \frac{(N-1)^{-\alpha+1} rTM}{N(1-\alpha)} \\
&\le \frac{rTM}{(1-\alpha)}.
\end{aligned} \tag{29}$$

Taking into account the equations (27) and (29), we obtain the main results of this section:

$$\begin{aligned}
\| \mathcal{E}_n \| \le \mathcal{O}\left(T\left(\frac{1}{N}\right)^{r\lambda_0} + (rT)^{d-1} \left(\frac{1}{N}\right)^{d-1} \right) \\
\times \exp\left(C\left(\frac{MrT}{(1-\alpha)} + \mathcal{O}(h^{1-\alpha}) \right) \right).
\end{aligned} \tag{30}$$

5.2. Discretised Discontinuous Collocation Method

Now, we can state an error bound for solutions of discretised discontinuous collocation method.

Theorem 5.5. Let $k \in \left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $\dfrac{\partial k}{\partial t} \in \left(\mathcal{C}^m(\mathcal{D})\right)^{\nu\times\nu}$, $m \in \mathcal{N}$, $f(t) = t^\beta g(t)$ where $g \in \left(\mathcal{C}^d([0,T])\right)^\nu$, $1 \le d \le m+1$, $d \in \mathcal{N}$, and $\beta \ge 1-\alpha$. Then the approximate solution u_h of the discretised discontinuous collocation method for system (1) with collocation parameters $0 < c_1 < c_2 < \cdots < c_m \le 1$ and the grading exponent $r \ge 1$ converges to the solution if the eigenvalues of the stability matrix $\Re$ be in the interval $[-1,1)$. Furthermore, the collocation error satisfies

$$\| y(t) - u_h(t) \| \le c \begin{cases} N^{-r(\alpha+\beta-1)}, & 1 \le r \le \dfrac{d-1}{\alpha+\beta-1}, \\ N^{-d+1}, & r \ge \dfrac{d-1}{\alpha+\beta-1}, \end{cases}$$

for a constant $c > 0$, and sufficiently large N.

Proof. By partitioning the domain of integral in (1) and substituting $s = t_l + vh_l$ we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-1} h_l \int_0^1 \frac{k(t, t_l + vh_l) y(t_l + vh_l)}{(t - t_l + sh_l)^\alpha} dv \\
&+ h_n \int_0^{\frac{t-t_n}{h_n}} \frac{k(t, t_n + vh_n) y(t_n + vh_n)}{(t - t_n + sh_n)^\alpha} dv = f(t).
\end{aligned} \tag{32}$$

for $n = 0, \cdots, N-1$ and $i = 1, \cdots, m.$ Applying Theorem 2.1 and Peano's theorem, we obtain

$$\begin{aligned}
&k(t, t_l + vh_l) y(t_l + vh_l) \\
&= \sum_{j=1}^m k(t, t_{l,j}) y(t_{l,j}) L_j(v) + h_l^{\lambda_l} R_l(t, v),
\end{aligned}$$

where λ_l is defined by (\ref{lambda}) and $R_l(t,v)$ for $l = 0, \cdots, N-1$ are uniformly bounded functions on $\mathcal{D}$. Setting $t = t_{n,i}$, and using (11), we obtain

$$\begin{aligned}
&\sum_{l=0}^{n-1} h_l \sum_{j=1}^m \int_0^1 \frac{L_j(v)}{(t_{n,i} - t_l + sh_l)^\alpha} dv k(t_{n,i}, t_{l,j}) y(t_{l,j}) \\
&+ h_n \int_0^{c_i} \frac{L_j(v)}{(t_{n,i} - t_n + sh_n)^\alpha} dv k(t_{n,i}, t_{n,j}) y(t_{n,j}) + \tilde{R}_{n,i} \\
&= f(t_{n,i}),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\tilde{R}_{n,i} = &\sum_{l=0}^{n-1} h_l \sum_{j=1}^m \int_0^1 \frac{h_l^{\lambda_l} R_l(t_{n,i}, v)}{(t_{n,i} - t_l + sh_l)^\alpha} dv \\
&+ h_n \int_0^{c_i} \frac{h_l^{\lambda_l} R_n(t_{n,i}, v)}{(t_{n,i} - t_n + sh_n)^\alpha} dv.
\end{aligned} \tag{34}$$

for $n = 0, \cdots, N-1$ and $i = 1, \cdots, m.$ Now, we can use the notation of previous sections to write the equation (33) in the form

$$\begin{aligned}
&\sum_{l=0}^{n-1} h_l \sum_{j=1}^m \int_0^1 b_{ij}(n, l, \alpha) k(t_{n,i}, t_{l,j}) y(t_{l,j}) \\
&+ h_n a_{ij}(\alpha) k(t_{n,i}, t_{n,j}) y(t_{n,j}) + \tilde{R}_{n,i} = f(t_{n,i}).
\end{aligned} \tag{35}$$

By subtracting equation (32) from equation (8) we have

$$\begin{aligned}
&\sum_{l=0}^{n-1} h_l \sum_{j=1}^m b_{ij}(n, l, \alpha) k(t_{n,i}, t_{l,j}) \hat{E}_{l,j} \\
&+ h_n a_{ij}(\alpha) k(t_{n,i}, t_{n,j}) \hat{E}_{n,j} + \tilde{R}_{n,i} = 0,
\end{aligned} \tag{36}$$

where $\hat{E}_{l,j} = y(t_{l,j} - \hat{U}_{l,j})$ for $l = 0, \cdots, N-1$ and $j = 1, \cdots, m.$ Rewriting (36) with n replaced by $n-1$ and $j = m$ and subtracting it from (36), we obtain

$$\sum_{l=0}^{n-2} h_l \sum_{j=1}^{m} \big(b_{ij}(n,l,\alpha)k(t_{n,i},t_{l,j}) - b_{mj}(n-1,l,\alpha)k(t_{n-1,m},t_{l,j})\big)\hat{E}_{l,j}$$
$$+h_{n-1}\sum_{j=1}^{m} b_{ij}(n,n-1,\alpha)k(t_{n,i},t_{n-1,j})\hat{E}_{n-1,j}$$
$$-h_{n-1}a_{mj}(\alpha)k(t_{n-1,m},t_{n-1,j})\hat{E}_{n-1,j} \tag{37}$$
$$+h_n a_{ij}(\alpha)k(t_{n,i},t_{n,j})\hat{E}_{n,j} + \tilde{R}_{n,i} - \tilde{R}_{n-1,m} = 0,$$

for $n = 0,\cdots,N-1$ and $i = 1,\cdots,m$, where it is straightforward to show that

$$\tilde{R}_{n,i} - \tilde{R}_{n-1,m} = \mathcal{O}\left(\frac{h_n}{h_{n-1}^{\alpha}}(h_0^{\lambda_0} + h_n^{d-1})\right).$$

Also, by Remark 5.3, we have

$$b_{ij}(n,l,\alpha)k(t_{n,i},t_{l,j}) - b_{mj}(n-1,l,\alpha)k(t_{n-1,m},t_{l,j}) = h_n\overline{k}_i(t_{l,j}).$$

Using mean value theorem for other terms and components of equation (37), there exist bounded matrix functions $\tilde{c}_i(s)$, for $i = 1,\cdots,4$ such that

$$\sum_{l=0}^{n-2} h_l h_n \sum_{j=1}^{m}\int_0^1 \overline{k}_i(t_{l,j})L_j(s)dsE_{l,j}$$
$$+h_{n-1}\sum_{j=1}^{m}\int_0^1 \frac{(k(t_n,t_n)+\tilde{c}_1(s)h_n+\tilde{c}_2(s)h_{n-1})L_j(s)}{h_{n-1}^{\alpha}(c_i\frac{h_n}{h_{n-1}}+1-s)^{\alpha}}dsE_{n-1,j}$$
$$+h_n\sum_{j=1}^{m}\int_0^{c_i}\frac{(k(t_n,t_n)+\tilde{c}_3(s)h_n)L_j(s)}{h_n^{\alpha}(c_i-s)^{\alpha}}dsE_{n,j}$$
$$-h_{n-1}\sum_{j=1}^{m}\int_0^{c_m}\frac{(k(t_n,t_n)+\tilde{c}_4(s)h_{n-1})L_j(s)}{h_{n-1}^{\alpha}(c_m-s)^{\alpha}}dsE_{n-1,j} \tag{38}$$
$$+\tilde{R}_{n,i} - \tilde{R}_{n-1,m} = 0,\ n = 0,\cdots,N-1,\ i = 1,\cdots,m.$$

The equation (38) is similar to (22) and hence the rest of the proof is similar to the lines after this equation.

6. Stability Matrix

We can use the well-known formula of interpolation

$$\sum_{j=1}^{m} c_j^i L_j(s) = s^i,\quad i = 0,1,\cdots,m-1,$$

to obtain

$$\sum_{j=1}^{m} c_j^i \int_0^b \frac{L_j(s)}{(d-s)^{\alpha}}ds = \int_0^b \frac{s^i}{(d-s)^{\alpha}}ds. \tag{39}$$

Therefore, defining following matrices

$$C := \begin{pmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_m & \dots & c_m^{m-1} \end{pmatrix}, \tag{40}$$

$$(D_1)_{i,j} := \int_0^1 \frac{s^{j-1}}{(c_i+1-s)^{\alpha}}ds,$$

$$(D_2)_{i,j} := \int_0^{c_m} \frac{s^{j-1}}{(c_m-s)^{\alpha}}ds,$$

and $(D_3)_{i,j} = \int_0^{c_i}\frac{s^{j-1}}{(c_i-s)^{\alpha}}ds$ for $i,j \in \{1,\cdots,m\}$, we can show that

$$\mathbf{A}_1 C = D_1,\ \mathbf{A}_2 C = D_2,\ \mathbf{A}_3 C = D_3. \tag{41}$$

The matrix C is a Vandermonde matrix, and is invertible. Thus, $\mathbf{A}_3$ is invertible if and only if D_3 be invertible.

The elements of D_1, D_2 and D_3 can be computed by the following formula

$$\int_0^b \frac{s^i}{(d-s)^{\alpha}}ds = \sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{i-j}}{i-j-\alpha+1}d^j(d^{i-j-\alpha+1} - (d-b)^{i-j-\alpha+1})$$

and for D_3 we have

$$D_3 = \begin{pmatrix} \frac{c_1^{-\alpha+1}}{-\alpha+1} & \dots & c_1^{m-\alpha}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{m-j-\alpha} \\ \vdots & \ddots & \vdots \\ \frac{c_m^{-\alpha+1}}{-\alpha+1} & \dots & c_m^{m-\alpha}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{m-1-j}}{m-j-\alpha} \end{pmatrix}.$$

Theorem 6.1. The matrices $\mathbf{A}_3$ and D_3 are invertible matrices.

Proof. By previous argument, it remains to prove that D_3 is invertible. It is not enough to show that the columns of D_3 are independents. Let $D_3(j)$ be the j-th column of the matrix D_3 and let $\sum_{j=1}^{m} d_j D_3(j) = 0$, where $d_j \in \mathcal{R}$. Therefore,

$$x^{1-\alpha}\sum_{j=1}^{m} d_j \sum_{k=0}^{j-1}\binom{j-1}{k}\frac{(-1)^{j-1-k}}{j-k-\alpha}x^{j-1} = 0$$

for $x = c_1,\cdots,c_m$. Hence, the polynomial

$$p(x) = \sum_{j=1}^{m} d_j \sum_{k=0}^{j-1}\binom{j-1}{k}\frac{(-1)^{j-1-k}}{j-k-\alpha}x^{j-1}$$

is of degree $m-1$, and has m roots, ($p(c_j) = 0$ for $j = 1,\cdots,m$) and is zero polynomial by fundamental theorem of algebra. Hence, $d_j = 0$, for $j = 1,\cdots,m$, which completes the proof.

Based on the Theorem 5.2, the convergence of the introduced collocation method depends on the eigenvalues of the stability matrix

$$\mathcal{R} = (\mathbf{A}_3)^{-1}(\mathbf{A}_2 - \mathbf{A}_1) \otimes \mathbf{I}_\nu .$$

Since, $\mathbf{A}_3$ is invertible, the stability matrix $\mathcal{R}$ is well-defined. Also, the matrices $\mathfrak{R}$ and the matrix $(\mathbf{A}_3)^{-1}(\mathbf{A}_2 - \mathbf{A}_1)$ have the same eigenvalues, and without loss of generality, we can redefine the stability matrix as $\mathfrak{R} := (\mathbf{A}_3)^{-1}(\mathbf{A}_2 - \mathbf{A}_1)$. Moreover, we can use equation (41) to obtain $C^{-1}\mathfrak{R}C = (D_3)^{-1}(D_2 - D_1)$.

Therefore, $\mathfrak{R}$ and $(D_3)^{-1}(D_2 - D_1)$ are similar and have the same eigenvalues. This fact can help us to compute the eigenvalues of $\mathfrak{R}$. For case $m = 1$, we have

$$(D_3)^{-1}(D_2 - D_1) = \frac{2c_1^{1-\alpha} - (1+c_1)^{1-\alpha}}{c_1^{1-\alpha}}.$$

For, $\alpha = 0$, it agrees with the results of [3] which is the Volterra integral equation of first kind. For $m \ge 2$, one can directly compute the eigenvalues of $(D_3)^{-1}(D_2 - D_1)$ for a given collocation points.

7. Numerical Experiments

We give some examples to show efficiency and effectiveness of the introduced methods. In these examples, we obtain the absolute error with respect to the parameters N and t, we denote it by

$$_i\epsilon_N = \max_{n=0,\ldots,N,\ j=1,\ldots,m} |_i\, u_h(t_{n,j}) - y_i(t_{n,j})|,$$

for $i = 1\cdots\nu$, where $y_i(t)$ and $_iu_h(t)$ are exact and numerical solutions of the i-th components, respectively. We report the approximation of convergent order using the formula $_i\rho_N = \log_2 \frac{_i\epsilon_N}{_i\epsilon_{2N}}$, $i = 1\cdots\nu$. We apply the following methods for the next examples:

Method 0. $m = 1$, and $c_1 = 0.5$.

Method 1. $m = 2$, $c_1 = 0.5$ and $c_2 = 1$.

Method 2. Roots of shifted Legendre polynomial of degree 2: $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$.

Method 3. Roots of shifted Chebyshev polynomial of degree 2: $c_1 = \frac{2-\sqrt{2}}{4}$, $c_2 = \frac{2+\sqrt{2}}{4}$.

Method 4. Roots of shifted Chebyshev polynomial of degree 3: $c_1 = \frac{2-\sqrt{3}}{4}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{2+\sqrt{3}}{4}$.

Method 5. Roots of shifted Legendre polynomial of degree 3: $c_1 = \frac{\sqrt{5}-\sqrt{3}}{2\sqrt{5}}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{\sqrt{5}+\sqrt{3}}{2\sqrt{5}}$.

Set $\lambda = \max_{i=1,2,3} \lambda_i$, where λ_i for $i = 1,2,3$ are the eigenvalues of stability matrix $\mathfrak{R}$. Table 1 shows the values of λ for methods 1-5, and different values of α. This insures us that the numerical solution of the corresponding methods is convergent to the exact solution, for given examples.

Table 1. *$\lambda = \max_{i=1,2,3} \lambda_i$, for methods 1-5 and different values of α.*

α	0.4	0.6	0.8	0.85	0.9	0.95
M. 1	0.45	0.66	0.66	0.89	0.93	0.96
M. 2	0.36	0.61	0.61	0.87	0.91	0.96
M. 3	0.41	0.63	0.63	0.87	0.92	0.96
M. 4	0.43	0.65	0.65	0.88	0.92	0.96
M. 5	0.42	0.64	0.64	0.88	0.92	0.96

Now, consider system (1) with

$$k(t,s) = \frac{1}{\Gamma(1-\alpha)}\begin{pmatrix} e^{t+s} & (s+t)e^{-s} \\ (s-t)e^{s} & e^{2t-s} \end{pmatrix}$$

and

$$f(t) = t^{\beta}\begin{pmatrix} \frac{\Gamma(\beta+\alpha)}{\Gamma(1+\beta)}e^{t} + \left(\frac{\Gamma(\beta+\alpha+2)}{\Gamma(3+\beta)} + \frac{\Gamma(\beta+\alpha+1)}{\Gamma(2+\beta)}\right)t^{2} \\ \frac{\Gamma(\beta+\alpha+1)}{\Gamma(2+\beta)}te^{2t} + \left(-\frac{\Gamma(\beta+\alpha)}{\Gamma(1+\beta)} + \frac{\Gamma(\beta+\alpha+1)}{\Gamma(2+\beta)}\right)t \end{pmatrix}.$$

Then, the exact solution of the two dimensional system (1) is

$$y(s) = \left(e^{-s}s^{\beta+\alpha-1}, e^{s}s^{\beta+\alpha}\right)^{T}.$$

We can construct the following examples by altering the parameters of the above system.

Example 7. 1. Let $T = 1$, $\beta = 1$ and $\alpha = \frac{2}{5}, \frac{4}{5}$, then apply the methods 1-3.

Example 7. 2. Let $T = 5$, $\beta = 1$ and $\alpha = \frac{17}{20}, \frac{18}{20}, \frac{19}{20}$, then apply the methods 4-5.

Example 7. 3. Let $T = 10$, $\beta = 1, 1.5, 2$ and $\alpha = 0.9$, then apply the methods 1-5.

By Theorem 5.2, we expect the error estimate

$$\| e_h(t) \| \le c \begin{cases} N^{-\alpha r}, & 1 \le r \le \frac{m}{\alpha}, \\ N^{-m}, & r \ge \frac{m}{\alpha+\beta-1}, \end{cases}$$

in Examples 7.1.-7.2. Tables, 2-5 confirm this error estimate.

Table 2. *The absolute error and corresponding convergent order in Example 1 with $r = 1$.*

	N	$\alpha = 0.4$		$\alpha = 0.8$	
M. 1		ϵ_N	ρ_N	ϵ_N	ρ_N
	64	1.030e-2	-	6.388e-4	-

	N	$\alpha = 0.4$		$\alpha = 0.8$	
	128	7.868e-3	0.388	3.697e-4	*0.787*
	256	5.987e-3	0.394	2.132e-4	*0.794*
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 2	64	1.153e-2	-	4.989e-4	-
	128	8.768e-3	0.395	2.878e-4	*0.794*
	256	6.657e-3	0.397	1.656e-4	*0.797*
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 3	64	1.227e-2	-	4.694e-4	-
	128	9.320e-3	0.396	2.705e-4	0.795
	256	7.072e-3	0.398	1.556e-4	0.798

Table 3. *The absolute error and corresponding convergent order in Example 1 with* $r = \frac{2}{\alpha}$.

	N	$\alpha = 0.4$		$\alpha = 0.8$	
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 1	64	1.177e-4	-	2.789e-5	-
	128	2.978e-5	*1.983*	7.240e-6	*1.946*
	256	7.492e-6	*1.991*	1.880e-6	1.946
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 2	64	1.299e-4	-	1.551e-5	-
	128	3.421e-5	1.925	3.938e-6	*1.978*
	256	8.776e-6	1.963	9.924e-7	1.989
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 3	64	1.624e-4	-	2.558e-5	-
	128	4.325e-5	1.908	6.798e-6	*1.912*
	256	1.120e-5	1.949	1.781e-6	1.932

Table 4. *The absolute error and corresponding convergent order in Example 2 with* $r = 1$.

	N	$\alpha = 0.85$		$\alpha = 0.95$	
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 4	32	3.281e-4	-	3.763e-5	-
	64	1.750e-4	*0.907*	1.759e-5	*1.097*
	128	9.689e-5	0.853	8.867e-6	0.988
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 5	32	4.236e-4	-	4.435e-5	-
	64	2.281e-4	*0.893*	2.024e-5	*1.132*
	128	1.259e-4	0.858	1.014e-5	0.997

Table 5. *The absolute error and corresponding convergent order in Example 2 with* $r = \frac{3}{\alpha}$.

	N	$\alpha = 0.85$		$\alpha = 0.95$	
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 1	32	2.677e-6	-	1.024e-6	-
	64	3.186e-7	*3.071*	1.239e-7	*3.046*
	128	4.064e-8	2.971	1.535e-8	3.013
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 2	32	2.359e-6	-	1.026e-6	-
	64	2.463e-7	*3.260*	1.140e-7	*3.169*
	128	2.992e-8	3.041	1.331e-8	3.098

Also, by Theorem 5.2, we expect the estimate

$$\| e_h(t) \| \le c \begin{cases} N^{-(\beta-0.1)r}, & 1 \le r \le \dfrac{m}{\beta - 0.1}, \\ N^{-m}, & r \ge \dfrac{m}{\alpha + \beta - 1}, \end{cases}$$

in Example 7.3. Tables, 6-7 show this estimate. In all tables, we reported estimate of the order for the component of the system which is less. Finally, these examples show that the obtained convergent result is optimal and can't be improved for our investigated class of SWSVIEFKs.

Table 6. *The absolute error and corresponding convergent order in Example 3 with* $r = 1$

	N	$\beta = 1$		$\beta = 2$	
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 1	256	1.635e-4	-	5.918e-5	-
	512	8.873e-5	0.882	1.601e-5	1.886
M. 2	256	1.179e-4	-	2.732e-5	-
	512	6.362e-5	0.890	7.496e-6	1.866
M. 3	256	1.079e-4	-	1.986e-5	-
	512	5.821e-5	0.890	5.334e-6	1.897
M. 4	256	3.494e-5	-	7.828e-7	-
	512	1.874e-5	0.899	2.131e-7	1.877
M. 5	256	1.858e-5	-	4.904e-7	-
	512	9.943e-6	0.902	1.337e-7	1.875

Table 7. *The absolute error and corresponding convergent order in Example 3 with* $r = \frac{m}{\beta - 0.1}$.

	N	$\beta = 1$		$\beta = 2$	
		ϵ_N	ρ_N	ϵ_N	ρ_N
M. 1	128	8.908e-7	-	1.597e-5	-
	256	2.112e-7	2.077	4.126e-6	1.953
M. 2	128	7.452e-6	-	6.391e-5	-
	256	1.891e-6	1.978	1.641e-5	1.962
M. 3	128	1.373e-5	-	1.053e-4	-
	256	3.710e-6	1.888	2.854e-5	1.883
M. 4	128	5.925e-8	-	1.697e-7	-
	256	8.340e-9	2.829	2.135e-8	2.990
M. 5	128	5.925e-8	-	1.697e-7	-
	256	8.340e-9	2.829	2.135e-8	2.990

8. Conclusion

A convergence analysis of the collocation methods for SWSVIEFKs on discontinuous piecewise polynomial spaces has been investigated. Based on this analysis, the order of the method does not change by increasing collocation parameters on uniform mesh. However, it can be increased up to m by using graded mesh. Our analysis stands on the eigenvalues of stability matrix. We obtained a closed form of this eigenvalues for case $m = 1$. However, for cases $m > 1$, we obtained the eigenvalues of stability matrix for prescribed collocation parameters.

References

[1] K. E. Atkinson, "The numerical solution of integral equations of the second kind," Vol. 4, Cambridge university press, 1997.

[2] H. W. Branca, "The nonlinear Volterra equation of abel's kind and its numerical treatment," Comput, 1978, 20, 307-324.

[3] H. Brunner, "Collocation methods for Volterra integral and related functional differential equations," Vol. 15, Cambridge University Press, 2004.

[4] H. Brunner, A. Pedas and G. Vainikko, “Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels,” SIAM Journal on Numerical Analysis, 2001, 39, 957-982.

[5] R. Cameron and S. McKee, “Product integration methods for second-kind Abel integral equations,” Journal of Computational and Applied Mathematics, 1984, 11, 1-10.

[6] P. Eggermont, “A new analysis of the trapezoidal-discretization method for the numerical solution of Abel-type integral equations,” Journal of Integral Equations and Applications, 1981, 3, 317-332.

[7] R. Kress, V. Maz'ya and V. Kozlov, “Linear integral equations,” Vol. 82, Springer, 1989.

[8] C. Lubich, “Fractional linear multistep methods for Abel-Volterra integral equations of the first kind,” IMA Journal of Numerical Analysis, 1987, 7, 97-106.

[9] A. Pedas and E. Tamme, “On the convergence of spline collocation methods for solving fractional differential equations,” Journal of Computational and Applied Mathematics, 2011, 235, 3502-3514.

[10] H. Te Riele and P. Schroevers, “A comparative survey of numerical methods for the linear generalized Abel integral equation,” ZAMM- Journal of Applied Mathematics and Mechanics, 1986, 66, 163-173.

[11] A. Saadatmandi and M. Dehghan, “A collocation method for solving Abel’s integral equations of first and second kinds,” Zeitschrift für Naturforschung A, 2008, 63, 752-756.

[12] B. Shiri, “Numerical solution of higher index nonlinear integral algebraic equations of Hessenberg type using discontinuous collocation methods,” Mathematical Modelling and Analysis, 2014, 19, 99-117.

[13] V. Volterra, “Sulla inversione degli integrali definiti,” Atti della Accademia delle scienze di Torino,1896, 31, 311-323.

[14] R. Weiss, “Numerical procedures for volterra integral equations,” Bulletin of the Australian Mathematical Society, 1973, 8, 477-478.

[15] R. Weiss, “Product integration for the generalized Abel equation,” Mathematics of Computation, 1972, 26, 177-190.

[16] R. Weiss and R. Anderssen, “A product integration method for a class of singular first kind Volterra equations,” Numerical Mathematics, 1971, 18, 442-456.