A fixed point theorem on reciprocally continuous self mapping under menger space

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To cite this article:

Neha Jain, Rajesh Shrivastava, K. Qureshi. A Fixed Point Theorem on Reciprocally Continuous Self Mapping under Menger Space. *American Journal of Applied Mathematics*. Vol. 3, No. 1, 2015, pp. 4-7. doi: 10.11648/j.ajam.20150301.12

Abstract: The Fixed Point Theorem had been proved on Reciprocally Continuous Self Mapping. In this paper the fixed point theorem on reciprocally continuous self mapping is proved under Menger Space.

Keywords: Fixed Point, Reciprocally Continuous, Compatible Maps, Self Maps, Complete Menger Space

1. Introduction

In 1964 Metric spaces were introduced by Gähler, the probabilistic metric spaces is an important part of stochastic Analysis, to develop the fixed point theory in such spaces. There are many results in fixed point theory in probabilistic metric space., since then there have been many fixed point theorems proved in metric spaces and as a generalization of metric spaces, there have been only a few results in fixed point theory.

A coincidence point theorem for multi valued mappings satisfying generalized Hicks contraction principle in Menger Space. A probabilistic metric space is introduced by Menger. Many fixed point results have been obtained for single valued in probabilistic metric Spaces. Fixed Point theorem is proved for multi-valued version and by using the and by using the notion of the function of non compactness, notion of the function of non compactness, A multi-valued generalization of the notion of a contraction and Fixed Point theorem are introduced in Hadzic generalized Fixed Point theorem for multi-valued in zikic proved a coincidence point theorem for three mappings which is a generalization of Hicks theorem.

Fixed point theory in Menger spaces is a developed branch of mathematics. Sehgal and Bharucha-Reid First introduced the contraction mapping principle in probabilistic metric spaces [Hadzic and Pap]

2. Preliminaries

2.1. Definition: [18]

A t-norm is a function $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions.

- $\Delta(1, a) = a$
- $\Delta(a, b) = \Delta(b, a)$
- $\Delta(c, d) \geq \Delta(a, b)$ whenever $c \geq a$ and $d \geq b$
- $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$

2.2. Definition

A mapping $F: R \rightarrow R^+$ is called a distribution if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) =$

1, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

2.3. Definition Menger Space [18]

A Menger Space is a triplet (M, F, Δ) where M is a non-empty set, F is a function defined on $M \times M$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied.

- $F_{x,y}(0) = 0$ for all $x, y \in M$,
- $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in M$, if

and only if $x=y$

iii. $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in M$, $s > 0$

and

iv. $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u), F_{x,y}(v))$ for all $u, v \geq 0$ and

$x, y, z \in M$

A sequence $\{X_n\} \subset M$ converges to some point $x \in M$ if for given $\varepsilon \in M$ if for given $\varepsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\varepsilon, \lambda}$ such that for all $n > N_{\varepsilon, \lambda}$.

$$F_{x_n, x}(x) > 1 - \lambda$$

2.4. Definition: (Cauchy Sequence)

A sequence $\{x_n\}$ in a Menger space (M, F, Δ) is called a Cauchy sequence if for each $\varepsilon \in (0, 1)$ and $t > 0$. There exists $\eta_0 \in N$ such that $F_{x_n, x_m}(t) > 1 - \varepsilon$ for all $m, n \geq \eta_0$. The Menger space (M, F, Δ) is said to be complete if every Cauchy sequence in M convergent.

Lemma 2.1

Let X be metric space and (X, F, Δ) be a Menger probabilistic metric space with metric d and let w be w -distance, t -norm in Δ on x . Let $\{x_n\}$ and $\{y_n\}$ be a sequence in X . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence in $(0, \infty)$ converging to 0, and let $x, y, z \in X$. Then, the following hold:

- If $w(x_n, y) \leq \alpha_n$ and $w(x_n, Z) \leq \beta_n$ for any $n \in N$, then $y=z$, in particular, if $w(x, y) = 0$ and $w(x, z) = 0$ then $y=z$,
- If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, Z) \leq \beta_n$ for $n \in N$, then $\{y_n\}$ converges to z ;
- If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
- If $w(y, x_n) \leq \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.

3. Main Result

Theorem

Let S, I, T, J be self maps of a complete Menger space (X, F, Δ) with $t\Delta t \geq t \forall t \in [0, 1]$

- $S(x) \subset J(x), T(x) \subset I(x)$

$$d_{S_x, T_y}^2 \leq \max \left\{ \phi(d_{I_x, J_y}), \phi(d_{I_x, S_x}), \phi(d_{I_x, J_y}), \phi(d_{J_y, T_y}), \right.$$

$$\left. \phi(d_{I_x, S_x}), \phi(d_{J_y, T_y}), \phi(d_{I_x, J_y}), \phi(d_{J_y, S_x}) \right\}$$

for $\forall x, y \in X$

- $SI=IS, ST=TS$

- SI or S is continuous

- The pair (S, I) (T, J) weak compatible and compatible.

Then S, I, T & J have unique common fixed point.

Proof

Let x_0 in X be arbitrary, choose a point x_1 in X such that $Sx_0 = Jx_1$. This can be done since $S(x) \subset J(x)$. Let x_2 be a point in X such that $Tx_1 = Ix_2$. This can be done since $T(x) \subset I(x)$.

In general we can choose $x_{2n}, x_{2n+1}, x_{2n+2}, \dots$ such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$, so that we obtain a sequence

$$Sx_0, Tx_1, Tx_3 \quad (1)$$

Taking condition (i), (ii) and (iii) as in Aage and Salunke [1]

$\{Sx_{2n}\}$ is a Cauchy sequence and consequently the sequence (1) is a Cauchy. The sequence (1) converges to a limit z in X . Hence the subsequence $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n+1}\} = \{Ix_{2n+2}\}$ also converges to the limit point z .

Suppose that the pair (S, I) is compatible pair of reciprocally continuous. By the definition of reciprocally continuous, there is a sequence $\langle x_n \rangle$ in X such that

$$Sx_{2n} \rightarrow z, Ix_{2n} \rightarrow z, \text{ then } SIx_{2n} \rightarrow Sz$$

$$ISx_{2n} \rightarrow Iz, n \rightarrow \infty \quad (2)$$

Since the pair (S, I) is compatible we have $Sx_{2n} \rightarrow z, Ix_{2n} \rightarrow z$, as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} (d_{SIx_{2n}, ISx_{2n}}) = 0 \quad (3)$$

Using (2) and (3) we get

$$d(Sz, Iz) = 0 \text{ or } Sz = Iz$$

$$\text{since } Sz = Iz$$

Now

$$d_{Sx_{2n+1}, Tx_{2n+1}}^2 \leq \max \left\{ \phi(d_{Iz, Jx_{2n+1}}), \phi(d_{Iz, Sz}), \phi(d_{Iz, Jx_{2n+1}}), \phi(d_{Jx_{2n+1}, Tx_{2n+1}}), \right. \\ \left. \phi(d_{Iz, Sz}), \phi(d_{Jx_{2n+1}, Tx_{2n+1}}), \phi(d_{Iz, Tx_{2n+1}}), \phi(d_{Jx_{2n+1}, Sz}) \right\}$$

$$Jx_{2n+1} \rightarrow z, Tx_{2n+1} \rightarrow z \text{ as } n \rightarrow \infty \text{ and } Iz = Sz, \text{ so letting } n \rightarrow \infty \text{ we get}$$

$$d_{Sz, z}^2 \leq \max \left\{ \phi(d_{Sz, z}), \phi(d_{Sz, z}), \phi(d_{Sz, z}), \phi(d_{z, z}), \right. \\ \left. \phi(d_{Sz, Sz}), \phi(d_{z, z}), \phi(d_{Sz, z}), \phi(d_{z, Sz}) \right\}$$

$$\text{i.e. } d_{Sz, z} \leq \phi(d_{Sz, z}) \leq d_{Sz, z} \text{ Hence}$$

$$\phi(d_{Sz, z}) = 0 \text{ i.e. } Sz = z \text{ Thus } Sz = Tz = z$$

Further since $S(x) \subset J(x)$, There is a point $w \in X$

so to $z = Sz = Jw$

Now we prove that $Jw = Tw$, Now

$$d_{S_z, w}^2 \leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z})\phi(d_{I_z, S_z}), \\ \phi(d_{I_z, J_z})\phi(d_{J_w, T_w}), \\ \phi(d_{I_z, S_z})\phi(d_{J_w, T_w}), \\ \phi(d_{I_z, T_w}), \phi(d_{J_w, S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{J_w, J_w})\phi(d_{J_w, J_w}), \phi(d_{J_w, J_w})\phi(d_{J_w, T_w}), \\ \phi(d_{J_w, J_w})\phi(d_{J_w, T_w}), \phi(d_{J_w, T_w}), \phi(d_{J_w, J_w}) \end{array} \right\}$$

so $d_{J_w, T_w}^2 \leq 0$ implies $d_{J_w, T_w} = 0$ hence

$$z = Jw = Tw$$

Since the pair (T,J) is compatible pair of reciprocally continuous. By the definition of reciprocally continuous, there is a sequence $\langle x_n \rangle$ in X such that

$$Tx_{2n} \rightarrow z, Jx_{2n} \rightarrow z$$

$$\text{Then } TJx_{2n} \rightarrow Tz, JT x_{2n} \rightarrow Jz \text{ as } n \rightarrow \infty \quad (4)$$

Since the pair (T,J) is compatible we have $Tx_{2n} \rightarrow z, Jx_{2n} \rightarrow z$ as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} d_{Tx_{2n}, JT x_{2n}} = 0 \quad (5)$$

Using (4) and (5) we get

$$T_{I_z, J_z} = 0 \text{ hence } Tz = Jz \text{ Now}$$

$$d_{z, Tz}^2 = d_{S_z, Tz}^2$$

$$\leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z})\phi(d_{I_z, S_z}), \phi(d_{I_z, J_z})\phi(d_{J_z, Tz}), \\ \phi(d_{I_z, S_z})\phi(d_{J_z, Tz}), \phi(d_{I_z, Tz}), \phi(d_{J_z, S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{z, J_z})\phi(d_{z, z}), \phi(d_{z, Tz})\phi(d_{Tz, Tz}), \\ \phi(d_{z, z})\phi(d_{Tz, Tz}), \phi(d_{z, Tz})\phi(d_{Tz, Tz}) \end{array} \right\}$$

$$= \phi(d_{Tz, z})\phi(d_{Tz, z})$$

implies that

$$d_{Tz, z} \leq \phi(d_{Tz, z}) \leq (d_{Tz, z})$$

Hence $\phi(d_{Tz, z}) = 0$ i.e. $Tz = z$ and $z = Tz = Jz$. so z is a common fixed point of S,I,J & T.

Uniqueness

Let z' be another common fixed point of S,I,J and T i.e. $z' = sz' = Iz' = Tz' = Jz'$, from condition (ii) we have

$$d_{z, z'}^2 = d_{S_z, Tz'}^2$$

$$\leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z'})\phi(d_{I_z, S_z}), \phi(d_{I_z, J_z'})\phi(d_{J_z', Tz'}) \\ \phi(d_{I_z, S_z})\phi(d_{J_z', Tz'}), \phi(d_{I_z, Tz'}), \phi(d_{J_z', S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{z, z'})\phi(d_{z, z}), \phi(d_{z, z'})\phi(d_{z', z'}), \\ \phi(d_{z, z'})\phi(d_{z', z'}), \phi(d_{z, z'}), \phi(d_{z', z}) \end{array} \right\}$$

Therefore $d_{z, z'} \leq \phi(d_{z, z'}) \leq d_{z, z'}$ i.e. $\phi(d_{z, z'}) = d_{z, z'}$ thus

$d_{z, z'} = 0$ i.e. $z' = z$. Hence the common fixed point is unique.

Hence Proved

References

- [1] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(1994), 395-399. MR1289545. ZBI 0843.54014
- [2] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, 90(1997), 365-368. MR1477836. Zbl 0917.54010.
- [3] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North-Holland, 1983. MRO790314. Zbl0546.60010.
- [4] B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, *J. Math. Anal. Appl.*, 301 (2005), 439-448. MR2105684 Zbl 1068, 54044.
- [5] Chang, Shi-sen, Huan, Nan-Jing, On the generalized 2-metric spaces and probabilistic 2-metric spaces with application to fixed point theory, *Math. Japon.* 34 1989, 885-900.
- [6] D. Mihet, Inegalitata triunghiului si puncte fixe in PM-spaii, Doctoral Thesis West of Timisoara, in English, (2001).
- [7] D. Mihet, Multi-valued generalizations of probabilistic contractions, *J. Math. Anal. Appl.*, 304(2005), 464-472. MR2126543 Zbl 1072-47066.
- [8] D. Mihet, On a theorem of O. Hadzic, *Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat.*
- [9] D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, *Fuzzy Sets and Systems*, 158(2007), 915-921. MR2302646. Zbl 117.54008.
- [10] E. Pap, O. Hadzic and R. Mesiar, A fixed point theorem in probabilistic metric space and an application, *J. Math. Anal. Appl.* 202 (1996), 433-449.
- [11] Gahler, S., 2-metrische Raume und ihre topologische Struktur, *Math. Nachr.*, 26 1963/1964. 115-148.
- [12] Hadzic, O., On Coincidence Point Theorem for Multivalued Mappings in Probabilistic Metric Spaces, *Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak., Ser. Mat.* 25, 1 1995, 1-7.
- [13] Hadzic, O., On Common Fixed Point Theorems in 2-metric spaces, *Univ. u Novom Sadu, Zb. Rad. Prirod-Mat. Fak., Ser. Mat.* 12, 1982, 7-18.
- [14] Hadzic, O., On the (E, A.) - topology of probabilistic locally convex spaces, *Glax. Mat.*, Vol. 13(33), 293-297.

- [15] Hicks, T.L., Fixed Point Theory in Probabilistic Metric Spaces, Univ. u Novom Saud, Zb. Rad. Prirod. - Mat. Fak., Ser. Mat. 13, 1 1983, 63-72.
- [16] K.Menger, Statistical metric, Proc Nat Acad Sci, USA, 28 (1942), 535-7.
- [17] Kaleva, O., Saikala, S., On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984), 215-229.
- [18] O. Hadzic and E. Pap, Fixed point theory in PM spaces, Dordrecht: Kluwer Academic Publishers, 2001.
- [19] O. Hadzic and E. Pap, A fixed point theorem for multi-valued mapping in probabilistic Metric space and an application in fuzzy metric spaces, Fuzzy Sets Syst, 127 (2002), 333-344.
- [20] O. Hadzic and E. Pap, A fixed point theorem for multivalued mappings in Probabilistic metric spaces and an application in fuzzy metric spaces, Fuzzy Sets and Systems, 127(2002), 333-344. MR1899066, Zbl 102.54025.
- [21] O. Hadzic and E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, 2001. MR1896451. Zbl0994.47077.
- [22] O. Hadzic, E. Pap, Fixed point theorem for multi-valued probabilistic $\wedge$ -contractions, Indian J. Pure Appl. Math, 25(8) (1994), 825-835.
- [23] R. Vasuki and P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets and Systems, 135 (2003), 415-417. MR 1979610 Zbl 1029.54012.
- [24] Radu, V, A family of deterministic metrics on Menger spaces, Sem. Teor. Prob. Apl. Univ. Timisoara, 78 (1985).
- [25] Singh, - Shyma-Lal, Talwar, - Rekha, Zeng, - Wen-Zhi, Common fixed point theorems in 2-Menger spaces and applications, Math-students 63 1994, no 1-4, 74-80.
- [26] Stojakovic, M., Ovcin, Z., Fixed Point Theorems and variational principle in fuzzy metric space, Fuzzy Sets and Systems 66 (1994) 353-356.
- [27] T.L. Hicks, Fixed point theory in probabilistic metric spaces, Zb. Rad. Prirod. Mat. Fak. Ser. Mat, 13 (1983), 63-72.
- [28] Tatjana Zikic-Dosenovic, A multi-valued Generalization of Hick's Co-contraction, Fuzzy Sets and Systems, 151(2005), 549-592. MR2126173 Zbl 1069.54025.
- [29] V.M. Sehgal and A.T. Bharucha-Reid, Fixed point of contraction mappings on PM space, Math Sys. Theory, 6(2) (1972), 97-100. MR0310858. Abl 0244.60004.
- [30] Zeng, - Wen -Zhi, Probabilistic 2-metric spaces, J. Math. Research Expo. 2 1987, 241-245.
- [31] Zikic-Dosenovic. T., A multi-valued generalization of Hick's C-contraction., Fuzzy sets Syst, 151 2005, 549-562.