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Review Article

Function as the Generator of Parametric T-norms

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Abstract: The method of constructing t-norms by generators consists in using a unary function (generator) to transform some known binary function (usually, addition or multiplication) into a T-norm. In order to allow using non-bijective generators, which do not have the inverse function, we have used the notion of pseudo-inverse function. Many families of related t-norms can be defined by an explicit formula depending on a parameter p. Firstly; some continuous and decreasing parametric functions have been selected. Then generate parametric T-norms by using those functions based on additive generator.

Keywords: Pseudo-inverse, Additive generators, Parametric T-norms, Yager's Product T_P^Y, Dombi's Product T_P^D, Aczel-Alsina T_P^{AA}, Frank Product T_P^Y, Schweizer and Sklar T_P^{SS}

1. Introduction

T-norms are generalization of the usual two-valued logical conjunction, studied by classical logic, for fuzzy logics. T-norms are also used to construct the intersection of fuzzy sets or as a basis for aggregation operators. In 1942, K. Menger introduced the concept of triangular norm generalizing the classical triangular inequality. In 1960, B. Schweizer and A. Sklar after revision of this work redefined the concept of triangular norm as an associative and commutative binary operation on which is generally accepted today. Since, the T - norms have become important tools in different contexts. They play a fundamental role in probabilistic metric spaces, probabilistic norms and scalar products, multiple-valued logic, fuzzy sets theory.

1.1. T-norms

Definition: A t-norm is a function $\boldsymbol{T}$: [0, 1] × [0, 1] → [0, 1] which satisfies the following properties:

$[\mathrm{T}_1]$: Monotonicity: $\mathrm{T(a,b) \le T(c,d)}$ if $\mathrm{a \le c}$ and $\mathrm{b \le d}$
$[\mathrm{T}_2]$: Commutativity: $\mathrm{T(a,b) = T(b,a)}$
$[\mathrm{T}_3]$: Associativity: $\mathrm{T\big(a, T(b,c)\big) = T(T(a,b),c)}$
$[\mathrm{T}_4]$: Boundary condition: $\mathrm{T(a,1) = a}$

1.2. Pseudo-inverse

Definition: Let J and L be closed subinterval of [0,∞].
Given a continuous mapping $f: J \to L$
Then the pseudo-inverse of f is a map $f^{(-1)}: L \to J$
(a) When f is strictly increasing, then it is defined by

$$f^{(-1)}(y) = \begin{cases} \min J \ if\ y\epsilon[minL, f(minJ)] \\ f^{-1}(y)\ if\ y\epsilon[f(minJ), f(\max \mathrm{L})] \\ \max J\ if\ y\epsilon[f(maxJ), maxL] \end{cases}$$

(b) When f is strictly decreasing, then it is defined by

$$f^{(-1)}(y) = \begin{cases} \max J\ if\ y\epsilon[minL, f(maxJ)] \\ f^{-1}(y)\ if\ y\epsilon[f(maxJ), f(\min \mathrm{J})] \\ \max J\ if\ y\epsilon[f(minJ), maxL] \end{cases}$$

If J=[0,1] and l=[0,∞] then this definition is equivalent to

$$f^{(-1)}(y) = \begin{cases} 0\ if\ y\epsilon[0, f(0)] \\ f^{-1}(y)\ if\ y\epsilon[f(0), f(0)] \\ 1\ if\ y\epsilon[f(1), \infty] \end{cases}$$

Where f is strictly increasing
Again

$$f^{(-1)}(y) = \begin{cases} 1 \; if \; y\epsilon[0, f(1)] \\ f^{-1}(y) \; if \; y\epsilon[f(1), f(0)] \\ 1 \; if \; y\epsilon[f(0), \infty] \end{cases}$$

Where f is strictly decreasing.

In both cases f^{-1} is ordinary inverse of$f$.

1.3. Additive Generators

Definition: If a t-norm *T* results from the latter construction by a function f which is right-continuous in 0, then f is called an additive generator of *T*. The construction of t-norms by additive generators is based on the theorem: Let$f$: [0, 1] → [0, +∞] be a strictly decreasing function such that $f(1) = 0$ and $f(x) + f(y)$ is in the range of f or equal to $f(0^+)$ or +∞ for all x, y in [0, 1]. Then the function *T*: $[0, 1]^2 \to [0, 1]$ defined as $T(x,y) = f^{(-1)}(f(x) + f(y))$ is a t-norm.

2. Generator of Parametric T-norms

Proposition 2.1: The function $f_P(x) = (1-x)^P$ is a continuous function at $x = a$ and decreasing function on (0,1]. Then The t-norm *Yager's Product* T_P^Y defined by $T_P^Y(x,y) = 1 - min\left(1, ((1-x)^P + (1-y)^P)^{\frac{1}{P}}\right)$ is additively generated by the function $f_P(x) = (1-x)^P; P > 0$.

Proof: Since x is in interval (0,1], so every points on the interval is a limit point of the function $f_P(x) = (1-x)^P$.

Let $x = a\epsilon(0,1]$ be any point, then by calculus method,

we get $\lim_{x\to a} f_P(x) = \lim_{x\to a}(1-x)^P = (1-a)^P$ and $f_P(a) = (1-a)^P$.

Therefore

$$\lim_{x\to a} f_P(x) = f_P(a).$$

Hence $f_P(x) = (1-x)^P$ is continuous at $x = a\epsilon(0,1]$.

Now we have to show that $f_P(x) = (1-x)^P$ is a decreasing function on (0,1].

For this let $x_1, x_2\epsilon(0,1]$, such that$x_1 \le x_2$.

Then

$$x_1 \le x_2.$$

$$\Rightarrow 1 - x_1 \ge 1 - x_2$$

$$\Rightarrow (1-x_1)^P \ge (1-x_2)^P$$

$$\Rightarrow f_P(x_1) \ge f_p(x_2).$$

Therefore $f_P(x) = (1-x)^P$ is a decreasing function on (0,1].

So, let $y = f_P(x) = (1-x)^P$

$$\Longrightarrow y = (1-x)^P$$

$$\Longrightarrow y^{\frac{1}{P}} = 1 - x$$

$$\Longrightarrow x = 1 - y^{\frac{1}{P}}.$$

So, we have

$$f_P^{(-1)}(x) = \begin{cases} 1 - x^P \,; x\epsilon[0,1] \\ 0; \; [1, \infty] \end{cases}$$

Now,

$$T_P^Y(x,y) = f_P^{(-1)}(f(x) + f(y))$$

$$= f_P^{(-1)}((1-x)^P + (1-y)^P)$$

$$= \begin{cases} 1 - ((1-x)^P + (1-y)^P)^{\frac{1}{P}}; (1-x)^P + (1-y)^P\epsilon[0,1] \\ 0; \; Othetwise \end{cases}$$

$$= 1 - min\left(1, ((1-x)^P + (1-y)^P)^{\frac{1}{P}}\right).$$

Therefore t-norm *Yager's Product*

$$T_P^Y(x,y) = 1 - min\left(1, ((1-x)^P + (1-y)^P)^{\frac{1}{P}}\right)$$

is additively generated by the function

$$f_P(x) = (1-x)^P; P > 0.$$

Proposition 2.2: The function $f_P(x) = \left(\frac{1-x}{x}\right)^P$ is a continuous function at $x = a$ and decreasing function on (0,1]. Then, the t-norm *Dombi's Product* T_P^D defined by

$$T_P^D(x,y) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^P + \left(\frac{1-y}{y}\right)^P\right)^{\frac{1}{P}}}$$

is additively generated by the function $f_P(x) = \left(\frac{1-x}{x}\right)^P; \; P > 0$.

Proof: Since x is in interval (0,1], so every points on the interval is a limit point of the function $f_P(x) = \left(\frac{1-x}{x}\right)^P$.

Let $x = a\epsilon(0,1]$ be any point, then by calculus method,

we get, $\lim_{x\to a} f_P(x) = \lim_{x\to a}\left(\frac{1-x}{x}\right)^P = \left(\frac{1-a}{a}\right)^P$ and $f_P(a) = \left(\frac{1-x}{x}\right)^P$.

Therefore

$$\lim_{x\to a} f_P(x) = f_P(a).$$

Hence $f_P(x) = \left(\frac{1-x}{x}\right)^P$ is continuous at $x = a\epsilon(0,1]$.

Now we have to show that $f_P(x) = \left(\frac{1-x}{x}\right)^P$ is a decreasing function on (0,1].

For this let $x_1, x_2\epsilon(0,1]$, such that$x_1 \le x_2$.

Then

$$x_1 \leq x_2.$$
$$\Rightarrow 1 - x_1 \geq 1 - x_2$$
$$\Rightarrow \left(\frac{1 - x_1}{x_1}\right)^P \geq \left(\frac{1 - x_2}{x_2}\right)^P$$
$$\Rightarrow f_P(x_1) \geq f_p(x_2).$$

Therefore $f_P(x) = \left(\frac{1-x}{x}\right)^P$ is a decreasing function on (0,1].

$$\text{So, let } y = f_P(x) = \left(\frac{1 - x}{x}\right)^P$$
$$\Rightarrow y = \left(\frac{1 - x}{x}\right)^P$$
$$\Rightarrow y^P = \frac{1 - x}{x}$$
$$\Rightarrow x + xy^P = 1$$
$$\Rightarrow x = \frac{1}{1 + y^P}$$

So, we have

$$f_P^{(-1)}(x) = \begin{cases} \dfrac{1}{1 + x^P}; \ x\epsilon[0,1] \\ 0; \ Otherwise \end{cases}$$

Now,

$$T_P^D(x, y) = f_P^{(-1)}(f(x) + f(y))$$
$$= f_P^{(-1)}\left(\left(\frac{1 - x}{x}\right)^P + \left(\frac{1 - y}{y}\right)^p\right)$$
$$= \frac{1}{1 + \left(\left(\frac{1 - x}{x}\right)^P + \left(\frac{1 - y}{y}\right)^P\right)^{\frac{1}{P}}}; \ \left(\frac{1 - x}{x}\right)^P + \left(\frac{1 - y}{y}\right)^p \epsilon[01].$$

Therefore the t-norm *Dombi's Product*

$$T_P^D(x, y) = \frac{1}{1 + \left(\left(\frac{1 - x}{x}\right)^P + \left(\frac{1 - y}{y}\right)^P\right)^{\frac{1}{P}}}$$

is additively generated by the function $f_P(x) = \left(\frac{1 - x}{x}\right)^P; \ P > 0.$

Proposition 2.3: The function $f_P(x) = (-\log x)^P$ is a continuous function at $x = a$ and decreasing function on (0,1]. Then The t-norm Aczel-Alsina T_P^{AA} defined by $T_P^{AA}(x, y) = e^{-(|\log x|^P + |\log y|^p)^{\frac{1}{P}}}$ is additively generated by the function $f_P(x) = (-\log x)^P$.

Proof: Since x is in interval (0,1], so every points on the interval is a limit point of the function $f_P(x) = (-\log x)^P$.

Let $x = a\epsilon(0,1]$ be any point, then by calculus method, we get, $\lim_{x\to a} f_P(x) = \lim_{x\to a}(-\log x)^P = (-\log a)^P$ and $f_P(a) = (-\log a)^P$.

Therefore

$$\lim_{x\to a} f_P(x) = f_P(a).$$

Hence $f_P(x) = (-\log x)^P$ is continuous at $x = a\epsilon(0,1]$.

Now we have to show that $f_P(x) = (-\log x)^P$ is a decreasing function on (0,1].

For this let $x_1, x_2\epsilon(0,1]$, such that$x_1 \leq x_2$.

Then

$$x_1 \leq x_2$$
$$\Rightarrow \log x_1 \leq \log x_2$$
$$\Rightarrow (-\log x_1)^P \geq (-\log x_2)^P$$
$$\Rightarrow f_P(x_1) \geq f_p(x_2).$$

Therefore $f_P(x) = (-\log x)^P$ is a decreasing function on (0,1].

$$\text{Proof: Let } y = f_P(x) = (-\log x)^P$$
$$\Rightarrow y = (-\log x)^P$$
$$\Rightarrow y^P = -\log x$$
$$\Rightarrow x = e^{-y^{\frac{1}{P}}}.$$

So, we have

$$f_P^{(-1)}(x) = \begin{cases} e^{-x^P}; \ x\epsilon[0,1] \\ 0; \ Otherwise \end{cases}$$

Now,

$$T_P^{AA}(x, y) = f_P^{(-1)}\big(f(x) + f(y)\big)$$
$$= f_P^{(-1)}((-\log x)^P + (-\log y)^P)$$
$$= e^{-(|\log x|^P + |\log y|^p)^{\frac{1}{P}}}; |\log x|^P + |\log y|^p\epsilon[0,1].$$

Therefore the t-norm *Aczel-Alsina* $T_P^{AA}(x, y) = e^{-(|\log x|^P + |\log y|^p)^{\frac{1}{P}}}$ is additively generated by the function $f_P(x) = (-\log x)^P$.

Proposition 2.4: The function $f_P(x) = \log\frac{P-1}{P^x-1}$ is continuous at $x = a$ and a decreasing function on (0,1]. Then the t-norm Frank Product T_P^Y is additively generated by the function $f_P(x) = \log\frac{P-1}{P^x-1}$.

Proof: Since x is in interval (0,1], so every points on the interval is a limit point of the function $f_P(x) = \log\frac{P-1}{P^x-1}$.

Let $x = a\epsilon(0,1]$ be any point, then we get

$$\lim_{x\to a} f_P(x) = \lim_{x\to a} \log\frac{P - 1}{P^x - 1} = \log\frac{P - 1}{P^a - 1}.$$

and

$$f_P(a) = \log\frac{P-1}{P^a-1}.$$

Therefore

$$\lim_{x\to a} f_P(x) = f_P(a).$$

Hence $f_P(x) = \log\frac{P-1}{P^x-1}$ is continuous at $x = a\epsilon(0,1]$.

Now we have to show that $f_P(x) = \log\frac{P-1}{P^x-1}$ is a decreasing function on (0,1].

For this let $x_1, x_2\epsilon(0,1]$, such that$x_1 \le x_2$.

Then

$$x_1 \le x_2$$

$$\Rightarrow P^{x_1} \le P^{x_2}$$

$$\Rightarrow P^{x_1} - 1 \le P^{x_2} - 1$$

$$\Rightarrow \frac{1}{P^{x_1}-1} \ge \frac{1}{P^{x_2}-1}$$

$$\Rightarrow \frac{P^a-1}{P^{x_1}-1} \ge \frac{P^a-1}{P^{x_2}-1}$$

$$\Rightarrow \log\frac{P^a-1}{P^{x_1}-1} \ge \log\frac{P^a-1}{P^{x_2}-1}$$

$$\Rightarrow f_P(x_1) \ge f_P(x_2).$$

Therefore $f_P(x) = \log\frac{P-1}{P^x-1}$ is a decreasing function on (0,1].

$$\text{Let } y = \log\frac{P-1}{P^x-1}$$

$$\Rightarrow \frac{P-1}{P^x-1} = e^y$$

$$\Rightarrow P^x - 1 = \frac{P-1}{e^y}$$

$$\Rightarrow P^x = \frac{P-1}{e^y} + 1$$

$$\Rightarrow x = \log_P\left(\frac{P-1}{e^y} + 1\right)$$

$$\Rightarrow x = \log_P\left(\frac{P-1+e^y}{e^y}\right)$$

So, we have

$$f^{(-1)}(x) = \begin{cases} \log_P\left(\frac{P-1+e^x}{e^x}\right); x \in [0, f(0)] \\ 0; x \in [f(0), \infty] \end{cases}$$

Now,

$$T_P^F(x,y) = f^{(-1)}(f(x) + f(y))$$

$$= f^{(-1)}\left(\log\frac{P-1}{P^x-1} + \log\frac{P-1}{P^y-1}\right)$$

$$= f^{(-1)}\left(\log\frac{P-1}{P^x-1}\frac{P-1}{P^y-1}\right)$$

$$= f^{(-1)}\left(\log\frac{(P-1)^2}{(P^x-1)(P^y-1)}\right)$$

$$= \log_P\left(\frac{P-1+\frac{(P-1)^2}{(P^x-1)(P^y-1)}}{\frac{(P-1)^2}{(P^x-1)(P^y-1)}}\right)$$

$$= \log_P\left(\frac{(P-1)(P^x-1)(P^y-1)}{(P-1)^2}\right)$$

$$= \log_P\left(1 + \frac{(P^x-1)(P^y-1)}{P-1}\right).$$

Hence the t-norm *Frank Product* T_P^Y is additively generated by the function $f_P(x) = \log\frac{P-1}{P^x-1}$.

Proposition 2.5: The function $f_P(x) = 1 - x^P$ is a continuous function at $x = a$ and decreasing function on (0,1]. Then the t-norm *Schweizer and Sklar* T_P^{SS} *defined by* $T_P^{SS}(x,y) = \left(max(0, a^P + b^P - 1)\right)^{\frac{1}{P}}$ is additively generated by the function $f_P(x) = 1 - x^P; P \ne 0$.

Proof: Since x is in interval (0,1], so every points on the interval is a limit point of the function $f_P(x) = 1 - x^P$.

Let $x = a\epsilon(0,1]$ be any point, then by calculus method, we get

$$\lim_{x\to a} f_P(x) = \lim_{x\to a}(1 - x^P) = 1 - a^P.$$

and

$$f_P(a) = 1 - a^P.$$

Therefore

$$\lim_{x\to a} f_P(x) = f_P(a).$$

Hence $f_P(x) = 1 - x^P$ is continuous at $x = a\epsilon(0,1]$.

Now we have to show that $f_P(x) = 1 - x^P$ is a decreasing function on (0,1].

For this let $x_1, x_2\epsilon(0,1]$, such that$x_1 \le x_2$.

Then

$$x_1 \le x_2$$

$$\Rightarrow x_1^P \le x_2^P$$

$$\Rightarrow 1 - x_1^P \ge 1 - x_2^P.$$

Therefore $f_P(x) = 1 - x^P$ is a decreasing function on (0,1].

Let $y = f_P(x) = 1 - x^P$

$$\Longrightarrow y = 1 - x^P$$

$$\Longrightarrow x^P = 1 - y$$

$$\Rightarrow \mathrm{x} = (1-y)^{\frac{1}{P}}$$

So, we have

$$f_P^{(-1)}(x) = \begin{cases} (1-x)^{\frac{1}{P}};\ x\epsilon[0,1] \\ 0; x\epsilon[1,\infty] \end{cases}$$

Now,

$$T_P^{SS}(x,y) = f_P^{(-1)}(f(x)+f(y))$$

$$= f_P^{(-1)}(1-x^P+1-y^P)$$

$$= f_P^{(-1)}(2-x^P-y^P)$$

$$= \begin{cases} (x^P+y^P-1)^{\frac{1}{P}}\ if(2-x^P-y^P)\epsilon[0,1] \\ 0\ Otherwise \end{cases}$$

$$= \left(max(0,a^P+b^P-1)\right)^{\frac{1}{P}}.$$

Therefore the t-norm Schweizer and Sklar

$$T_P^{SS}(x,y) = \left(max(0,a^P+b^P-1)\right)^{\frac{1}{P}}$$

is additively generated by the function $f_P(x) = 1-x^P; P \neq 0$.

3. Conclusion

In this paper, to find the different types of parametric T-norms strictly decreasing parametric function has used. The parametric T-norms depend on the range of parameter p. We can apply T-norms for optimization under fuzzy constraints. There are vast field in business sectors for strategic management portfolio analysis such as growth strategy, leadership strategy, industry attractiveness, industry maturity etc.

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