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Applications of the exp(-Φ(ξ))-Expansion Method to Find Exact Traveling Wave Solutions of the Benney-Luke Equation in Mathematical Physics

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Abstract: In this article, we construct the traveling wave solutions involving parameters of nonlinear evolutions equations via the Benney-Luke equation using the exp(-Φ(ξ))-expansion method. The traveling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions. When the parameters are taken special values, the solitary waves are derived from the traveling waves. The proposed method is direct, concise elementary and effective and can be used for many other nonlinear evolutions equations.

Keywords: Exp(-Φ(ξ))-Expansion Method, Benney-Luke Equation, Nonlinear Evolution Equations, Traveling Wave Solution

1. Introduction

The nonlinear evolution equations (NLEEs) that are studied in theoretical physics, especially in the context of wave phenomena leads to various forms of wave solutions. They are solitary waves, shock wave, cnoidal waves, snoidal waves and various other types. These waves appear in various scenarios in daily real life situations. For example, solitons appear in the propagation of pulse through optical fibers while shock waves appear in the supersonic jet flow. Another example is where cnoidal waves appear in shallow water waves although an extremely rare phenomena.

In recent years, several direct methods for finding the explicit traveling wave solutions to nonlinear evolution equations (NLEEs) have been proposed such as the extended tanh-method [1], the extended tanh-function method[2], the variational iteration method[3], Exp-function method[4], the complex hyperbolic-function method[5], the extended F-expansion method[6], the generalized Riccati equation rational expansion method[7], the Sub-ODE method[8], the *(G′/G)*-Expansion Method[9,10,11, 12, 13,14], the ansatz method[15], Outline of the Tanh-Coth Method [16], Cauchy Problem [17,18,19] and so on.

The aim of this paper was to apply the exp(-Φ(ξ))-expansion method [20,21] to construct the new exact traveling wave solutions for nonlinear evolution equations in mathematical physics via the Benney-Luke equation.The organization of the paper is as follows: In section 2, a description of the main steps of the exp(-Φ(ξ))-expansion method for finding the traveling wave solutions of nonlinear equations are given. In section 3, we apply this method to the nonlinear Benney-Luke equation; in section 4 we discuss the results and discussion of the traveling wave solutions. Finally concluding remarks are presented in section 5.

2. Description of the exp(-Φ(ξ))-Expansion Method

In this section, we will describe the algorithm of the exp(-Φ(ξ))-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a non linear equation in two independent variables *x* and *t* is given by,

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots\ldots\ldots\ldots) = 0, \quad x \in R, t > 0 \quad (2.1)$$

where $u(\xi) = u(x,t)$ is an unknown function, P is a polynomial of u(x ,t) and its partial derivatives in which the highest order derivatives and non linear terms are involved.

Step 1.Combining the independent variables x and t into one variables $\xi = x \pm Vt$, we suppose that

$$u(x,t) = u(\xi) \quad \xi = x \pm Vt \tag{2.2}$$

where $\omega \in R - \{0\}$ is the velocity of relative wave mode.

The traveling wave transformation equation (2.2) permits us to reduce equation (2.1) to the following ordinary differential equation (ODE)

$$F(u, u', u'', \ldots\ldots\ldots) = 0 \tag{2.3}$$

Where F is a polynomial in $u(\xi)$ and its derivatives, whereas $u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. Suppose the traveling wave solution of Eq. (2.3) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} A_i \left(\exp(-\Phi(\xi))\right)^i \tag{2.4}$$

Where A_i $(0 \le i \le N)$ are constants to be determined, such that $A_N \neq 0$, and $\Phi = \Phi(\xi)$ satisfies the following ODE,

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda\,. \tag{2.5}$$

Eq. (2.5) gives the following solutions:

Family 1: When $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right) \tag{2.6}$$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \coth\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right) \tag{2.7}$$

Family 2: When $\lambda^2 - 4\mu < 0, \mu \neq 0,$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right) \tag{2.8}$$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \cot\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right) \tag{2.9}$$

Family 3: When $\lambda^2 - 4\mu > 0, \mu = 0,\ \lambda \neq 0$,

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + k)) - 1}\right) \tag{2.10}$$

Family 4: When $\lambda^2 - 4\mu = 0,\ \mu \neq 0, \lambda \neq 0$,

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi + k) + 2)}{\lambda^2(\xi + k))}\right) \tag{2.11}$$

Family 5: When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$\Phi(\xi) = \ln(\xi + k) \tag{2.12}$$

where k is an arbitrary constant and $A_N, \cdots\cdots, V, \lambda, \mu$ are constants to be determined later, $A_N \neq 0$, the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

Step 3. We substitute Eq. (2.4) into (2.3) and then we account the function exp(-Φ(ξ)). As a result of this substitution, we get a polynomial of exp(-Φ(ξ)). We equate all the coefficients of same power of exp(-Φ(ξ)) are equal to zero. This procedure yields a system of algebraic equations whichever can be solved to find $A_1, A_2, \cdots\cdots, V, \lambda, \mu$. Substituting the values of $A_1, A_2, \cdots\cdots, V, \lambda, \mu$ into Eq. (2.4) along with general solutions of Eq. (2.5) completes the determination of the solution of Eq. (2.1).

3. Application

In this section, we will make use of the exp(-Φ(ξ))-expansion method to find the exact solitary wave solutions to the Benney-Luke equation. Let us consider the Benney-Luke equation in the form

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0 \tag{3.1}$$

Where α and β are positive number such that $\alpha - \beta = \sigma - 1/3$ is a Sobolev type equation and studied for a very long time. The dimensionless parameter σ is named the Bond number, which captures the effects of surface tension and gravity force and is a formally valid approximation for describing two-way water wave propagation in presence of surface tension [22].

Using the traveling wave variable $\xi = x - Vt$, Eq. (3.1) converts into the following ODE for $u(x,t) = u(\xi)$,

$$(V^2 - 1)u'' + (\alpha - \beta V^2)u^{(4)} - 3Vu'u'' = 0 \tag{3.2}$$

Eq. (3.2) is integrable, therefore integrating with respect to ξ once and choosing the integration constant to zero, we obtain

$$(V^2 - 1)u' + (\alpha - \beta V^2)u^3 - \frac{3}{2}Vu'^2 = 0 \tag{3.3}$$

Where, primes denote the differentiation with regard to ξ. By balancing the highest order term u'^2 and nonlinear term of the highest order u^3, we obtain $N=1$. Therefore, the exp(-Φ(ξ))-expansion method admits the solution of the Eq.(3.3) in the form

$$u(\xi)=A_0+A_1(\exp(-\varphi(\xi))),\quad A_1\neq 0 \tag{3.4}$$

By substituting the Eq.(2.5) and (3.4) into the Eq.(3.3) and equating the coefficient of $(\exp(-\Phi(\xi)))^i$, are equal to zero, yielding a set of algebraic equations as follows:

$$-A_1V^2\mu-\alpha A_1\mu\lambda^2+2\beta V^2A_1\mu^2-\frac{3}{2}VA_1^2\mu^2+\beta V^2A_1\mu\lambda^2+A_1\mu-2\alpha A_1\mu^2=0 \tag{3.5}$$

$$-8\alpha A_1\mu\lambda+A_1\lambda-\alpha A_1\lambda^3+\beta V^2A_1\lambda^3-3VA_1^2\mu\lambda+8\beta V^2A_1\mu\lambda-A_1V^2\lambda=0 \tag{3.6}$$

$$-8\alpha A_1\mu-A_1V^2+7\beta V^2A_1\lambda^2-7\alpha A_1\lambda^2-3VA_1^2\mu+A_1-\frac{3}{2}VA_1^2\lambda^2+8\beta V^2A_1\mu=0 \tag{3.7}$$

$$-3VA_1^2\lambda+12\beta V^2A_1\lambda-12\alpha A_1\lambda=0 \tag{3.8}$$

$$-6\alpha A_1-\frac{3}{2}VA_1^2+6\beta V^2A_1=0 \tag{3.9}$$

Solving the algebraic Eq. (3.5) to (3.9), we obtain a set of solution as follows:

$$\left.\begin{aligned}&V=\pm\frac{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}{1+4\beta\mu-\beta\lambda^2},\quad A_0=A_0,\\&A_1=\pm\frac{4(\alpha-\beta)}{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}\end{aligned}\right\}(3.10)$$

Where λ and μ are arbitrary constants.

By substituting Eq.(3.10)into Eq.(3.4), we have

$$u(\xi)=A_0\pm\frac{4(\alpha-\beta)}{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}(\exp(-\varphi(\xi))) \tag{3.11}$$

Where

$$\xi=x\pm\frac{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}{1+4\beta\mu-\beta\lambda^2}t$$

Now substituting Eq. (2.6) to Eq. (2.12) into Eq. (3.11) respectively, the traveling wave solutions of the Eq.(3.1) are obtained as follows:

Family 1: When $\lambda^2-4\mu>0,\mu\neq 0$,

$$u_{1,2}(x,t)=A_0\pm\frac{Q}{\left(-\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+k))-\lambda\right)},$$

$$u_{3,4}(x,t)=A_0\pm\frac{Q}{\left(-\sqrt{\lambda^2-4\mu}\coth(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+k))-\lambda\right)},$$

Where

$$\xi=x\pm\frac{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}{1+4\beta\mu-\beta\lambda^2}t,$$
$$Q=\frac{8(\alpha-\beta)\mu}{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}$$

Family 2: When $\lambda^2-4\mu<0,\mu\neq 0$,

$$u_{5,6}(x,t)=A_0\pm\frac{Q}{\left(\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+k))-\lambda\right)}$$

$$u_{7,8}(x,t)=A_0\pm\frac{Q}{\left(\sqrt{4\mu-\lambda^2}\cot(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+k))-\lambda\right)}$$

Where

$$\xi=x\pm\frac{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}}{1+4\beta\mu-\beta\lambda^2}t,$$
$$Q=\frac{8(\alpha-\beta)\mu}{\sqrt{(1+4\beta\mu-\beta\lambda^2)\,(1+4\,\alpha\mu-\alpha\lambda^2)}},$$

Family 3: When $\lambda^2-4\mu>0,\mu=0,\ \lambda\neq 0$,

$$u_{9,10}(x,t)=A_0\pm Q\,(\exp\,(\lambda(\xi+k))-1),$$

Where

$$\xi=x\pm\frac{\sqrt{(1-\beta\lambda^2)\,(1-\alpha\lambda^2)}}{1-\beta\lambda^2}t,$$
$$Q=\frac{4(\alpha-\beta)\lambda}{\sqrt{(1-\beta\lambda^2)\,(1-\alpha\lambda^2)}},$$

Family 4: When $\lambda^2-4\mu=0,\ \mu\neq 0,\lambda\neq 0$,

$$u_{11,12}(x,t) = A_0 \pm \frac{2(\alpha-\beta)\lambda^2(\xi+k)}{\lambda(\xi+k)+2},$$

Where $\xi = x \pm t$,

Family 5: When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$u_{13,14}(x,t) = A_0 \pm \frac{4(\alpha-\beta)}{\xi+k},$$

Where $\xi = x \pm t$,

4. Results and Discussion

In this section, we describe the physical explanation and graphical representation of the solutions of the Benney-Luke equation.

4.1. Explanation

The solution $u(x,t)$ to the solitary wave Eq.(3.1) play an important role for describing different types of wave propagation of pulses through optical fibers while shock waves appear in the supersonic jet flow. The Eq.(3.1) is given not only more new multiple explicit solutions but also many types of exact traveling wave solutions. The exact traveling wave solutions are obtained from the explicit solutions by choosing the particular value of the physical parameters. So we can appropriate values of the parameters to obtained exact solutions. There are various types of traveling wave solutions that are particular interest in solitary wave theory. In this research work, some important traveling wave solutions are described and presented graphically.

The solution $u_1(x,t)$ is presented the kink type solitonsolution. Kink solitons are rise from one asymptotic state at $\xi \to -\infty$ to another asymptotic state at $\xi \to +\infty$. These solitons are referred to as topological solitons. The Fig. 1 has been shown the shape of the solution $u_1(x,t)$ for $\alpha = 6, \beta = 3, \lambda = 8, \mu = 1.5, A_0 = 1, k = 0.5$ within the interval $-3 \le x,t \le 3$. The solution $u_5(x,t)$ is presented the periodic soliton solution for the various values of the physical parameters. The Fig. 2 has been shown the shape of the solution $u_5(x,t)$ for $\alpha = 7, \beta = 1, \lambda = -1, \mu = 1, A_0 = 0$ $k = 2$ within the interval $-10 \le x,t \le 10$. For the values of $\alpha = 5, \beta = 2, \lambda = 1, \mu = 0, A_0 = 3, k = 0$ within the interval $-10 \le x,t \le 10$, the solution $u_9(x,t)$ is presented the soliton profile which is shows in Fig. 3. The solution $u_{11}(x,t)$ is also presented the singular kink type soliton solution which is shows in Fig. 4 for $\alpha = 8, \beta = 3, \lambda = 1, k = 0.8, A_0 = 0$ within the interval $-10 \le x,t \le 10$. Finally, Fig. 5 shows an exact singular kink type solitary wave profile corresponding to $u_{13}(x,t)$ with fixed parameters $\alpha = 7, \beta = 3,$ $k = 2, A_0 = 2$ within the interval $-10 \le x,t \le 10$.

4.2. Graphical Representation

In this sub section, the graphical representations of the solutions are given below in the figures (Fig. 1-5) with the aid of mathematical software Maple 13.

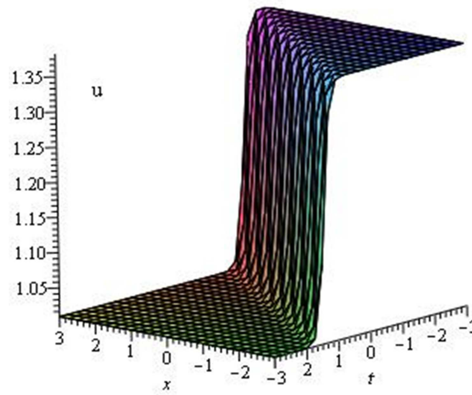

Fig. 1. Kink type soliton profile, Shape of $u_1(x,t)$ *when* $\alpha = 6, \beta = 3, \lambda = 8, \mu = 1.5$ $A_0 = 1, k = 0.5$ *with* $-3 \le x,t \le 3$

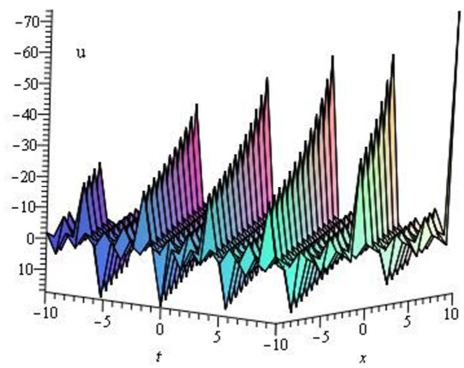

Fig. 2. Periodic type soliton profile, Shape of $u_5(x,t)$ *when* $\alpha = 7, \beta = 1, \lambda = -1, \mu = 1$ $A_0 = 0, k = 2$ *with* $-10 \le x,t \le 10$.

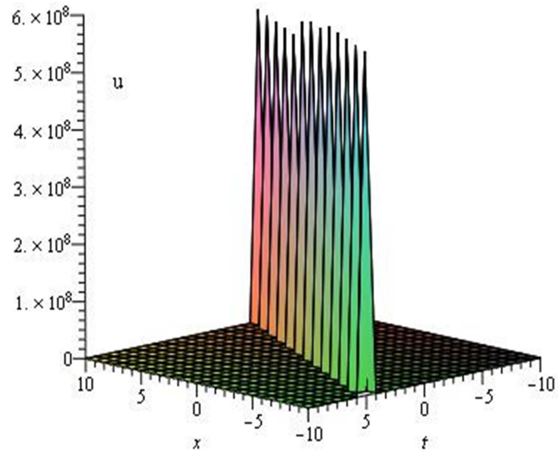

Fig. 3. soliton profile, Shape of $u_9(x,t)$ *when* $\alpha = 5, \beta = 2, \lambda = 1, \mu = 0$ $A_0 = 3, k = 0$ *with* $-10 \le x,t \le 10$.

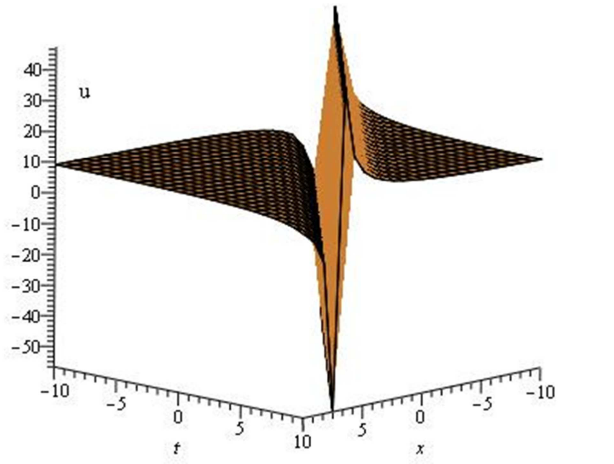

Fig. 4. *Singular kink typesoliton profile, Shape of* $u_{11}(x,t)$ *when* $\alpha=8,\beta=3,\lambda=1$ $A_0=0,k=0.8$ *with* $-10\le x,t\le 10$.

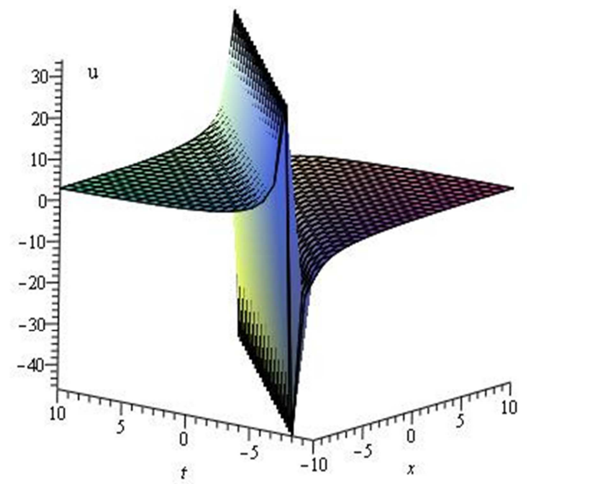

Fig. 5. *Singular kink typesoliton profile, Shape of* $u_{13}(x,t)$ *when* $\alpha=7,\beta=3,A_0=2,k=2$ *with* $-10\le x,t\le 10$.

5. Conclusion

In this paper, the exp(-Φ(ξ))-expansion method has been successfully applied to construct new traveling wave solutions for the Benney-Luke equation. This study shows that the exp(-Φ(ξ))-expansion method is quite efficient and practically well suited for use in finding exact solutions for the problems considered here. Being concise and effective, the exp(-Φ(ξ))-expansion method can also be used to many other nonlinear equations.

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