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On the survival of insurance company's investment with consumption under power and exponential utility functions

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Abstract: In this paper, the survival of insurance company's investment with consumption is investigated under power and exponential utility functions. We take the risk reserve of an insurance company to follow Brownian motion with drift and tackle an optimal portfolio selection problem of the company. The investment case considered was insurance company that trades two assets: the money market account (bond) growing at a linear rate r and a risky stock with an investment behavior in the presence of a stochastic cash flow or a risk process, continuously in the economy.Under these functions, we obtained the optimal strategies. It is discovered that both utility functions are alike.

Keywords: Stochastic Optimal Control, Company's Investment With Consumption, Power Utility Function, Exponential Utility Function

1. Introduction

To approximate the risk process of an insurance company by Brownian motion, consider a sequence of risk process $R_n(t)$ defined in the following way;

$$R_n(t) = u_n + c_n t - \sum_{i=1}^{N(t)} Y_k{}^{(n)}, \qquad (1)$$

where$u_n$is the initial risk reserve of the insurance company, c_n is the gross risk premium per unit time paid by the policy holders and the sequence $\{Y_k{}^{(n)}: k \in N\}$,describes the consecutive claim sizes. Assume also that $E\left(Y_k{}^{(n)}\right) = \mu_n$and $var\left(Y_k{}^{(n)}\right) = \sigma_n{}^2$.

The point process $N = \{N(t) : t \geq 0\}$ counts claims appearing up to time t, that is

$$N(t) = \max\{k: \sum_{i=1}^{k} T_i \leq t\}, \qquad (2)$$

where$\{T_k: k \in N\}$ is an identically independent sequence of non negative random variables describing the times between the arriving claims with $E(\{T_k) = \frac{1}{\lambda} \geq 0$.

If T_k are exponentially distributed then $N(t)$ is a Poison process with intensity λ.

The sequence of classical reserve processes converges weakly to a stochastic process of the form;

$$R_t = u + \Gamma + \sigma \lambda^{\frac{1}{2}} B_t^{(1)}. \qquad (3)$$

Where $\Gamma = (\Gamma_t)_{t \geq 0}$ with $\Gamma_t = (c - \lambda\mu)t$and $(B_t^{(1)})_{t \geq 0}$ is a standard Brownian motion (Iglehart , 1969). In recent years, many authors have reached a significant plateau in modeling the probability of ruin of the insurance company (see for example;Azcue and Muler,2009; Bai and Liu, 2007; Bayraktar and Young, 2008;Gaier and Grandits, 2002). On the other hand, Oksendal and Sulem (2002) considered an investor who consumes from a bank account and has the opportunity at any time to transfer funds between two assets, with the objective to maximize the cumulative expected utility of consumption over planning horizon.

Kostadinova (2007) considered a stochastic model for the wealth of an insurance company which has the possibility to invest into a risky asset and a risk-less asset under constant mix strategy and provided an approximation of the optimal investment strategy,that maximizes the expected wealth of the insurance company under the risk constraint on the Value-at-Risk.

When the growing rate of return is a linear function of t, and in the case of no consumption cost, Osu and Ihedioha (2013a, 2013b) obtained a strategy that optimizes the probability of achieving a given upper wealth level before hitting a given lower level. They established among others that, the optimized investment in the assets and the optimal value function are dependent on horizon and the wealth. It is recommended that the managers of the assets of the insurance company should take into consideration this horizon dependency when making policy decisions.

In this paper however, the survival of insurance company's investment with consumption is investigated under power and exponential utility functions. Under these functions, we obtained the optimal strategies. It is discovered that they are alike.

2. The Model

Adapting the formulation of Osu and Ihedioha (2012), we assume that insurance company trades two assets continuously in the economy. The first asset is the money market account (bond) growing at a rate r_t that is linear function of time ($r_t = \varepsilon + \sigma t$), instead of a constant as in Wokiyi(2012). $r_t = \varepsilon + \sigma t$, *($\alpha > 0, 0 \leq \sigma \leq 1$) is a decreasing (or an increasing) linear* function of $tast \to \infty$ .The parameter$\varepsilon$is the initial investment on the money market account which determines the speed of a mean-reversion to the stationary level. $\sigma$is the acceleration coefficient which is the volatility (variance) of the process and is proportional to the level of the interest rate. It decreases as the interest rate $r_t \to 0$. The equation governing the dynamics of the money market account (bond) is given as;

$$dB_t = (\varepsilon + \sigma t)B_t dt. \tag{4}$$

We assume that there is only one risky stock available for investment (e.g a mutual fund) whose price at any time t will be denoted by S_t. We will also assume that the price process of the risky stock follows the geometric Brownian motion:

$$dS_t = S_t dZ_t \ , \tag{5}$$

whereZ_t is a Brownian motion with drift $\mu$and diffusion parameter σ, that is, $dZ_t = \mu dt + \sigma dB_t^{(2)}$, where μ and σ are constants and $B_t^{(2)}: t \geq 0$ is a standard Brownian motion.

In classical theory of risk, the true net claims process say $\{R_t\}$ is usually modeled as;

$$R_t = u + ct - \sum_{i=1}^{N(t)} Y_i. \tag{6}$$

where u is the initial risk reserve, c is the premium income rate per unit time, N_t is the number of claims up to time t usually modeled as a stationary renewal process with rate λ and Y_i is the size of thei^{th} claim with $\{Y_i: i \geq 1\}$ assumed to be an identically independent sequence as shown in the previous section, $\alpha = c - \mu\lambda$ and $\beta^2 = \sigma^2\lambda$ and these can also be written as $\alpha = c\lambda E(Y_1)$and$\beta^2 = \lambda E({Y_1}^2)$,
so the parameter α can be understood as the relative safety loading of the claims process.

We are concerned with investment behavior in the presence of a stochastic cash flow or a risk process which we will denote by $R_t : t \geq 0$ which describes a Brownian motion with drift$\alpha$and diffusion parameter σ that is R_t satisfies the stochastic differential equation;

$$dR_t = \alpha dt + \beta dB_t^{(1)} \tag{7}$$

Whereα and$\beta$are constants (with $\beta \geq 0$).

We also allow the two Brownian motions to be correlated and we denote their correlation coefficient byρthat is $E(B_t^{(1)} B_t^{(2)}) = \rho t$. We will not consider the uninteresting case of ρ^2, in which case there would be only one source of randomness in the model.

The company is allowed to invest its surplus in the risky stock and we will denote the total amount of money invested in the risky stock at time t under an investment policy π as π_t where $\{\pi_t\}$ is a suitable admissible adapted control process, that is, π_t is a non-anticipative function and satisfies for any T,almost surely.

$$\int_0^T {\pi_t}^2 \mathrm{dt} < \infty \ , \tag{8}$$

We assume that W_t is the total wealth of an insurance company. We also assume that the insurance company allocates its wealth as follows: Let $\pi_t$be the total amount of the company's wealth that is invested in risky assets and remaining balance $(W_t - \pi_t)$ be invested in a risk-less asset (bond/market).

We note that $\pi_t$may become negative, which is to be interpreted as short selling a stock. The amount invested in the bond, $W_t - \pi_t$ may also be negative, and this amounts to borrowing at the interest rate r. For any policy π, the total wealth process of an insurance company evolves according to the stochastic differential equation as;

$$dW_t^\pi = \pi_t \frac{dS_t}{S_t} + (W_t - \pi_t)\frac{dB_t}{B_t} + dR_t. \tag{9}$$

Substituting the expressions for S_t, B_t and R_t, the stochastic differential equation for the wealth process of the company then reduces to;

$$dW_t^\pi = \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha\right]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}. \tag{10}$$

Assuming $B_t^{(1)}$ and $B_t^{(2)}$ are correlated standard Brownian motions, with correlation coefficient ρ ,the quadratic variation of the wealth process is;

$$d < W >_t = ({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)dt \tag{11}$$

Definition: A control process π_t is said to be admissible for an initial endowment $w \geq 0$ if the wealth process generated by the stochastic differential equation (10) satisfies, $W_t \geq 0; 0 \leq t \leq T$; almost surely. Then the quadratic variation of the wealth process is given by;

$$d < W >_t = dW_t^\pi . W_t^\pi = ({\pi_t}^2\sigma^2)\rho dt + 2\pi_t \sigma\beta\rho dt + \beta^2 B_t^{(1)} B_t^{(2)}, \quad (12)$$

for

$$dB_t^{(1)} dB_t^{(1)} = dB_t^{(2)} dB_t^{(2)} = dt,\ dtdB_t^{(1)} = dtdB_t^{(2)} = dtdt = 0 \text{and } dB_t^{(1)} dB_t^{(2)} = \rho dt,$$

since $B_t^{(1)}$ and $B_t^{(2)}$ are correlated Brownian motions, with correlations coefficient ρ.

If $\rho^2 \neq 1$, this model is incomplete in a very strong sense in that the random cash flow or the random endowment R_t, can not be traded on the security market, and therefore the risk to the investor cannot be eliminated under any circumstance. We put no constraints on the control π_t except for the particular case where the possibility of borrowing is not allowed, we allow $\pi < 0$as well as $\pi_t > W_t^\pi$. In the first instance the company is shorting stock while in the second instance the company borrows money to invest long in the stock.

The company can always borrow money for as long as it has a positive net worth, that is, $W_t^\pi > 0$ and we don't allow the company to borrow money once it's bankrupt and thus the possibility of ruin is of real concern.

Suppose the investor has a power utility function, the Arrow-Pratt measure of relative risk aversion (RRA) or coefficient of relative risk averse is defined as;

$$R(w) = \frac{U''(w)}{U'(w)}, \quad (13)$$

where$w$ is the wealth level of an investor and a, k are constants. We consider a special case where the utility function is of the form,

$$U(w) = \frac{w^{1-\gamma} - q}{1-\gamma} , \quad (14)$$

which has a constant relative risk averse parameter c. The motivation to use power utility stems from the fact that power utility functions with a constant relative risk averse are related to survival as well as growth objectives that may be taken up by a prospective investor.

Consider a discrete time and space ordinary investor $(\alpha = \beta = 0)$, that is no external risk process facing favorable investment and then when the investor has an exponential utility function say $U(w) = -e^{-\theta w}$ and is interested in maximizing the utility of his terminal fortune at a fixed terminal time, the optimal policy is to invest a fixed constant. Such a strategy is asymptotically optimal in general for the criteria of maximizing the probability of ruin for some value θ.

A stronger form of the conjecture in continuous time for a more complicated model was proved by Browne (1995).

He showed that the policy that maximizes exponential utility of terminal wealth at a fixed time is exactly equivalent to the policy that minimizes the probability of ruin for a specific value of θ.

Suppose now that the investor is interested in maximizing the utility of his wealth say at time T. The utility function is $U(w)$ and satisfies $U' > 0$and $U'' < 0$.

Let $V(t, w)$ denote the maximal utility attainable by the investor from the state w at time t.

That is, $V(t, w) = sup_\pi E(U(W_t^\pi)|W_t^\pi = w$ and let $\pi_t^* : \{0 \leq t \leq T\}$ denote the optimal investment policy.

We suppose now that the investor has an exponential utility function: $U(w) = \lambda e - \frac{\gamma}{\theta} e^{-\theta w}$

where $\theta > 0$. Exponential utility implies constant absolute risk averse, with coefficient of absolute risk aversion equal to a constant: $-\frac{U''(w)}{U'(w)} = \theta$.

In the standard model of one risky asset and one risk-free asset, this implies that the optimal holding of a risky asset is independent of the level of initial wealth; thus on the margin any additional wealth would be allocated totally to the additional holdings of the risk-free asset.

The most straight forward implication of increasing or decreasing the relative risk averse, and the ones that motivate a focus on these concepts, occur in the context of forming a portfolio with one risky asset and risk free asset. If an investor experiences an increase in wealth he will choose to increase (or keep unchanged or decrease) the fraction of the portfolio held in the risky asset if the relative risk averse is decreasing (or constant, or increasing).

The insurance company's problem can therefore be written as:

$$\left.\begin{matrix} sup_\pi\{\mathcal{A}^\pi V(t,w)\} = 0 \\ V(T,w) = U(w) \end{matrix}\right\},$$

where

$$V(t,w) = sup_\pi E^{(t,w)}[U(W_T^\pi)] \quad (16)$$

subject to:

$$dW_t^\pi = \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha\right]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}.$$

2.1. The Problem

The insurance company chooses optimal investment strategies so as to maximize the final wealth at a deterministic time T.

Define the value function at time T as;

$$J(W,t;T) = sup_\pi E(U(W_t^\pi)|W_t^\pi = w , \quad (17)$$

subject to:

$$dW_t^\pi = \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha - c_t\right]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}$$

Assumption 1*:* The insurance company makes intermediate consumption decision on the admissible consumption space, which satisfies $\int_0^t |c_s| ds < \infty, \forall t \in [0,T]$.

Assumption 2*:* Consumption is made through the money market account (bond).

The problem then becomes:

$$JJ(W,t;\,T) = sup_{\pi}E\left[\int_0^T e^{-\rho\tau}\frac{c_\tau{}^{1-\gamma}-q}{1-\gamma}d\tau + e^{-\rho T}\frac{w_T{}^{1-\gamma}-q}{1-\gamma}\right] \quad (18a)$$

subject to:

$$dW_t^{\pi} = \left[w(\varepsilon+\sigma t) + \pi_t\big(\mu-(\varepsilon+\sigma t)\big) + \alpha - c_t\right]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}$$

The value function should also satisfy the terminal condition:

$$J(W,T;\,T) = \frac{w_T{}^{1-\gamma}-q}{1-\gamma}. \quad (19a)$$

In this case and under the exponential utility, given the two assumptions above, the insurance company's problem becomes;

$$J(W,t;\,T) = sup_{\pi}E\left[\int_0^T e^{-\rho\tau}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta c_\tau}\right)d\tau + e^{-\rho T}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w_T}\right)\right]$$

$$dJ = \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial w}dw + \frac{1}{2}\frac{\partial^2 J}{\partial w^2}(dw)^2 \ , \quad (24)$$

applying this to the Bellman equation, we get the corresponding HJB equation:

$$\frac{C^{1-\gamma}-q}{1-\gamma} + J_t + J_W\big(w(\varepsilon+\sigma t) + \pi_t\big(\mu-(\varepsilon+\sigma t)\big) + \alpha - c_t\big) + \frac{1}{2}(\pi_t{}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)J_{WW} - \vartheta J = 0. \quad (25)$$

Assuming the insurance company is trying to maximize the probability of beating a given benchmark by some percentage before going below it by another percentage, this objective is related to the case of manager who stands to receive a bonus achieving the benchmark by a predetermined percentage.

The formalization of this problem takes the form; let $J^*(w)$ denote the maximal probability of beating the benchmark when starting from state w before being beaten by it. That is, let $W_0 = w$ and let $p \text{ and } q$ be given constants with $p < w < q$ such that;

$$J(w) = sup_{\pi}P_w(\tau_q^{\pi} < \tau_p^{\pi}). \quad (26)$$

The HJB equation(25);

$$\frac{C^{1-\gamma}-q}{1-\gamma} + V_t + \left[w(\varepsilon+\sigma t) + \pi_t\big(\mu-(\varepsilon+\sigma t)\big) + \alpha - c_t\right]V_W + \frac{1}{2}\left[(\pi_t{}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)\right]V_{WW} - \vartheta J = 0,$$

will now be subject to the boundary conditions; $V(p) = 0$; $V(q) = 1$ for $p < w < q$. Since $V(w)$ in this case is independent of time, the equation above reduces to;

$$\frac{c^{1-\gamma}-q}{1-\gamma} + \left[w(\varepsilon+\sigma t) + \pi_t\big(\mu-(\varepsilon+\sigma t)\big) + \alpha - c_t\right]J_W + \frac{1}{2}\left[(\pi_t{}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)\right]J_{WW} - \vartheta J = 0, \quad (27)$$

for $p < w < q$.

Let there exist a classical solution V, guided by the nature of the objective function, restriction and the terminal condition,that satisfies $V_{WW} < 0$ given by;

$$J(w) = g\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right], where\ g\ is\ a\ constant, \quad (28)$$

such that;

$$\left.\begin{aligned} J_W &= gkw^{-\gamma} \\ J_{WW} &= -cgkw^{-\gamma-1} \end{aligned}\right\}. \quad (29)$$

The new HJB equation

$$\frac{c^{1-\gamma}-q}{1-\gamma} + [w\varepsilon + \pi_t(\mu-\varepsilon) + \alpha - c_t]gw^{-\gamma} - \frac{\gamma}{2}(\pi_t{}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)gw^{-\gamma-1} - \vartheta g\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right] = 0. \quad (30)$$

The optimal consumption is obtained using the first order condition on c as follows; $gw^{-\gamma} = c_t^{-\gamma}$

$$c_t^* = g^{\frac{-1}{\gamma}}w \quad . \quad (31)$$

Substituting this optimal value c_t^* into the HJB equation (30), we get;

$$\frac{g^{\frac{-(1-\gamma)}{\gamma}}w^{1-\gamma}}{1-\gamma} - \frac{q}{1-\gamma} + [w\varepsilon + \pi_t(\mu-\varepsilon) + \alpha]gw^{-\gamma} - g^{\frac{-(1-\gamma)}{\gamma}}w^{1-\gamma} - \frac{\gamma}{2}(\pi_t{}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)gw^{-\gamma-1} - \vartheta g\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right] = 0.$$

So that

$$\frac{\gamma w^{1-\gamma}}{1-\gamma}g^{\frac{-(1-\gamma)}{\gamma}} = \left\{\begin{aligned} &[w\varepsilon + \pi_t(\mu-\varepsilon) + \alpha] - \frac{\gamma}{2}(\pi_t{}^2\sigma^2 + \beta^2 + \\ &2\sigma\rho\beta\pi_t)w^{-\gamma-1} - \vartheta\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right]w^{-\gamma} + x' \end{aligned}\right\}g \quad (32)$$

Applying the first order condition on π_t to get the optimal value of π_t, we have;

$(\mu-\varepsilon)\,gw^{-\gamma} - \frac{\gamma}{2}(2\pi_t\sigma^2 + 2\sigma\rho\beta)gw^{-\gamma-1} = 0$, from which

$$\pi_t^* = \left[\frac{(\mu-\varepsilon)}{\gamma\sigma^2} - \frac{\beta\rho}{\sigma}\right]. \quad (33)$$

Rearranging (32), we get,

$$g^{\frac{\gamma-1}{\gamma}} = \frac{D}{A}g \quad (34)$$

where, $A = \frac{w^{1-\gamma}-q}{1-\gamma}$, and $D = \frac{[w\varepsilon + \pi_t(\mu-\varepsilon)+\alpha] - \frac{\gamma}{2}({\pi_t}^2\sigma^2+\beta^2+2\sigma\rho\beta\pi_t)w^{-\gamma-1} - \vartheta\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right]w^{-\gamma}+x'}{}$.

So ;

$$g = s^{-\gamma}. \quad (35)$$

Where $s = \frac{D}{A}$.

Therefore;

$$J(w) = s^{-\gamma}\left[\frac{w^{1-\gamma}-q}{1-\gamma}\right] \quad (36)$$

The application of the boundary conditions yields;

$J(l) = s^{-\gamma}\left[\frac{l^{1-\gamma}-q}{1-\gamma}\right] = 0,$

implying,

$$q = l^{1-\gamma} \quad (37)$$

and $J(m) = s^{-\gamma}\left[\frac{m^{1-\gamma}-q}{1-\gamma}\right] = 1,$

implying,

$$s^{-\gamma} = \frac{1-\gamma}{m^{1-\gamma}-l^{1-\gamma}} \quad (38)$$

The optimal value function is then given as;

$$J^*(w) = \frac{(1-\gamma)}{[m^{1-\gamma}-l^{1-\gamma}]}\left[\frac{w^{1-\gamma}-l^{1-\gamma}}{(1-\gamma)}\right]$$

$$= \left[\frac{w^{1-\gamma}-m^{1-\gamma}}{m^{1-\gamma}-l^{1-\gamma}}\right], \quad (39)$$

which satisfies the boundary conditions.

3.2. The Case of Exponential Utility Function

In this case, the value function is;

$$J(w) = \varphi\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right). \quad (40)$$

The insurance company's problem becomes;

$$J(W,t;\,T) = sup_{\pi}E\left[\int_0^T e^{-\rho\tau}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta C_\tau}\right)d\tau + e^{-\rho T}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w_T}\right)\right] \quad (41)$$

subject to:

$$dW_t^{\pi} = \left[w(\varepsilon+\sigma t) + \pi_t\big(\mu - (\varepsilon+\sigma t)\big) + \alpha - c_t\right]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}$$

The value function should satisfy the terminal condition: $J(W,T;\,T) = \lambda - \frac{\gamma}{\theta}e^{-\theta W_T}$.

The HJB equation becomes;

$$\varphi\left(\lambda - \frac{\gamma}{\theta}e^{-\theta c}\right) + [\mu\pi_t + (w-\pi_t)(\varepsilon+\sigma t)\alpha - c_t]J_W + \frac{[(\mu\pi_t)^2+\beta^2+2\sigma\beta\rho\pi_t]}{2}J_{WW} - \vartheta J = 0. \quad (42)$$

The optimal consumption here is obtained using the first order condition, thus; $\gamma e^{-\theta c} - J_W = 0$

$$c^* = \ln\left(\frac{\gamma}{J_W}\right)^{\frac{1}{\theta}}. \quad (43)$$

Considering the nature of the objective function, the restriction and the terminal condition, let;

$G(w,t;\,T) = \left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right]f(t;T)$, where $f(t;T)$ is a function of time, be the new value function, then with;

$$G_t = f'\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right];\, G_w = \gamma e^{-\theta w}f;\, G_{ww} = -\theta\gamma e^{-\theta w}f$$

The HJB equation becomes;

$$\lambda - \frac{\gamma}{\theta}e^{-\theta c} + f'\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right] + [\mu\pi_t + (w-\pi_t)(\varepsilon+\sigma t)\alpha - c_t]\gamma e^{-\theta w}f$$

$$-\theta\gamma e^{-\theta w}\frac{[(\mu\pi_t)^2+\beta^2+2\sigma\beta\rho\pi_t]}{2}f - \vartheta\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right]f = 0. \quad (44)$$

Differentiating with respect to π_t and simplifying yields;

$$[\mu - (\varepsilon+\sigma t)]\gamma e^{-\theta w}f = \frac{1}{2}[2\pi_t\sigma^2 + 2\sigma\rho\beta)]\theta\gamma e^{-\theta w}f.$$

$$\pi_t^* = \left[\frac{[\mu-(\varepsilon+\sigma t)]}{\theta\sigma^2} - \frac{\beta\rho}{\sigma}\right]. \quad (45)$$

This is independent of the wealth at hand unlike the case of power utility which is dependent on the wealth.

Lemma1: The optimal value function is given by;

$$G(w,t;\,T) = D'\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right]e^{-\left[\frac{b(T-t)}{a}+\frac{c}{2a}(T^2-t^2)\right]}\int_t^T e^{-\left[b(T-\tau)+\frac{c}{2}(T^2-\tau^2)\right]}d\tau$$

Proof:

On simplifying equation (44); we get;

$$af' + (b+ct)f = d \quad (46)$$

where,

$a = \left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right]$; $(b+ct) = [\mu\pi_t + (w-\pi_t)(\varepsilon+\sigma t)\alpha - c_t]\gamma e^{-\theta w} - \theta\gamma e^{-\theta w}\frac{[(\mu\pi_t)^2+\beta^2+2\sigma\beta\rho\pi_t]}{2} - \vartheta\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right]$ and $d = \lambda - \frac{\gamma}{\theta}e^{-\theta c}$,

a first order linear differential equation with integrating factor;

$$R(t) = e^{\int_t^T \frac{(b+ct)}{a}d\tau}$$

$$R(t) = e^{\frac{b(T-t)}{a}+\frac{c(T^2-t^2)}{2a}} \quad (47)$$

Therefore, the solution to the linear differential equation (46) is;

$$R(t)f = \int_t^T R(t).\frac{d}{a}d\tau\, e^{\frac{b(T-t)}{a}+\frac{c(T^2-t^2)}{2a}}f$$

$$= \int_t^T e^{\frac{b(T-\tau)}{a}+\frac{c(T^2-\tau^2)}{2a}}.\frac{d}{a}d\tau$$

$$f = D'e^{-\left(\frac{b(T-t)}{a}+\frac{c(T^2-t^2)}{2a}\right)}\int_t^T e^{\frac{b(T-\tau)}{a}+\frac{c(T^2-\tau^2)}{2a}}d\tau, \qquad (48)$$

for which when $t \to T; f \to 1$ and,

$$G(w,t;\,T) = \left[\lambda-\frac{\gamma}{\theta}e^{-\theta w}\right]f(t;T)$$

$$= \left[\lambda-\frac{\gamma}{\theta}e^{-\theta w}\right]D'e^{-\left(\frac{b(T-t)}{a}+\frac{c(T^2-t^2)}{2a}\right)}\int_t^T e^{\frac{b(T-\tau)}{a}+\frac{c(T^2-\tau^2)}{2a}}d\tau.$$

So, the optimal value function is then given as;

$$G(w,t;\,T)$$

$$= D'\left[\lambda-\frac{\gamma}{\theta}e^{-\theta w}\right]e^{-\left[\frac{b(T-t)}{a}+\frac{c}{2a}(T^2-t^2)\right]}\int_t^T e^{-\left[\frac{b}{a}(T-\tau)+\frac{c}{2a}(T^2-\tau^2)\right]}.$$

4. Conclusion

In this paper, the problem of optimizing investment returns and probability of survival of an insurance company with time–varying rate of return was dealt with for two utility functions (power and exponential utility functions). The main emphasis has been on how the two utility functions affect the insurance company's portfolio selection given investment choices. Also, the work investigated how utility functions affect the probability of survival of the insurance investor. The proportions for optimizing the probability of survival are all observed to be constant proportions of the investor's total wealth. So, utility optimization is related to probability of survival optimization for both utility functions.

Furthermore, it was observed that the optimal value function of the probability of survival under both utility functions, are same.

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