Fixed Point Theorem in 2 – Metric spaces of implicit Relations

Rajesh Shrivastava¹, Neha Jain², K. Qureshi³

¹Deptt .of Mathematics, Govt. Science and comm. College Banazeer Bhopal(M.P) India

²Research Scholar, Govt. Science and comm. College Banazeer Bhopal(M.P) India

³Additional Director, Higher Education Deptt. Govt. of M.P., Bhopal (M.P) India

Email address:

capricone.neha@gmail.com(N. Jain)

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Abstract: In this paper present on Fixed point Theorem in 2-metric spaces .A concept which has been in focus recent times. The result is supported with an example.

Keywords: Metric Spaces, Fixed Points, Complete Metric Spaces, Cauchy Sequence

1. Introduction

Theorem: -

let (X ,d) and (Y ,ρ)be a complete metric space .if T:X→Y and S :Y →X satisfying the Conditions :

1.1. [Fisher [1]]

$$\begin{aligned} d(Sy, STx) &\leq c \max\{d(x, Sy), d(x, STx), \rho(y, Tx)\} \\ \rho(Tx, TSy) &\leq c \max\{\rho(y, Tx), \rho(y, TSy), d(x, Sy)\} \end{aligned}$$

for all x in X and y in Y, where $0 \leq c < 1$,

1.2. [Popa [3]]

$$\begin{aligned} d^2(Sy, STx) &\leq c_1 \max\{\rho(y, Tx)d(x, Sy), \rho(y, TSy) \\ &\quad d(x, STx), d(x, Sy)d(x, STx)\} \\ \rho^2(Tx, TSy) &\leq c_2 \max\{d(x, Sy) \rho(y, Tx), d(x, STx) \\ &\quad \rho(y, TSy), \rho(y, Tx) \rho(y, TSy)\} \end{aligned}$$

for all x in X and y in Y, where $0 \leq c_1, c_2 < 1$.

1.3. [(Nešić[2])]

$$\begin{aligned} M_1(x, y) &= \{d^p(x, Sy), d^p(x, STx), \rho^p(y, Tx)\} \\ M_2(x, y) &= \{\rho^p(y, Tx), \rho^p(y, TSy), d^p(x, Sy)\} \end{aligned}$$

for all x in X, y in Y and $p = 1, 2, 3, \dots$

let R^+ be the set of nonnegative real numbers, and let

$F_i : R^+ \rightarrow R^+$ be a mapping such that

$F_i(0) = 0$ and F_i is continuous at 0 for $i = 1, 2$. If T and S satisfying the inequalities

$$d^p(Sy, STx) \leq c_1 \max M_1(x, y) + F_1(\min M_1(x, y))$$

$$\rho^p(Tx, TSy) \leq c_2 \max M_2(x, y) + F_2(\min M_2(x, y))$$

for all x in X and y in Y, where $0 \leq c_1, c_2 < 1$.

Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y.

Futher, $Tz = w$ and $Sw = z$.

2. Main Results

The theorem that we are attempting to prove generalizes that Fisher [1], Nešić[2], Popa [3]theorem using an implicit relations. Let $\Phi_3^{(m)}$ be the set of continuous functions with 3 variables

$\varphi: [0, \infty)^3 \rightarrow [0, \infty)$ satisfying the properties:

a. φ is non decreasing in t_1, t_2, t_3 .

b. $\varphi(t, t, t) \leq t^m \quad m \in \mathbb{N}$

Some examples of such functions are as follows:

Example 2.1 $\psi(t_1, t_2, t_3) = \min\{t_1^p, t_2^p, t_3^p\}$

Example 2.2 $\psi(t_1, t_2, t_3) = \min\{t_1 t_2, t_1 t_3, t_2 t_3\}$

Example 2.3 $\psi(t_1, t_2, t_3) = \min\{t_1, t_2, t_3\}$

Example 2.4 $\varphi(t_1, t_2, t_3) = \max\{t_1^p, t_2^p, t_3^p\}$,
with $m = p$

Example 2.5 $\varphi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$,
with $m = 1$ etc.

Let F be a set of continuous functions $F: [0, \infty) \rightarrow [0, \infty)$ with $F(0) = 0$ (for example $F(t) = t^q, q > 0$).

3. Main theorem

Theorem 3.1

Let (X, d) , (Y, ρ) be two complete metric spaces and $T: X \rightarrow Y$, $S: Y \rightarrow X$ two mappings. Let $\emptyset \in \Phi_3^{(m)}$, $\psi_i \in \Psi_3$, $F_i \in f$ for $i = 1, 2$. if for some $k \in [0, 1)$, the following inequalities are satisfied:

$$d^m(Sy, STx) \leq k\varphi_1(d(x, STx), \rho(y, Tx)) + F_1(\psi_1(d(x, STx), \rho(y, Tx))) \dots \dots \dots (1)$$

$$\rho^m(Tx, TSy) \leq k\varphi_2(\rho(y, TSy), d(x, Sy)) + F_2(\psi_2(\rho(y, TSy), d(x, Sy))) \dots \dots \dots (2)$$

for all $x \in X$, $y \in Y$ and some $m = 1, 2, \dots$, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further $T_\alpha \in \beta$ and $S_\beta \in \alpha$.

Proof :- let $x_0 \in X$ be an arbitrary point. we define the sequences (x_n) and (y_n) in X and Y respectively as follows:

$$x_n = (ST)^n x_0, \quad y_n = Tx_{n-1}, \quad n = 1, 2, 3, \dots$$

Let us prove that the sequences (x_n) and (y_n) are Cauchy sequences. We assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1} \forall n \in \mathbb{N}$, because otherwise if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n , we could put $\alpha = x_n$ and $\beta = y_n$.

denote

$$d_n = d(x_n, x_{n+1}), \quad \rho_n = \rho(y_n, y_{n+1}), \quad n = 1, 2, \dots$$

By the inequality (2) for $x = x_{n-1}$ and $y = y_n$ we get:

$$\begin{aligned} \rho^m(y_n, y_{n+1}) &= \rho^m(Tx_{n-1}, TSy_n) \\ &\leq k\varphi_2(\rho(y_n, TSy_n), d(x_{n-1}, Sy_n)) \\ &\quad + F_2(\psi_2(\rho(y_n, TSy_n), d(x_{n-1}, Sy_n))) \\ \rho^m(y_n, y_{n+1}) &\leq k\varphi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)) \\ &\quad + F_2(\psi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n))) \end{aligned}$$

or

$$\begin{aligned} \rho_n^m &\leq k\varphi_2(\rho_n, d_{n-1}) + F_2(\psi_2(\rho_n, d_{n-1})) \\ &= k\varphi_2(\rho_n, d_{n-1}) \dots \dots \dots (3) \end{aligned}$$

For the coordinates of the point (ρ_n, d_{n-1}) we have:

$\rho_n \leq d_{n-1}, \forall n \in \mathbb{N}$, because; in case that

$\rho_n > d_{n-1}$ for some n , if we replace the coordinates with ρ_n and apply the property (a) and (b) of φ_2 we get

$$\rho_n^m \leq k\varphi_2(\rho_n, \rho_n) \leq k\rho_n^m$$

This is impossible since $0 \leq k < 1$.

Replacing on the right hand side of (3), the coordinates with ρ_{n-1} and applying properties (a) and (b) of φ_2 we get:

$$\begin{aligned} \rho_n^m &\leq k\varphi_2(d_{n-1}, d_{n-1}) \leq kd_{n-1}^m \\ \text{Thus } \rho_n &\leq \sqrt[m]{k} d_{n-1} \dots \dots (4) \end{aligned}$$

By the inequality (1) for $x = x_n$ and $y = y_n$ we get

$$d^m(x_n, x_{n+1}) = d^m(Sy_n, STx_n)$$

$$\begin{aligned} &\leq k\varphi_1(d(x, STx), \rho(y, Tx)) \\ &\quad + F_1(\psi_1(d(x, STx), \rho(y, Tx))) \\ &\leq k\varphi_1[d(x_n, x_{n+1}), \rho(y_n, y_{n+1})] \\ &\quad + F_1(\psi_1(d(x_n, x_{n+1}), \rho(y_n, y_{n+1}))) \end{aligned}$$

Or

$$\begin{aligned} d_n^m &\leq k\varphi_1(d_n, \rho_n) + F_1(\psi_1(d_n, \rho_n)) \\ &= k\varphi_1(d_n, \rho_n) \end{aligned}$$

In similar, we get:

$$d_n^m \leq k\varphi_1(\rho_n, \rho_n) \leq k\rho_n^m$$

Thus we will obtain:

$$d_n \leq \sqrt[m]{k} \rho_n$$

By this inequality and (4) we get:

$$\begin{aligned} d_n &\leq \sqrt[m]{k} (\sqrt[m]{k} d_{n-1}) \\ &\leq \sqrt[m]{k} d_{n-1} \dots \dots (5) \end{aligned}$$

by the inequalities (4) and (5), using the mathematical induction, we obtain:

$$\begin{aligned} d(x_n, x_{n+1}) &= d_n \leq q^{n-1} d_1 \\ \rho(y_n, y_{n+1}) &= \rho_n \leq q^{n-1} \rho_1 \text{ Where } q = \sqrt[m]{k} < 1 \end{aligned}$$

Thus the sequences (x_n) and (y_n) are Cauchy sequences, since the Metric space (X, d) , (Y, ρ) complete metric spaces we have:

$$\lim_{n \rightarrow \infty} x_n = \alpha \in X, \quad \lim_{n \rightarrow \infty} y_n = \beta \in Y$$

by (1) for $y = \beta$ and $x = x_n$ we get:

$$\begin{aligned} d^m(S\beta, x_{n+1}) &\leq k\varphi_1(d(x_n, x_{n+1}), \rho(\beta, y_{n+1})) \\ &\quad + F_1(\psi_1(d(x_n, x_{n+1}), \rho(\beta, y_{n+1}))) \end{aligned}$$

letting $n \rightarrow \infty$, we get:

$$\begin{aligned} d^m(S\beta, \alpha) &\leq k\varphi_1(0, 0) + F(0) \\ &\leq k\varphi_1(0, 0) \end{aligned}$$

Or

$$d^m(S\beta, \alpha) = 0 \Leftrightarrow S\beta = \alpha \dots \dots (6)$$

by (2) for $x = x_n$ and $y = \beta$ we get:

$$\begin{aligned} \rho^m(y_{n+1}, TS\beta) &\leq k\varphi_2(\rho(\beta, TS\beta), d(x_n, S\beta)) \\ &\quad + F_2(\psi_2(\rho(\beta, TS\beta), d(x_n, S\beta))) \end{aligned}$$

letting $n \rightarrow \infty$, and using (6) we get:

$$\begin{aligned} \rho^m(\beta, TS\beta) &\leq k\varphi_2(\rho(\beta, TS\beta), 0) + F(0) \\ \text{or} \end{aligned}$$

$$\rho^m(\beta, TS\beta) \leq k\rho^m(\beta, TS\beta) \Leftrightarrow TS\beta = \beta \dots (7)$$

by (6) and (7) it follows:

$$\begin{aligned} TS\beta &= T\alpha = \beta \\ ST\alpha &= S\beta = \alpha \end{aligned}$$

Thus we proved that the points α, β are fixed points of ST and TS respectively.

UNIQUENESS

suppose that ST has a second distinct Fixed point α' in X .

By (1) for $x = \alpha'$ and $y = T\alpha$ we get :

$$d^m(ST\alpha, ST\alpha') \leq k\varphi_1(d(\alpha', ST\alpha'), \rho(T\alpha, T\alpha')) + F_1(\psi_1(0, \rho(T\alpha, T\alpha')))$$

Or

$$d^m(\alpha, \alpha') \leq k\varphi_1(d(0, \rho(T\alpha, T\alpha')) + F_1(0)) \leq k\rho^m(T\alpha, T\alpha') \quad \dots \dots (8)$$

By (2) for $x = \alpha'$ and $y = T\alpha$ we obtain :

$$\rho^m(T\alpha', T\alpha) \leq k\varphi_2(\rho(T\alpha, T\alpha), d(\alpha', \alpha)) + F_2(\psi_2(0, d(\alpha, \alpha')))$$

or

$$\rho^m(T\alpha', T\alpha) \leq k\varphi_2(\rho(0, d(\alpha', \alpha))) \leq kd^m(\alpha', \alpha) \quad \dots \dots (9)$$

By (8) and (9) we get :

$$d^m(\alpha, \alpha') \leq k^2 d^m(\alpha, \alpha')$$

it follows

$$d(\alpha, \alpha') = 0$$

Thus we have again $\alpha = \alpha'$. in same way , it is proved the inequality of β ■

Hence Proved

Theorem 3.2

Let (X, d) , (Y, ρ) be two complete metric spaces and $T : X \rightarrow Y$, $S : Y \rightarrow X$ two mappings . Let $\emptyset \in \Phi_3^{(m)}$, $\psi_i \in \Psi_3$, $F_i \in f$ for $i = 1, 2$. if for some $k \in [0, 1)$, the following inequalities are satisfied :

$$d^m(Sy, STx) \leq k\varphi_1(d(x, Sy), d(x, STx), \rho(y, Tx), \frac{d(x, Sy) d(x, STx) \{1 + \rho(y, Tx)\}}{1 + d(x, Sy)}) + F_1(\psi_1(d(x, Sy), d(x, STx), \rho(y, Tx), \frac{d(x, Sy) d(x, STx) \{1 + \rho(y, Tx)\}}{1 + d(x, Sy)})) \quad \dots \dots (1)$$

$$\rho^m(Tx, TSy) \leq k\varphi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy), \frac{\rho(y, Tx) d(x, Sy) \{1 + \rho(y, TSy)\}}{1 + \rho(y, Tx)}) + F_2(\psi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy), \frac{\rho(y, Tx) d(x, Sy) \{1 + \rho(y, TSy)\}}{1 + \rho(y, Tx)})) \quad \dots \dots (2)$$

for all $x \in X$, $y \in Y$ and some $m = 1, 2, \dots$, then ST has a unique fixed point $\alpha \in X$ and

TS has a unique fixed point $\beta \in Y$. Futher $T\alpha \in \beta$ and

$$S\beta \in \alpha .$$

Proof :- let $x_0 \in X$ be an arbitrary point .

we define the sequences (x_n) and (y_n) in X and Y respectively as follows :

$$x_n = (ST)^n x_0 , \quad y_n = Tx_{n-1} , \quad n = 1, 2, 3, \dots$$

Let us proof that the sequences (x_n) and (y_n) are Cauchy sequences .

We assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1} \forall n \in \mathbb{N}$,

because otherwise if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n , we could put $\alpha = x_n$ and $\beta = y_n$.

$$\text{denote } d_n = d(x_n, x_{n+1}) , \quad \rho_n = \rho(y_n, y_{n+1}) ,$$

$$n = 1, 2, \dots$$

By the inequality (2) for $x = x_{n-1}$ and $y = y_n$ we get :

$$\begin{aligned} \rho^m(y_n, y_{n+1}) &= \rho^m(Tx_{n-1}, TSy_n) \\ &\leq k\varphi_2(\rho(y_n, Tx_{n-1}), \rho(y_n, TSy_n), d(x_{n-1}, Sy_n), \frac{\rho(y_n, Tx_{n-1}) d(x_{n-1}, Sy_n) \{1 + \rho(y, TSy_n)\}}{1 + \rho(y_n, Tx_{n-1})}) \\ &\quad + F_2(\psi_2(\rho(y_n, Tx_{n-1}), \rho(y_n, TSy_n), d(x_{n-1}, Sy_n), \frac{\rho(y_n, Tx_{n-1}) d(x_{n-1}, Sy_n) \{1 + \rho(y, TSy_n)\}}{1 + \rho(y_n, Tx_{n-1})})) \\ \rho^m(y_n, y_{n+1}) &\leq k\varphi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), d(x_{n-1}, x_n), \frac{\rho(y_n, y_n) d(x_{n-1}, x_n) \{1 + \rho(y_n, y_{n+1})\}}{1 + \rho(y_n, y_n)}) \\ &\quad + F_2(\psi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), d(x_{n-1}, x_n), \frac{\rho(y_n, y_n) d(x_{n-1}, x_n) \{1 + \rho(y_n, y_{n+1})\}}{1 + \rho(y_n, y_n)})) \end{aligned}$$

or

$$\rho_n^m \leq k\varphi_2(0, \rho_n, d_{n-1}, 0) + F_2(\psi_2(0, \rho_n, d_{n-1}, 0)) = k\varphi_2(0, \rho_n, d_{n-1}, 0) \quad \dots \dots (3)$$

For the coordinates of the point (ρ_n, d_{n-1}) we have : $\rho_n \leq d_{n-1}$, $\forall n \in \mathbb{N}$, because ; in case that

$\rho_n > d_{n-1}$ for some n , if we replace the coordinates with ρ_n and apply the property (a) and (b) of φ_2 we get

$$\rho_n^m \leq k\varphi_2(\rho_n, \rho_n, \rho_n, \rho_n) \leq k\rho_n^m$$

This is impossible since $0 \leq k < 1$.

Replacing on the right hand side of (3) , the coordinates with ρ_{n-1} and applying properties (a) and (b) of φ_2 we get :

$$\rho_n^m \leq k\varphi_2(d_{n-1}, d_{n-1}, d_{n-1}, d_{n-1}) \leq kd_{n-1}^m$$

Thus

$$\rho_n \leq \sqrt[m]{k} d_{n-1} \quad \dots \dots (4)$$

By the inequality (1) for $x = x_n$ and $y = y_n$ we get

$$d^m(x_n, x_{n+1}) = d^m(Sy_n, STx_n)$$

$$\begin{aligned}
&\leq k\varphi_1(d(x_n, Sy_n), d(x_n, STx_n), \rho(y_n, Tx_n), \\
&\quad \frac{d(x_n, Sy_n) d(x_n, STx_n) \{1 + \rho(y_n, Tx_n)\}}{1 + d(x_n, Sy_n)}) \\
&\quad + F_1(\psi_1(d(x_n, Sy_n), d(x_n, STx_n), \rho(y_n, Tx_n), \\
&\quad \frac{d(x_n, Sy_n) d(x_n, STx_n) \{1 + \rho(y_n, Tx_n)\}}{1 + d(x_n, Sy_n)})) \\
&\leq k\varphi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \\
&\quad \frac{d(x_n, x_n) d(x_n, x_{n+1}) \{1 + \rho(y_n, y_{n+1})\}}{1 + d(x_n, x_n)}) \\
&\quad + F_1(\psi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \\
&\quad \frac{d(x_n, x_n) d(x_n, x_{n+1}) \{1 + \rho(y_n, y_{n+1})\}}{1 + d(x_n, x_n)}))
\end{aligned}$$

Or

$$\begin{aligned}
d_n^m &\leq k\varphi_1(0, d_n, \rho_n, 0) + F_1(\psi_1(0, d_n, \rho_n, 0)) \\
&= k\varphi_1(0, d_n, \rho_n, 0)
\end{aligned}$$

In similar, we get :

$$d_n^m \leq k\varphi_1(\rho_n, \rho_n, \rho_n, \rho_n) \leq k\rho_n^m$$

Thus we will obtain :

$$d_n \leq \sqrt[m]{k} \rho_n$$

By this inequality and (4) we get :

$$d_n \leq \sqrt[m]{k} (\sqrt[m]{k} d_{n-1}) \leq \sqrt[m]{k} d_{n-1} \quad \dots\dots (5)$$

by the inequalities (4) and (5), using the mathematical induction, we obtain :

$$d(x_n, x_{n+1}) = d_n \leq q^{n-1} d_1$$

$$\rho(y_n, y_{n+1}) = \rho_n \leq q^{n-1} d_1$$

$$\text{Where } q = \sqrt[m]{k} < 1$$

Thus the sequences (x_n) and (y_n) are Cauchy sequences, since the Metric space (X, d) and (Y, ρ) complete metric spaces we have :

$$\lim_{n \rightarrow \infty} x_n = \alpha \in X, \lim_{n \rightarrow \infty} y_n = \beta \in Y$$

by (1) for $y = \beta$ and $x = x_n$ we get :

$$\begin{aligned}
d^m(S\beta, x_{n+1}) &\leq k\varphi_1(d(x_n, S\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \\
&\quad \frac{d(x_n, S\beta) d(x_n, x_{n+1}) \{1 + \rho(\beta, y_{n+1})\}}{1 + d(x_n, S\beta)}) \\
&\quad + F_1(\psi_1(d(x_n, S\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \\
&\quad \frac{d(x_n, S\beta) d(x_n, x_{n+1}) \{1 + \rho(\beta, y_{n+1})\}}{1 + d(x_n, S\beta)}))
\end{aligned}$$

letting $n \rightarrow \infty$, we get :

$$\begin{aligned}
d^m(S\beta, \alpha) &\leq k\varphi_1(d(\alpha, S\beta), 0, 0, 0) + F(0) \\
&\leq k\varphi_1(d(\alpha, S\beta), 0, 0, 0) \leq kd^m(\alpha, S\beta)
\end{aligned}$$

Or

$$d^m(S\beta, \alpha) = 0 \Leftrightarrow S\beta = \alpha \quad \dots\dots\dots (6)$$

by (2) for $x = x_n$ and $y = \beta$ we get :

$$\begin{aligned}
\rho^m(y_{n+1}, TS\beta) &\leq k\varphi_2(\rho(\beta, Tx_n), \rho(\beta, TS\beta), d(x_n, S\beta), \\
&\quad \frac{\rho(\beta, Tx_n) d(x_n, S\beta) \{1 + \rho(\beta, TS\beta)\}}{1 + \rho(\beta, Tx_n)}) \\
&\quad + F_2(\psi_2(\rho(\beta, Tx_n), \rho(\beta, TS\beta), d(x_n, S\beta), \\
&\quad \frac{\rho(\beta, Tx_n) d(x_n, S\beta) \{1 + \rho(\beta, TS\beta)\}}{1 + \rho(\beta, Tx_n)})) \\
\rho^m(y_{n+1}, TS\beta) &\leq k\varphi_2(\rho(\beta, y_{n+1}), \rho(\beta, TS\beta), d(x_n, S\beta), \\
&\quad \frac{\rho(\beta, y_{n+1}) d(x_n, S\beta) \{1 + \rho(\beta, TS\beta)\}}{1 + \rho(\beta, y_{n+1})}) \\
&\quad + F_2(\psi_2(\rho(\beta, y_{n+1}), \rho(\beta, TS\beta), d(x_n, S\beta), \\
&\quad \frac{\rho(\beta, y_{n+1}) d(x_n, S\beta) \{1 + \rho(\beta, TS\beta)\}}{1 + \rho(\beta, y_{n+1})}))
\end{aligned}$$

letting $n \rightarrow \infty$, and using (6) we get :

$$\begin{aligned}
\rho^m(\beta, TS\beta) &\leq k\varphi_2(0, \rho(\beta, TS\beta), 0, 0) + F(0) \\
&\text{or}
\end{aligned}$$

$$\begin{aligned}
\rho^m(\beta, TS\beta) &\leq k\rho^m(\beta, TS\beta) \Leftrightarrow TS\beta = \beta \\
&\dots\dots\dots (7)
\end{aligned}$$

by (6) and (7) it follows :

$$TS\beta = T\alpha = \beta$$

$$ST\alpha = S\beta = \alpha$$

Thus we proved that the points α, β are fixed points of ST and TS respectively.

UNIQUENESS

suppose that ST has a second distinct fixed point α' in X.

By (1) for $x = \alpha'$ and $y = T\alpha$ we get :

$$\begin{aligned}
d^m(ST\alpha, ST\alpha') &\leq k\varphi_1(d(\alpha', ST\alpha), d(\alpha', ST\alpha'), \rho(T\alpha, T\alpha'), \\
&\quad \frac{d(\alpha', ST\alpha) d(\alpha', ST\alpha') \{1 + \rho(T\alpha, T\alpha')\}}{1 + d(\alpha', ST\alpha)}) \\
&\quad + F_1(\psi_1(d(\alpha', ST\alpha), d(\alpha', ST\alpha'), \rho(T\alpha, T\alpha'), \\
&\quad \frac{d(\alpha', ST\alpha) d(\alpha', ST\alpha') \{1 + \rho(T\alpha, T\alpha')\}}{1 + d(\alpha', ST\alpha)})) \\
d^m(ST\alpha, ST\alpha') &\leq k\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), 0) \\
&\quad + F_1(\psi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), 0))
\end{aligned}$$

Or

$$\begin{aligned}
d^m(\alpha, \alpha') &\leq k\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), 0) \\
&\quad + F_1(0) \\
&\leq k\rho^m(T\alpha, T\alpha') \quad \dots\dots\dots (8)
\end{aligned}$$

By (2) for $x = \alpha'$ and $y = T\alpha$ we obtain :

$$\begin{aligned}
\rho^m(T\alpha', T\alpha) &\leq k\varphi_2(\rho(T\alpha, T\alpha'), \rho(T\alpha, T\alpha), d(\alpha', \alpha), \\
&\quad \frac{\rho(T\alpha, T\alpha') d(\alpha', \alpha) \{1 + \rho(T\alpha, T\alpha)\}}{1 + \rho(T\alpha, T\alpha')}) \\
&\quad + F_2(\psi_2(\rho(T\alpha, T\alpha'), \rho(T\alpha, T\alpha), d(\alpha', \alpha), \\
&\quad \frac{\rho(T\alpha, T\alpha') d(\alpha', \alpha) \{1 + \rho(T\alpha, T\alpha)\}}{1 + \rho(T\alpha, T\alpha')})) \\
\rho^m(T\alpha', T\alpha) &\leq k\varphi_2(\rho(T\alpha, T\alpha'), 0, d(\alpha', \alpha), \\
&\quad \frac{\rho(T\alpha, T\alpha') d(\alpha', \alpha) \{1 + 0\}}{1 + \rho(T\alpha, T\alpha')}) \\
&\quad + F_2(\psi_2(\rho(T\alpha, T\alpha'), 0, d(\alpha', \alpha), \\
&\quad \frac{\rho(T\alpha, T\alpha') d(\alpha', \alpha) \{1 + 0\}}{1 + \rho(T\alpha, T\alpha')}))
\end{aligned}$$

or

$$\begin{aligned}
 \rho^m(T\alpha', T\alpha) &\leq k\varphi_2\left(\rho(T\alpha', T\alpha), 0, d(\alpha, \alpha')\right), \\
 &\quad \frac{\rho(T\alpha, T\alpha')d(\alpha', \alpha)}{1+\rho(T\alpha, T\alpha')} \\
 &\quad + F_2(0) \\
 &\leq kd^m(\alpha, \alpha')\rho^m(T\alpha', T\alpha) \\
 &\leq k\varphi_2\left(\rho(T\alpha', T\alpha), 0, d(\alpha, \alpha')\right), \\
 &\quad \frac{\rho(T\alpha, T\alpha')d(\alpha', \alpha)}{1+\rho(T\alpha, T\alpha')} \\
 &\leq kd^m(\alpha, \alpha') \\
 &\quad \dots\dots\dots(9)
 \end{aligned}$$

By (8) and (9) we get :

$$d^m(\alpha, \alpha') \leq k^2 d^m(\alpha, \alpha')$$

it follows

$d(\alpha, \alpha') = 0$ Thus we have again $\alpha = \alpha'$. in same way, it is proved the inequality of β ■

Hence Proved

References

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