American Journal of Applied Mathematics
2018; 6(1): 20-22
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20180601.14
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Hamiltonicity of Mycielski Graphs

Shuting Cheng[1], Dan Wang[1, *], Xiaoping Liu[2]

[1]College of Mathematics and System Sciences, Xinjiang University, Urumqi, P. R. China
[2]Department of Mathematics, Xinjiang Institute of Engineering, Urumqi, P. R. China

Email address:
dangxjwang@163.com (Dan Wang)
[*]Corresponding author

To cite this article:
Shuting Cheng, Dan Wang, Xiaoping Liu. Hamiltonicity of Mycielski Graphs. *American Journal of Applied Mathematics*. Vol. x, No. x, 2018, pp. x-x. doi: 10.11648/j.ajam.20180601.14

Received: January 28, 2018; **Accepted**: February 16, 2018; **Published**: March 16, 2018

Abstract: Fisher, McKenna, and Boyer showed that if a graph G is hamiltonian, then its Mycielski graph $\mu(G)$ is hamitonian. In this note, it was shown that for a bipartite graph G, if its mycielski graph $\mu(G)$ is hamiltonian, then G has a Hamilton path.

Keywords: Bipartite Graphs, Mycielski Graph, Hamilton Cycle, Walk

1. Introduction

All graphs considered in this paper are finite and simple. For notation and terminology not defined here, we refer to West [16]. Mycielski [13] found a kind of construction to create triangle-free graphs with large chromatic numbers. For a graph G on vertices $V=\{v_1,v_2,\cdots,v_n\}$, the Mycielski graph $\mu(G)$ of G is defined as

$$V(\mu(G))=X\cup Y\cup\{z\}=\{x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z\}$$

z is adjacent to every y_i, and

if $v_iv_j\in E(G)$, then $x_ix_j, y_iy_j\in E(\mu(G))$.

For example $\mu(K_2)=C_5$, an $\mu^2(K_2)$ is known as the Grötsch graph, see Figure 1. Mycielski showed that $\mu^k(K_2)$ is triangle-free and has chromatic number $k+2$. In general, for a graph G (not necessarily a triangle-free graph), $\chi(\mu(G))=\chi(G)+1$, where $\chi(G)$ denotes the chromatic number of G.

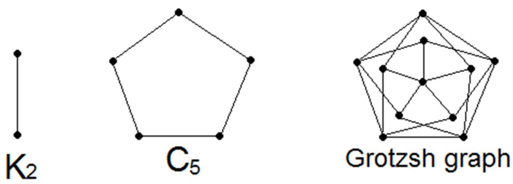

Figure 1. *$\mu(K_2)=C_5$ and Grötsch graph.*

Recently, a number of papers are devoted to the various parameters of Mycielski graphs, such as the fractional chromatic number [9], circular chromatic number [2, 5, 8, 10, 12], connectivity [1, 6]. Total chromatic number [3], hub number [11], covering number [14], total weight choosability [15], Hamilton-connectivity [7].

A spanning cycle (resp. path) of a graph is called Hamilton cycle (resp. Hamiton path). A graph is said to be *hamiltonian* if it has a Hamilton cycle. In general, it is NP-hard to decide whether a graph has a Hamilton cycle or not. Our main objective is to search best possible condition for a graph G whose Mycielski graph is hamiltonian. Fisher, McKenna, and Boyer [4] obtained the following results.

Theorem 1.1. ([4]) If G is hamiltonian, then $\mu(G)$ is hamiltonian.

Theorem 1.2. ([4]) If G is not connected, then $\mu(G)$ is not hamiltonian.

Theorem 1.3. ([4]) If G has at least two pendent vertices, then $\mu(G)$ is not hamiltonian.

One might suspect that the converse of Theorem 1.1 is true.

Conjecture 1.4. For a graph G, if $\mu(G)$ is hamiltonian, then G has a Hamilton cycle.

But, the conjecture is not true, see the graph $\Theta_{4,4,4}$ in Figure 2. Clearly, $\mu(\Theta_{4,4,4})$ is not hamiltonian by Lemma 2.1. However, a Hamilton cycle of $\mu(\Theta_{4,4,4})$ is given.

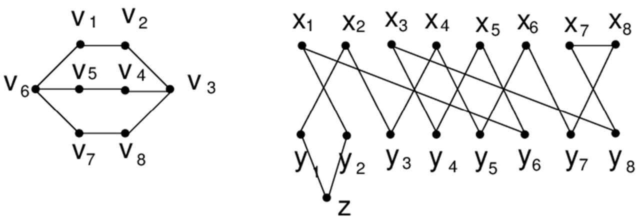

Figure 2. *$\Theta_{4,4,4}$ and a Hamilton cycle of $\mu(\Theta_{4,4,4})$.*

In this note, we show the converse of Theorem 1.1 is almost true for bipartite graphs. Precisely, we show that for a biparite graph G, if the mycielski graph $\mu(G)$ of a graph G is hamiltonian, then G has a Hamilton path.

2. Main Theorem

For a graph G, $c(G)$ denotes the number of components of G. The following is the well-known 1-tough condition for the hamiltonicity of graphs.

Lemma 2.1. If G is hamiltonian, then for any nonempty subset $S \subseteq V(G)$, $c(G-S) \leq |S|$.

A bipartite graph $G = G[B,W]$ is said to be *balanced* if $|B| = |W|$.

Lemma 2.2. Let G be a bipartite graph. If $\mu(G)$ is hamiltonian, then G is balanced.

proof. Let (B,W) be the bipartition of G, and let B', W' be the corresponding vertices in $\mu(G)$, and z be the special vertex. By contradiction, suppose that G is not balanced, and that $|B| > |W|$. Take $S = \{z\} \cup W \cup W'$. Then $c(\mu(G) - S) = |B \cup B'| = 2|B| > 2|W| + 1$, which contradicts the fact of Lemma 2.1. This shows that $|B| = |W|$.

Lemma 2.3. For a graph G, $\mu(G)$ is hamiltonian if and only if G has a walk P with two distinct end vertices v_1 and v_2, say, with the following additional properties:

(1) every vertex of G appear precisely twice on P, and

(2) there exists an edge e of P, such that $P - e$ is divided into two walks P_1 and P_2, and if for every vertex $v \in V(G)$, if v appear twice on some P_i, then the length of $P[v,v]$ is odd; and otherwise, the length of $P[v,v]$ is even.

Proof. First we show the necessity. Let C be a Hamilton cycle of $\mu(G)$. Then $C - z$ is a path of $\mu(G)$ joining two vertices y_1, y_2, say, in Y. Recall that y_1 and y_2 correspond to v_1 and v_2 in G, respectively.

Walking along on the path $C - z$ from y_1 to y_2, one obtain a walk P in G, with the property that *(1)* every vertex of G appear twice on P. Let e be the unique edge of P, which joins two vertices in X, and let P_1 and P_2 be the two walks results from P deleting e. Let v be a vertex of G, and assume that v corresponds to a vertex x of X and a vertex y of Y. Then the walk $P[v,v]$ corresponds to the path of $(C-z)[x,y]$, and thus they have the same length. By the definition of Mycielski graph, Y is an independent set of $\mu(G)$, the length of $(C-z)[x,y]$ is odd if and only if v appear twice either on P_1 or P_2. This proves *(2)*.

If there is a walk in G with the property in the assumption of the lemma, one could find a Hamilton path in $\mu(G) - z$ with two end vertices in Y by going along on the walk, and thereby connecting them to z, we are able to obtain a Hamilton cycle of $\mu(G)$.

Now we are ready to present our main theorem.

Theorem 2.4. *Let G be a bipartite graph. If $\mu(G)$ is hamiltonian, then G has a Hamilton path.*

Proof. Let C be a Hamilton cycle of $\mu(G)$. Then, as in proof of Lemma 2.3, one could get a walk P with the properties described in the assumption in Lemma 2.3. Let e, P_1, P_2 be those, as given in Lemma 2.3. We claim that both P_1 and P_2 are Hamilton paths of G. Note that there is no vertex of G appear on P_1 or P_2 twice. Since, otherwise, by Lemma 2.3, the length of $P_i[v,v]$ is odd if there is such a vertex v in G. It implies that G has an odd cycle, which contradicts our assumption that G is bipartite. Since every vertex of G appear on P twice, and by our claim that every vertex of G appear on P_i at most once, we conclude that every vertex of G appear on P_i precisely once, and thus both P_1 and P_2 are Hamilton paths of G.

3. Conclusion

In this short note, it was shown for a biparite graph G, if the mycielski graph $\mu(G)$ of a graph G is hamiltonian, then G has a Hamilton path. We conclude with posing the following two conjectures.

Conjecture 3.1. *For a graph G, if $\mu(G)$ is hamiltonian, then G has a Hamilton path.*

For a positive integer k, let us define $\mu^k(G) = \mu(\mu^{k-1}(G))$, where $\mu^1(G) = \mu(G)$.

Conjecture 3.2. *For a graph G without isolated vertices, there exists a natural number k_0, such that $\mu^k(G)$ is hamiltonian for all $k \geq k_0$.*

References

[1] R. Balakrishnan, S. F. Raj, Connectivity of the Mycielskian of a graph, Discrete Math. 308 (2008) 2607-2610.

[2] G. J. Chang, L. Huang, X. Zhu, Circular chromatic number of Mycielski's graph, Discrete Math. 205 (1999) 23-37.

[3] M. Chen, X. Guo, H. Li, L. Zhang, Total chromatic number of generalized Mycielski graph., Discrete Math. 334 (2014) 48–51.

[4] D. C. Fisher, P. A. McKenna, E. D. Boyer, Hamiltonicity, diameter, domination, packing, and biclique partitions of Mycielski' graphs, Discrete Appl. Math. 84 (1998) 93-105.

[5] D. Hajibolhassan, X. Zhu, The circular chromatic number an Mycielski construction, J. Graph Theory 44 (2003) 106-115.

[6] L. Guo, X. Guo, Connectivity of the Mycielskian of a digraph, Appl. Math. Lett. 22 (2009) 1622-1625.

[7] W. Jarnicki, W. Myrvold, P. Saltzman, S. Wagon, Properties, proved and conjectured, of Keller, Mycielski, and queen graphs, Ars Math. Contemp. 13 (2017) 427–460.

[8] P. C. B. Lam, W. Lin, G. Gu, Z. Song, Circular chromatic number and a generalization of the construction of Mycielski, J. Combin. Theory Ser. B 89 (2003) 195-205.

[9] M. Larsen, J. Propp. D. Ullman, The fractional chromatic number of Mycielski's graphs, J. Graph Theory 19 (1995) 411-416.

[10] D. Liu, Circular chromatic number for iterated Mycielski graphs, Discrete Math. 285 (2004) 335-340.

[11] X. Liu, Z. Dang, B. Wu, The hub number, girth and Mycielski graphs, Inform. Process. Lett. 114 (2014) 561–563.

[12] H. Liu, Circular chromatic number and Mycielski graphs, Acta Math. Sci. 26B (2) (2006) 314-320.

[13] J. Mycielski, Sur le coloriage des graphs, Colloq. Math. 3 (1955) 161-162.

[14] H. P. Patil, R. Pandiya, On the total graph of Mycielski graphs, central graphs and their covering numbers, Discuss. Math. Graph Theory 33 (2013) 361–371.

[15] Y. Tang, X. Zhu, Total weight choosability of Mycielski graphs, J. Comb. Optim. 33 (2017) 165–182.

[16] D. B. West, Introduction to Graph Theory, Second Edition, Prentice Hall, Upper Saddle River, NJ (2001).