Analytical expressions for commensal-host ecological model: Homotopy perturbation method

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Abstract: In this paper a mathematical commensal-host ecological model with replenishment rate for both species is discussed. This model is characterized by a pair of first order non-linear coupled differential equations. The non-linear coupled system-equations are solved analytically by using Homotopy perturbation method. Further, our results are compared with the previous work and a satisfactory agreement is noted.

Keywords: Mathematical Model, Commensal, Host, Replenishment Rate, Non-Linear Differential Equations, Homotopy Perturbation Method

1. Introduction

The analytical solutions for the first order non-linear coupled differential equations are not possible directly because of intractability of non-linear terms. Techniques like perturbation to linearizing the non-linear coupled differential equations to give qualitative nature of solutions in treatises of Cushing [3], Phanikumar Kapoor [4], Meyer [6], Phanikumar and Pattabhi Ramacharyulu [7].

The previous investigation presented [1] a four stage recursive procedure to give an approximate solution for a first order non-linear coupled differential equations and same is applied to solve the commensal-host ecological model with replenishment rate for both the species. A similar algorithm was earlier developed by Langlosis and Rivlin [5] to obtain approximate solution for the flows of slightly visco-elastic fluids. Srinivas [8] and Bhaskara Rama Sarma [2] adopted this algorithm for a general two species eco system.

The purpose of this paper is to derive the approximate analytical solution of the populations of the two species for all values of the dimensionless parameters using the Homotopy perturbation method.

2. The Basic Model Equation

The model equations are obtained as

\[ \frac{dN_1(t)}{dt} = a_1 N_1(t) - a_{11} N_1^2(t) + a_{12} N_1(t) N_2(t) + h_1 \]  
\[ \frac{dN_2(t)}{dt} = a_2 N_2(t) - a_{22} N_2^2(t) + h_1 \]  

With the initial conditions

\[ N_1(0) = N_{10}; \quad N_2(0) = N_{20} \]  

where \( N_1(t) \) and \( N_2(t) \) are the population of the species \( S_1 \) and \( S_2 \) respectively at time \( t \), \( a_1 \) and \( a_2 \) are the natural growth rates of species \( S_1 \) and \( S_2 \) respectively, \( a_{11} \) and \( a_{22} \) are the rates of decrease of species \( S_1 \) and \( S_2 \) respectively due to limited resources, \( a_{12} \) is the inhibition coefficient, \( h_1 \) and \( h_2 \) are the replenishment rates of \( S_1 \) and \( S_2 \) respectively. All the coefficients \( a_1,a_2,a_{11},a_{12},a_{22}, h_1 \) and \( h_2 \) are positive values.

3. Solution of the Non-Linear Initial Value Problem Using the Homotopy Perturbation Method (HPM)

Recently, many authors have applied the Homotopy perturbation method (HPM) to solve the non-linear problem in physics and engineering sciences [9-12]. This method is also used to solve some of the non-linear problem in physical
sciences [13-15]. This method is a combination of Homotopy in topology and classic perturbation techniques. Ji-Huan He used to solve the Lighthill equation [13], the Duffing equation [14] and the Blasius equation [15-16]. The HPM is unique in its applicability, accuracy and efficiency. The HPM uses the imbedding parameter $p$ as a small parameter, and only a few iterations are needed to search for an asymptotic solution. Using this method [17-19], we can obtain solution to the eqns. (1) - (3) as follows:

$$N_1(t) = \left(k_0 e^{a_1 t} - k_1\right) + A e^{a_1 t} + \left[k_1 a_1^2 t - a_1 k_1 u_1 \right] + \left[2 a_1 k_0 t - a_1 k_0 u_1 \right] e^{a_1 t}$$

$$N_2(t) = \left[u_0 e^{a_2 t} - u_1\right] + \left[a_22 / a_2\right] \left[u_0^{2} - u_1^{2}\right] e^{a_2 t} - \left[u_0^{2} - u_1^{2}\right] e^{a_2 t}$$

where

$$k_0 = N_{10} \frac{b_1}{a_1}, \quad k_1 = \frac{b_2}{a_2}, \quad u_0 = N_{20} - \frac{b_2}{a_2}, \quad u_1 = \frac{b_2}{a_2}$$

$$A = \left[\frac{a_1 k_1 a_1 + a_1 k_2^2 - a_1 k_1^2}{a_1} + a_1 k_0 a_1\right] - \left[\frac{a_22 k_0 a_0}{a_2}\right]$$

4. Results and Discussions

The variation of $N_1$ and $N_2$ verses time $t$ in the interval $[0, 1]$ is computed analytically employing Homotopy perturbation method for a wide range of values of the characterizing the parameters $a_1$, $a_2$, $a_1$, $a_2$ and $a_2$, as shown in Table-1. From Fig.1-8, it is observed that when the growth rate of host increases the $N_2$ curves are falling with increasing steepness. Both $N_1$ and $N_2$ are decreases with time $t$ because of utilization of energy. For $a_{12} < 1$, weak commensalism of host over commensal $N_2$ falls with slower rate than $N_1$. However with increasing growth rate of host steepness increase for the $N_2$ curves and falling rate is decreased comparison with $N_1$. With increasing $a_{22}$ steepness decreases.

For $a_{12} = 1$, the steepness of $N_2$ increases with $a_2$ however falling of $N_2$ is faster along with increasing $a_{22}$ than $N_1$. For $a_{12} > 1$, strong commensalism of host over commensal it is observed that $N_1$ increasing to some extent and then start falling. This tendency is observed at slow rate in case of $a_{12} = 1$. Because of growth rate of the species $S_1$, $N_1$ falls much faster than $N_2$. However this fall becomes slower in case of $a_{22} > 1$.

5. Conclusion

Investigate some relation-chains between the species such as commensal-host ecological model with replenishment rate between two species ($S_i, S_j$) with the population relations. The present paper deals with an investigation on analytical approach of a typical two species syn eco-system. The system of coupled non-linear differential equations of populations of two species $S_1$ and $S_2$ has been solved analytically using Homotopy perturbation method. Our analytical results are compared with previous work and the results are good agreement with the previous work. This method can be extended to solve a mathematical model of a population of three species and four species with Syn-Ecological system.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$N_1$</td>
<td>Population of the specie $S_1$</td>
</tr>
<tr>
<td>$N_2$</td>
<td>Population of the specie $S_2$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>Natural growth rate of the specie $S_1$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>Natural growth rate of the specie $S_2$</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>Rate of decrease of the specie $S_1$</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>Rate of decrease of the specie $S_2$</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>Inhibition coefficient</td>
</tr>
<tr>
<td>$k_1$</td>
<td>Replenishment rate of the specie $S_1$</td>
</tr>
<tr>
<td>$k_2$</td>
<td>Replenishment rate of the specie $S_2$</td>
</tr>
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</table>

Appendix A

Basic Concept of the Homotopy Perturbation Method (HPM) [9-16]

To explain this method, let us consider the following function:

$$D_f(u) - f(r) = 0, \quad r \in \Omega$$

with the boundary conditions of

$$B_j(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma$$

where $D_j$ is a general differential operator, $B_j$ is a boundary operator, $f(r)$ is a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$. In general, the operator $D_j$ can be divided into a linear part $L$ and a non-linear part $N$. The eqn. (A.1) can therefore be written as
\[ L(u) + N(u) - f(r) = 0 \quad \text{(A.3)} \]

By the Homotopy technique, we construct a Homotopy \( v(r, p) : \Omega \times [0,1] \to \mathbb{R} \) that satisfies
\[
H(v, p) = (1 - p)[L(v) - L(u_0)] + p[D_0(v) - f(r)] = 0. \tag{A.4}
\]

\[
H(v, p) = L(v) - L(u_0) + p[L(v) - f(r)] = 0. \tag{A.5}
\]

where \( p \in [0,1] \) is an embedding parameter, and \( u_0 \) is an initial approximation of eqn. (A.1) that satisfies the boundary conditions. From eqns. (A.4) and (A.5), we have
\[
H(v,0) = L(v) - L(u_0) = 0 \quad \text{(A.6)}
\]
\[
H(v,1) = D_0(v) - f(r) = 0 \quad \text{(A.7)}
\]

When \( p=0 \), the eqns. (A.4) and (A.5) become linear equations. When \( p=1 \), they become non-linear equations. The process of changing \( p \) from zero to unity is that of changing the initial approximation of eqn. (A.1): \( v = v_0 + pv_1 + p^2v_2 + ... \quad \text{(A.8)} \)

Setting \( p = 1 \) results in the approximate solution of the eqn. (A.1):
\[
(1 - p)\left[ d(N_{10} + pN_{11} + p^2 N_{12} + ... \right) dt \right] - a_1 (N_{10} + pN_{11} + p^2 N_{12} + ...) - h_1 = 0 \quad \text{(B.5)}
\]
\[
(1 - p)\left[ d(N_{20} + pN_{21} + p^2 N_{22} + ... \right) dt \right] - a_2 (N_{20} + pN_{21} + p^2 N_{22} + ...) - h_2 = 0 \quad \text{(B.6)}
\]

Comparing the coefficients of like powers of \( p \) in (B.5) and (B.6) we get
\[
p^0: \frac{dN_{10}}{dt} = a_1 N_{10} - h_1 = 0 \quad \text{(B.7)}
\]
\[
p^0: \frac{dN_{20}}{dt} = a_2 N_{20} - h_2 = 0 \quad \text{(B.8)}
\]
\[
p^1: \frac{dN_{11}}{dt} = a_1 N_{11} + a_1 N_{12}^2 - a_1 N_1 N_2 = 0 \quad \text{(B.9)}
\]
\[
p^1: \frac{dN_{21}}{dt} = a_2 N_{21} + a_{22} N_2^2 = 0 \quad \text{(B.10)}
\]

The analytical solutions of (B.1) and (B.2) is
\[
N_1 = N_{10} + pN_{11} + p^2 N_{12} + ... \quad \text{(B.3)}
\]
\[
N_2 = N_{20} + pN_{21} + p^2 N_{22} + ... \quad \text{(B.4)}
\]

Substituting the eqns. (B.3) and (B.4) in (B.1) and (B.2) respectively we get
\[
N_{10} = N_{20} \quad \text{(B.11)}
\]
\[
N_{1i}(0) = N_{2i}(0), i = 1,2,3...... \quad \text{(B.12)}
\]

Solving the eqns. (B.7) - (B.10) and using the boundary conditions (B.11) and (B.12) we obtain the following results:
\[
N_{10}(t) = k_0 e^{dt} - k_1 \quad \text{(B.13)}
\]

This is the basic idea of the HPM.

**Appendix B**

**Solution of the Non-Linear Differential eqns. (1)-(3) Using the Homotopy Perturbation Method[9-19]**

In this Appendix, we indicate how the eqns. (4) and (5) are derived in this paper. To find the solution of eqns. (1) – (3), we construct the Homotopy as follows:
\[
(1 - p)\left[ \frac{dN_1}{dt} - a_1 N_1 - h_1 \right] + p\left[ \frac{dN_1}{dt} - a_1 N_1 - h_1 \right] = 0 \quad \text{(B.1)}
\]
\[
(1 - p)\left[ \frac{dN_2}{dt} - a_2 N_2 - h_2 \right] + p\left[ \frac{dN_2}{dt} - a_2 N_2 - h_2 \right] = 0 \quad \text{(B.2)}
\]

The solutions of (B.1) and (B.2) is
\[
N_1 = N_{10} + pN_{11} + p^2 N_{12} + ... \quad \text{(B.3)}
\]
\[
N_2 = N_{20} + pN_{21} + p^2 N_{22} + ... \quad \text{(B.4)}
\]
Where $k_0$, $k_1$, $u_0$, $u_1$ and $A$ are defined in the text (6) and (7).

According to the HPM, we can conclude that

$$N_1 = \lim_{p \to 1} N_1(t) = N_0 + N_1$$  \hspace{1cm} (B.17)

$$N_2 = \lim_{p \to 1} N_2(t) = N_{02} + N_{21}$$  \hspace{1cm} (B.18)

After putting the eqns. (B.13) and (B.14) into an eqn. (B.17) and (B.15) and (B.16) into an eqn. (B.18), we obtain the solutions in the text (4) - (7).

Appendix C

Fig 1. The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

Fig 2. The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

Fig 3. The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.
**Fig 4.** The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

**Fig 5.** The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

**Fig 6.** The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

**Fig 7.** The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

**Fig 8.** The populations $N_1(t)$ and $N_2(t)$ versus the time $t$. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.
The authors are thankful to Shri. S. Natanagopal, Secretary, The Madura College Board, Dr. R. Murali, The Principal and Mr. S. Muthukumar, Head of the Department of Mathematics, The Madura College (Autonomous), Madurai, Tamil Nadu, India for their constant encouragement.

References


Table 1. Numerical values of the dimensionless parameters for previous and present work.

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<th>Homotopy Perturbation Method</th>
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