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Reachable sets for autonomous systems of differential equations and their topological properties

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Abstract: The initial value problems for autonomous systems of differential equations are the main object of this paper. Different variants of the concept reachable sets for the solutions of such systems are introduced. Several conditions for their existence are found and some properties are studied.

Keywords: Autonomous Differential Equations, Reachable Sets

1. Introduction

The main goal of this paper is the initial value problem for a system of autonomous differential equations. Let us consider a nonempty set Φ in the phase space G of such system. If a trajectory of the system crosses the set Φ then the starting point x_0 of the trajectory is called a starting (initial) point of reachability and Φ is called a set of reachability. Some basic properties of the starting points set of reachability are studied in the paper.

The results obtained are applied in the investigation of differential equations with variable - time impulses. The impulsive systems are several types, depending on the way of determining the impulsive moments. The obtained results concern the impulsive systems in which the impulsive moments coincide with the moments at which the trajectory crosses the impulsive set (in this case – reachable set Φ). Such type impulsive systems are discussed in [4] - [7]. There are many applications of impulsive equations (see [1], [4], [8] - [14]).

2. Preliminaries

The Euclidean norm and dot product in R^n are denoted by $\|.\|$ and $\langle .,. \rangle$ respectively. For the points $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in R^n, we have

$$\langle x, y\rangle = x^1y^1 + x^2y^2 + \cdots + x^ny^n,$$

$$\|x\| = (\langle x, x\rangle)^{1/2} = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}.$$

The Euclidean distance between nonempty sets X and Y, where $X, Y \subset R^n$, is denoted by

$$\rho(X, Y) = inf\{\|x - y\|;\ x \in X, y \in Y\}.$$

An open ball with center $x_0 \in R^n$ and radius $\delta = const > 0$ is denoted by

$$B_\delta(x_0) = \{x \in R^n;\ \|x - x_0\| < \delta\}.$$

$\bar{X}$ and ∂X are notations for the closure and boundary of the set X.

Consider the following initial value problem

$$\frac{dx}{dt} = f(x),\ x(0) = x_0, \tag{1}$$

where function $f: G \to R^n$,set $G \subset R^n$, $G \neq \emptyset$ and G is a domain (open connected set); $x_0 \in G$. Let $x(t;\ x_0)$ be the solution of problem (1). Let $\gamma(\theta, x_0)$ be the trajectory of system (1) which lies between the points $x(0;\ x_0) = x_0$ and $x(\theta;\ x_0)$, where $\theta \in R$. It is given by

$$\gamma(\theta, x_0) = \begin{cases} \{x = x(t;\ x_0);\ 0 \le t < \theta\}, & \text{if } \theta \ge 0, \\ \{x = x(t;\ x_0);\ \theta < t \le 0\}, & \text{if } \theta < 0. \end{cases}$$

In particular,

$$\gamma(\infty, x_0) = \{x = x(t;\ x_0);\ 0 \le t < \infty\}$$

and

$$\gamma(-\infty, x_0) = \{x = x(t;\ x_0); -\infty < t \le 0\}.$$

Definition 2.1.

If:

1. $X_0^+ \subset G,\ \Phi \subset G,\ X_0^+ \neq \emptyset$ *and* $\Phi \neq \emptyset$.
2. *For each point* $x_0 \in X_0^+$, *the solution* $x(t;\, x_0)$ *of the initial value problem (1) is defined and is unique in the interval* $[0, \infty)$.
3. *It is valid*

$$(\forall x_0 \in X_0^+)(\exists \theta = \theta(x_0) > 0)\colon x(\theta;\, x_0) \in \Phi.$$

Then:

- *The set* Φ *is said to be positive reachable from set* X_0^+ *via system (1);*
- X_0^+ *is said to be a starting set of positive reachability of* Φ *via system (1);*
- *Every point* $x_0 \in X_0^+$ *is said to be a starting point of positive reachability of* Φ *via system (1).*

By analogy, we define the concepts of negative reachable set from the set X_0^- via system (1), starting set of negative reachability of Φ and starting point of negative reachability of Φ via system (1).

The terms introduced above will be applied only to system (1) in the next research, so we will omit this detail. For convenience, we denote by X_0^+ and X_0^- the sets of all starting points of positive and negative reachability, respectively. Finally, the set $X_0 = X_0^- \cup X_0^+ \cup \Phi$ is said to be an initial set of reachability.

Assume that

$$\Phi = \{x \in D;\ \varphi(x) = 0\} \subset G,$$

where $D \subset G$, D is a domain and the function $\varphi\colon D \to R$.

The following conditions are introduced:

H1. *There exists a constant* $C_{Lip} > 0$, *such that*

$$(\forall x', x'' \in G) \Rightarrow \|f(x') - f(x'')\| \leq C_{Lip}\|x' - x''\|.$$

H2. For each point $x_0 \in G$, the solution of problem (1) exists and is unique for all t in R.

H3. The function $\varphi \in C^1[D, R]$. The set $\Phi = \{x \in D;\ \varphi(x) = 0\} \neq \emptyset$ and

$$(\forall x \in \Phi) \Rightarrow \langle grad\varphi(x), f(x)\rangle > 0.$$

H4. The *set* Φ *is connected.*

Further, the function $\phi\colon \Delta \rightarrow R$, where $\phi(t) = \varphi\big(x(t; x_0)\big)$ and $\Delta = \{t \in R;\ x(t; x_0) \in D\}$ will be used repeatedly. If the point $x_0 \in D$, then the function ϕ is defined in a neighborhood of 0, i.e. $0 \in \Delta$. It is clear that $\phi(0) = \varphi\big(x(0; x_0)\big) = \varphi(x_0)$ in this case.

3. Main Results

Theorem 3.1.

Assume that:

1. *The condition H2 holds.*
2. *The set* Φ *is positive reachable from the set* X_0^+ *and point* $x_0 \in X_0^+$.

Then the trajectory $\gamma(\theta, x_0) \subset X_0^+$, where the positive constant θ is chosen such that $x(\theta;\, x_0) \in \Phi$ and $x(t;\, x_0) \notin \Phi$ for $0 \leq t < \theta$.

Proof. Consider an arbitrary point x_{t^*} on the curve $\gamma(\theta, x_0)$. For example, $x_{t^*} = x(t^*;\, x_0) \in \gamma(\theta, x_0) \in G$ for $0 \leq t^* < \theta$. We have

$$\theta - t^* > 0 \text{ and } x(\theta - t^*; x_{t^*}) = x(\theta;\, x_0) \in \Phi.$$

Therefore $x_{t^*} \in X_0^+$.

The theorem is proved.

Corollary 3.1.

Assume that:

1. *The condition H2 holds.*
2. *The set* Φ *is negative reachable from the set* X_0^- *and point* $x_0 \in X_0^-$.

Then the trajectory $\gamma(\theta, x_0) \subset X_0^-$, *where the negative constant* θ *is determined so that* $x(\theta;\, x_0) \in \Phi$ *and* $x(t;\, x_0) \notin \Phi$ *for* $\theta < t \leq 0$.

Theorem 3.2.

Assume that:

1. *The condition H2 holds.*
2. *The function* $\varphi \in C[D, R]$.
3. *The set* Φ *is positive reachable from the set* X_0^+.

Then $X_0^+ \neq \emptyset$.

Proof. Let the point $x_0 \in \Phi$, i.e. $\varphi(x_0) = 0$. Since the function φ is defined in the open set D and the point $x(0;\, x_0) = x_0 \in \Phi \subset D$, it follows that

$$(\exists \delta = const > 0)\colon \phi = \varphi\big(x(t;\, x_0)\big)\colon [-\delta, \delta] \longrightarrow R.$$

In other words, the point $x(-\delta;\, x_0) = x_{-\delta} \in D \subset G$. In this way we have

$$(\exists x_{-\delta} \in X_0^+)(\exists \delta = const > 0)\colon$$

$$x(\delta;\, x_{-\delta}) = x(0;\, x_0) = x_0 \in \Phi.$$

This means that $x_{-\delta} \in X_0^+$.

The theorem is proved.

Corollary 3.2.

Assume that:

1. *The condition H2 holds.*
2. *The function* $\varphi \in C[D, R]$.
3. *The set* Φ *is negative reachable from the set* X_0^-.

Then $X_0^- \neq \emptyset$.

Theorem 3.3.

Assume that:

1. *The condition H2 holds.*
2. *The function* $\varphi \in C[D, R]$.
3. *The set* Φ *is positive reachable from the set* X_0^+.

Then $\Phi \subset \overline{X_0^+}$.

Proof. Let x_0 be an arbitrary point from Φ. Then from Definition 2.1 it follows that $x_0 \in G$. We have:

1. $(\exists \delta = const > 0)\colon \phi(t) = \varphi\big(x(t;\, x_0)\big) \in C[-\delta, \delta]$;

2. Then from Theorem 3.1 and Theorem 3.2 it follows that

$$(\forall t;\ -\delta \le t < 0): x(t;\ x_0) \in X_0^+;$$

3. $x(0; x_0\ ) = x_0$;
4. It is valid

$$\lim_{t\to 0, -\delta<t<0} \phi(t) = \phi(0)$$

$$\Leftrightarrow \lim_{t\to 0, -\delta<t<0} \varphi\big(x(t; x_0)\big) = \varphi\big(x(0; x_0)\big) = \varphi(x_0)$$

$$\Leftrightarrow \lim_{t\to 0, -\delta<t<0} x(t; x_0) = x_0.$$

From the four relations above, we obtain that point $x_0 \in \overline{X_0^+}$, i.e. $\Phi \subset \overline{X_0^+}$.

The theorem is proved.

Corollary 3.3.

Assume that:

1. *The condition H2 holds.*
2. *The function $\varphi \in C[D, R]$.*
3. *The set Φ is negative reachable from the set X_0^-.*

Then $\Phi \subset \overline{X_0^-}$.

Theorem 3.4.

Assume that:

1. The conditions H1, H2 and H3 hold.

2. The set Φ is positive reachable from the set X_0^+.

Then X_0^+ is open set.

Proof. If $X_0^+ = G$, the statement of this theorem is trivial. Let $X_0^+ \subset G, X_0^+ \neq \emptyset$ and $X_0^+ \neq G$. Assume that the point $x_0 \in X_0^+$. Using condition 2 of the theorem and Definition 2.1, we have

$$(\exists\theta = \theta(x_0) > 0):$$

$$x_\theta = x(\theta;\ x_0) \in D \text{ and } \varphi\big(x(\theta;\ x_0)\big) = 0.$$

As $x(\theta;\ x_0) \in D$ and φ is defined in the open set D, it follows that the function ϕ is defined in a neighborhood of the point θ. Therefore,

$$(\exists\delta_\theta = const > 0): \phi: [\theta - \delta_\theta, \theta + \delta_\theta] \to R.$$

We have

$$\phi(\theta) = \varphi\big(x(\theta;\ x_0)\big) = 0. \tag{2}$$

Furthermore, it is valid

$$\frac{d}{dt}\phi(\theta) = \frac{d}{dt}\varphi\big(x(\theta; x_0)\big)$$

$$= \langle grad\varphi\big(x(\theta;\ x_0)\big), f\big(x(\theta; x_0)\big)\rangle > 0. \tag{3}$$

From (2) and (3), we have

$$(\exists t', t''; \theta - \delta_\theta < t' < \theta < t'' < \theta + \delta_\theta):$$

$$\phi(t') < 0,\ \ \phi(t'') > 0.$$

This means that

$$x(t'; x_0) \in D,\ \ x(t'';\ x_0) \in D,$$

$$\varphi\big(x(t';\ x_0)\big) < 0,\ \varphi\big(x(t'';\ x_0)\big) > 0. \tag{4}$$

From (4) and using the continuity of function φ, it follows that

$$(\exists\delta_\varphi, 0 < \delta_\varphi < \delta_\theta): \tag{5}$$

- $B_{\delta_\varphi}\big(x(t';\ x_0)\big) \subset D$;
- $B_{\delta_\varphi}\big(x(t'';\ x_0)\big) \subset D$;
- $\Big(\forall x \in B_{\delta_\varphi}\big(x(t';\ x_0)\big)\Big) \Rightarrow \varphi(x) < 0$;
- $\Big(\forall x \in B_{\delta_\varphi}\big(x(t'';\ x_0)\big)\Big) \Rightarrow \varphi(x) > 0$.

According to the theorem of continuous dependence of the solutions of differential equations on the initial condition (see Theorem 7.1 of Chapter 1 of [2]), it follows that:

$\big(\exists\delta_{x_0} = const > 0\big): \big(\forall x_0^*,\ \|x_0^* - x_0\| < \delta_{x_0}\big) \Rightarrow$

$- x_0^* \in G$;

$-\ \ \|x(t; x_0^*) - x(t;\ x_0)\| < \delta_\varphi$, when $0 \le t \le \theta + \delta_\theta$.

From the last inequality, for $t = t'$ and $t = t''$, we obtain

$$\|x(t'; x_0^*) - x(t';\ x_0)\| < \delta_\varphi$$

and

$$\|x(t''; x_0^*) - x(t'';\ x_0)\| < \delta_\varphi\ .$$

By (5), we conclude that

$$x(t'; x_0^*) \in B_{\delta_\varphi}\big(x(t';\ x_0)\big) \Rightarrow \varphi\big(x(t'; x_0^*)\big) < 0$$

and

$$x(t''; x_0^*) \in B_{\delta_\varphi}\big(x(t'';\ x_0)\big) \Rightarrow \varphi\big(x(t''; x_0^*)\big) > 0.$$

Consider the function $\phi^*: \Delta^* \to R$, where $\phi^*(t) = \varphi\big(x(t; x_0^*)\big)$ and $\Delta^* = \{t \in R;\ x(t; x_0^*) \in D\}$. By means of two inequalities above, we get the conclusion $\phi^*(t') < 0$ and $\phi^*(t'') > 0$. Then

$$(\exists\theta^* = \theta^*(x_0^*), t' < \theta^* < t''):$$

$$\varphi\big(x(\theta^*; x_0^*)\big) = \phi(\theta^*) = 0,$$

i.e. $x(\theta^*;\ x_0^*) \in \Phi$. This implies that the point $x_0^* \in X_0^+$. Therefore, the set X_0^+ is open.

The theorem is proved.

Corollary 3.4.

Assume that:

1. The conditions H1, H2 and H3 hold.

2. The set Φ is negative reachable from the set X_0^-.

Then X_0^- is open set.

Theorem 3.5.

Assume that:
1. The conditions H1, H2 and H3 hold.
2. The set Φ is reachable from the sets X_0^+ and X_0^-.
Then the set $X_0 = X_0^- \cup X_0^+ \cup \Phi$ is open.

Proof. Let x_0 be an arbitrary point in X_0. By Theorem 3.4 and Corollary 3.4, we establish that $X_0^- \cup X_0^+$ is open set. Assuming $x_0 \in X_0^- \cup X_0^+$, it follows that x_0 is an inner point of X_0, so in this case the theorem is proved.

Let $x_0 \in \Phi$. We have

$$\phi(0) = 0 \text{ and } \frac{d}{dx}\phi(0) > 0.$$

Therefore,

$(\exists\theta = const > 0)$:

- $\phi(-\theta) < 0$ and $x(-\theta; x_0) \in X_0^+ \cap D$;
- $\phi(\theta) > 0$ and $x(\theta; x_0) \in X_0^- \cap D$.

Since $X_0^- \cap D$ and $X_0^+ \cap D$ are open sets, we have

$$(\exists\varepsilon = const > 0):$$

$$B_\varepsilon\big(x(-\theta;\, x_0)\big) \subset X_0^+ \cap D \text{ and } B_\varepsilon\big(x(\theta;\, x_0)\big) \subset X_0^- \cap D.$$

From the theorem of continuous dependence, it follows that

$(\exists\delta = const > 0)$:

- $B_\delta(x_0) \subset D$;
- $\big(\forall x_0^* \in B_\delta(x_0)\big) \Rightarrow \|x(t;\, x_0^*) - x(t;\, x_0)\| < \varepsilon,$

when $-\theta \le t \le \theta$.

In particular, for $t = -\theta$, we have

$$\|x(-\theta;\, x_0^*) - x(-\theta;\, x_0)\| < \varepsilon$$

$$\Leftrightarrow x(-\theta;\, x_0^*) \in B_\varepsilon\big(x(-\theta;\, x_0)\big)$$

$$\Rightarrow x(-\theta, x_0^*) \in X_0^+ \cap D \Rightarrow \varphi\big(x(-\theta, x_0^*)\big) < 0$$

$$\Leftrightarrow \phi^*(-\theta) < 0,$$

where the function $\phi^*(t) = \varphi\big(x(t;\, x_0^*)\big)$, for $x(t;\, x_0^*) \in D$. In the same way, we obtain

$$\varphi\big(x(\theta;\, x_0^*)\big) > 0 \;\Leftrightarrow\; \phi^*(\theta) > 0.$$

From the continuity of function ϕ^*, we find

$$(\exists\theta^*;\, -\theta < \theta^* < \theta): \phi^*(\theta^*) = 0$$

$$\Leftrightarrow \varphi\big(x(\theta^*; x_0^*)\big) = 0.$$

From the last equality, we conclude that one of the following statements is valid:

- If $\theta^* < 0$, it follows that the set Φ is negative reachable from the point x_0^* i.e. $x_0^* \in X_0^-$;
- If $\theta^* > 0$, we have $x_0^* \in X_0^+$;
- If $\theta^* = 0$, then $x_0^* \in \Phi$ is satisfied.

Thus, we find that $x_0^* \in X_0^- \cup X_0^+ \cup \Phi = X_0$.

Finally, for the point $x_0 \in \Phi$, it is shown that

$$(\exists\delta > 0): \big(\forall x_0^* \in B_\delta(x_0)\big)$$

$$\Rightarrow (x_0^* \in X_0) \Leftrightarrow B_\delta(x_0) \subset X_0.$$

The latter indicates that the point x_0 is an inner point of X_0.

The theorem is proved.

Theorem 3.6.

Assume that:
1. The conditions H1-H4 hold.
2. The set $\Phi$is reachable from the sets X_0^- and X_0^+.
Then the set $X_0 = X_0^- \cup X_0^+ \cup \Phi$ is connected.

Proof. Given two arbitrary points $x_{01}, x_{02} \in X_0$. Consider the case $x_{01}, x_{02} \in X_0^+$. The other cases are treated similarly.

From Definition 2.1, it follows that:

$$(\exists\theta_1 = \theta_1(x_{01}) > 0): x_{\theta_1} = x(\theta_1;\, x_{01}) \in \Phi$$

and

$$(\exists\theta_2 = \theta_2(x_{02}) > 0): x_{\theta_2} = x(\theta_2;\, x_{02}) \in \Phi.$$

Since Φ is a connected set, there exists a continuous curve $\Gamma(x_{\theta_1}, x_{\theta_2})$ with endpoints x_{θ_1} and x_{θ_2}, such that

$$\Gamma(x_{\theta_1}, x_{\theta_2}) \subset \Phi \subset X_0. \tag{6}$$

Consider the curve

$$\Upsilon = \gamma(\theta_1; x_{01}) \cup \Gamma(x_{\theta_1}, x_{\theta_2}) \cup \gamma(\theta_2;\, x_{02}).$$

From Theorem 3.2 and inclusion (6), it follows that $\Upsilon \subset X_0$ Obviously, the points $x_{01}, x_{02} \in \Upsilon$. Since

$$x_{\theta_1} = x(\theta_1;\, x_{01}) \in \gamma(\theta_1;\, x_{01}) \cap \Gamma(x_{\theta_1}, x_{\theta_2})$$

and

$$x_{\theta_2} = x(\theta_2;\, x_{02}) \in \gamma(\theta_2; x_{02}) \cap \Gamma(x_{\theta_1}, x_{\theta_2}),$$

we find that the curve Υ is continuous. So, we establish that the set X_0 is connected.

The theorem is proved.

The following corollary is obtained using Theorems 3.2, 3.5 and 3.6.

Corollary 3.5.

Assume that:
1. The conditions H1-H4 hold.
2. The set Φ is reachable from the sets X_0^- and X_0^+.
Then the set X_0 is nonempty domain.

4. Applications

Example 4.1.

The Lotka-Volterra (LV) model describes the dynamics of an isolated community of type prey-predator without external influences fairly adequately. The corresponding initial value problem has the form:

$$\frac{dm}{dt} = \dot{m} = f_m(m, M) = m(r_1 - q_1 M), \quad (7)$$

$$\frac{dM}{dt} = \dot{M} = f_M(m, M) = -M(r_2 - q_2 m), \quad (8)$$

$$m(0) = m_0,\ M(0) = M_0, \quad (9)$$

where:

- $m = m(t) > 0$ and $M = M(t) > 0$ are the prey and predator biomasses, respectively at the moment $t \geq 0$;
- $r_1 = const > 0$ and $r_2 = const > 0$ are specific growth factors, relevant to the first species (prey) and the second (predator),respectively;
- $q_1 = const > 0$ and $q_2 = const > 0$ are the coefficients indicating interspecies competition. In the common case, they are different for the prey and predator;
- $m_0 > 0$ and $M_0 > 0$ are the prey and predator biomasses at the initial moment $t = 0$.

It is known that, the initial value problem (7), (8), (9) possesses:

1. An unstable (saddle) stationary point $(0, 0)$.
2. A stable stationary point $(m_{00}, M_{00}) = \left({r_2}/{q_2}, {r_1}/{q_1}\right)$.
3. A first integral of the form

$$U(m, M) = q_1 M + q_2 m - r_1 \ln M - r_2 \ln m$$

$$+ r_1\left(\ln {r_1}/{q_1} - 1\right) + r_2\left(\ln {r_2}/{q_2} - 1\right)$$

$$= W(m, M) - W(m_{00}, M_{00}),$$

where

$$W(m, M) = q_1 M + q_2 m - r_1 \ln M - r_2 \ln m;$$

4. For any point

$$(m, M) \in R^+ \times R^+, (m, M) \neq (m_{00}, M_{00}),$$

the inequality $U(m, M) > 0$ is valid. It is fulfilled $U(m_{00}, M_{00}) = 0$;

5. For any constant $c \geq 0$, the implicitly defined curve

$$\gamma_c = \{(m, M): U(m, M) = c\}$$

is a trajectory of system (7), (8) with properly chosen initial condition ($U(m_0, M_0) = c$ is sufficient);

6. For any constant $c > 0$, the set

$$D_c = \{(m, M): U(m, M) < c\}$$

is a simply connected domain located in $R^+ \times R^+$ with contour $\partial D_c = \gamma_c$;

7. For any constant $c > 0$, it is fulfilled $(m_{00}, M_{00}) \in D_c$;
8. If $0 < c_1 < c_2$, then $\gamma_{c_1} \subset D_{c_2}$.

Let c_G be an arbitrary positive constant. We define the phase space of system (7), (8) as follows: $G = D_{c_G}$. Let the function $\varphi(m, M) = M - M_{00}$ be defined in the domain

$$D = (I_m \times R^+) \cap G,$$

where the endpoints of the open interval

$$I_m = \{m;\ m_{min} < m < m_{max}\}$$

satisfy the inequalities

$$m_{00} = \frac{r_2}{q_2} < m_{min} < m_{max},$$

$$W(m_{max}, M_{00}) - W(m_{00}, M_{00}) < c_G$$

$$\Leftrightarrow q_2 m_{max} - r_2 \ln m_{max} < r_2\left(1 - \ln {r_2}/{q_2}\right) + c_G$$

The reachable set Φ has the form

$$\Phi = \langle (m, M);\ M = M_{00}, m \in I_m \rangle$$

We will show that the system considered satisfies the conditions H1-H4. Actually, the conditions H2 and H4 are verified immediately. The condition H1 follows from the continuous differentiability of the right hand side of system (7), (8) in $R^+ \times R^+$ and the fact that the closure of phase space $\bar{G} \subset R^+ \times R^+$ is compact. For $(m, M) \in \Phi$, we have

$$\langle grad\varphi(m, M), f(m, M) \rangle$$

$$= \langle (0, 1), \left(f_m(m, M_{00}), f_M(m, M_{00})\right) \rangle$$

$$= -M_{00}(r_2 - q_2 m) = M_{00} q_2 (m - m_{00})$$

$$\geq r_2(m_{min} - m_{00}) > 0,$$

Whereby, it is shown that condition H3 is valid. Therefore, system (7), (8) satisfies the propositions proved in the previous section. More precisely, the initial sets X_0^-, X_0^+ and X_0 are nonempty and open. Beside this X_0 is connected. We derive that

$$X_0^- = X_0^+ = X_0 \setminus \Phi = \left(D_{c_{max}} \setminus D_{c_{min}}\right) \setminus \Phi,$$

Where $c_{min} = U(m_{min}, M_{00})andc_{max} = U(m_{max}, M_{00})$.

Example 4.2.

Consider the Volterra-Gause-Witt (VGW) model, which describes the properties of predator-prey type of interaction between two species provided without external influences. The populations of both species are isolated. We have

$$\frac{dm}{dt} = r_1 m\left(k_1 - \frac{m}{k} - g_2(M)\right), \quad (10)$$

$$\frac{dM}{dt} = -r_2 M\left(k_2 - g_1(m)\right), \quad (11)$$

$$m(0) = m_0,\ M(0) = M_0, \quad (12)$$

where:

- $m = m(t) > 0$ and $M = M(t) > 0$ are the prey and predator biomasses, respectively at the moment $t \geq 0$;
- $r_1 = const > 0$ and $r_2 = const > 0$ are specific growth factors. They show the inherent capacity (rate) of the mass increasing, corresponding to each of the species of

community;

- $k_1 = const > 0$ and $k_2 = const > 0$ are coefficients indicating the capacity of the environment. Upon reaching these values of the biomass community, the direction of growth of the respective type is changed;

- k is a positive coefficient, indicating the level of saturation of the prey. It is clear that for $k \longrightarrow \infty$, the VGW model is transformed into the LV model;

- The functions $g_1, g_2: R^+ \longrightarrow R^+$ are analytic and they render an account the level of interspecies competition;

- $m_0 > 0$ and $M_0 > 0$ are the biomass quantities of the prey and predator at the initial moment $t = 0$.

We will explore the concrete realization of system (10), (11). Assume that $r_1 = r_2 = {}^1\!/_2$; $k_1 = k_2 = 1$; $k = {}^3\!/_2$; $g_1(m) = {}^3\!/_{20}\, m$; $g_2(M) = M$.The system becomes

$$\frac{dm}{dt} = \frac{m}{2}\left(1 - \frac{2m}{3} - M\right), \tag{13}$$

$$\frac{dM}{dt} = -\frac{M}{2}\left(1 - \frac{3m}{20}\right). \tag{14}$$

Introduce the function $\varphi(m, M) = {}^1\!/_2 - M$, where $(m, M) \in (0,1) \times (0,1) = D$. It is clear that the reachable set is $\Phi = \left\{(m, M); M = \frac{1}{2}, 0 < m < 1\right\}$. Verifying the conditions H1-H4 is simple. Indeed, the right hand side f of system (13), (14) is continuously differentiable in $R \times R$ and therefore, also in $G = (0,1) \times (0,2)$. Since G is a bounded domain, then f is a Lipchitz function in G. The conditions H1 and H2 are valid. Obviously, the set Φ is connected, i.e. the condition H4 is fulfilled. The function φ is differentiable in D. Moreover for $(m, M) \in \Phi$, we have

$$\begin{aligned} &\langle grad\ \varphi(m, M), f(m, M)\rangle \\ &= \left\langle (0, -1), \left(\frac{m}{4} - \frac{m^2}{3}, -\frac{1}{4} + \frac{3m}{80}\right)\right\rangle \\ &= \frac{1}{4} - \frac{3m}{80} > \frac{17}{80} > 0, \end{aligned}$$

whence, condition H3 is valid.

Using the results, obtained from the previous section, we conclude that the initial sets X_0^-, X_0^+ and X_0 are nonempty and open. Moreover, X_0 is connected.

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