Stability Study of a Holling-II Type Model and Leslie-Gower Modified with Diffusion and Time Delays in Dimension 3

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Abstract: This current paper investigates a predator-prey model from Holling-II type and Leslie Gower modified with diffusion and two time delays in dimension three. Firstly, we demonstrate that its solutions are positive and globally bounded. Secondly, we study the local stability of six equilibria points of from one is located within the relevant domain. Under certain conditions, it reveals that among the equilibria points, some are locally stable. Finally, we focus on the global stability of the positive interior equilibrium point. We show that the global stability set out due to the lack of time delays is kept until a certain threshold value of time delays above which a change in the stability is observed. Thus, the global convergence analysis towards the positive interior equilibrium point demonstrate the importance and impacts of the time delay in the stability of our model.

Keywords: Holling-2, Leslie-Gower, Boundedness, Lyapunov’s Functional, Equilibrium Point, Local Stability, Global Stability, Time Delay

1. Introduction

In the tropic network, the predator-prey link is characterized by the dynamic interaction between prey and predator populations. This interaction that can bind three and probably many species is the extension of the one that links two species [1]. This study deals with three-species food chain model. It is a dimension three model describing a population of preys $U_1$; that constitutes the only food of the predators’ population $U_2$. This predator called intermediary is also the prey of another upper predator named super predator $U_3$.

The model takes into account the diffusion in predator-prey interactions and reflects the opportunity for each species present to move in a given space. For mathematical and simplification reasons, we select a limited open set $\Omega$, from which we assume that the migration flows towards its boundary $\partial\Omega$ is null. Predators and preys density is also supposed null from the exterior of the chosen domain. We get the following model by adding the term of diffusion to the given model in the article [2]:

$$
\begin{align*}
\frac{\partial U_1(S,X)}{\partial t} &= \delta_1 \Delta U_1(S,X) + (a_0 - b_0 U_1(S,X)) - \frac{v_1 U_2(S,X)}{U_1(S,X) + d_0} U_1(S,X), \\
\frac{\partial U_2(S,X)}{\partial t} &= \delta_2 \Delta U_2(S,X) + (-a_1 + \frac{v_1 U_1(S,X)}{U_1(S,X) + d_0}) U_2(S,X), \\
\frac{\partial U_3(S,X)}{\partial t} &= \delta_3 \Delta U_3(S,X) + (c_3 - \frac{v_1 U_1(S,X)}{U_1(S,X) + d_0}) U_3(S,X), \\
U_1(0,X) &> 0, U_2(0,X) > 0, U_3(0,X) > 0,
\end{align*}
$$

where $X \in \Omega, S > 0$

$$(a_0, a_1, b_0, c_3, d_0, d_2, d_3, v_0, v_1, v_2, v_3) \in (\mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}_0^+) \times \mathbb{R}^+$

are ecological parameters [2], $U_1(S,X), U_2(S,X)$ and $U_3(S,X)$ respectively indicate not only the densities of both prey and intermediary predator but also that of the super predator at instant $S$ and position $X$,
shown that if, for every specy, the instantaneous intraspecific competition (instantaneous negative feedback) dominates the total competition due to delayed intraspecific competition and interspecific competition, then, the positive equilibrium point of the system remains globally and asymptotically stable. Most of the global stability or convergence results appearing so far for delayed ecological systems, require that the instantaneous negative feedbacks dominate both delayed feedback and interspecific interactions. Such requirement is rarely met in real systems when feedbacks are generally delayed. The model studied by Nindjin and al did not have any term of diffusion [2,7]. Therefore, the issue of mobility of the species in Ω has not been tackled. This approach is not always consistent with both ecological and biological realities. We introduce a term of diffusion to take into account the mobility of species to complement their works. The resulting model seems to be more realistic.

2. Presentation of the Model

Thus, this study is based on the following model:

\[
\begin{align*}
\frac{dU_1(S,X)}{ds} & = \delta_1 \Delta U_1(S,X) + (a_0 - b_0)U_1(S - \tilde{r}_1, X) - \nu_1 U_2(S, X) - \frac{v_1 U_1(S, X)}{U_1(S, X) + a_0} U_1(S, X), \\
\frac{dU_2(S,X)}{ds} & = \delta_2 \Delta U_2(S,X) + \left(-a_1 + \frac{v_1 U_1(S, X)}{U_1(S, X) + a_0}\right) U_2(S, X) - b_1 U_2(S - \tilde{r}_2, X) - \nu_2 U_3(S, X) U_2(S, X), \\
\frac{dU_3(S,X)}{ds} & = \delta_3 \Delta U_3(S,X) + \left(c_3 - \frac{v_3 U_3(S, X)}{U_2(S, X) + d_3}\right) U_3(S, X), \\
U_1(\theta, \cdot) & = \phi_1(\theta, \cdot), U_2(\theta, \cdot) = \phi_2(\theta, \cdot), U_3(\theta, \cdot) = \phi_3(\theta, \cdot), \\
U_1(0, \cdot) & > 0, U_2(0, \cdot) > 0, U_3(0, \cdot) > 0, \\
\phi & = (\phi_1, \phi_2, \phi_3) \in C([\tilde{r}; 0] \times \bar{\Omega}; \mathbb{R}^3), \\
\frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = \frac{\partial U_3}{\partial n} & = 0 \quad \text{on} \quad ]0; +\infty[ \times \partial \Omega,
\end{align*}
\]

Where \( X \in \Omega, S > 0, \theta \in [-\tilde{r}; 0] \) with

\[ \tilde{r} = \max \{\tilde{r}_1; \tilde{r}_2\} \]

In (2), the focus has not only been on the Laplacian operator which expresses the diffusion but especially, the time delays \( \tilde{r}_1 \) and \( \tilde{r}_2 \) which indicate respectively a time of recruitment for the prey \( U_1 \) and the intermediary predator \( U_2 \) [2]. In other words, the length of time necessary for these species to take part in the research of food, the procreation or occupation of space.

Furthermore, the insertion of both time delays and the term of diffusion is a relevant approach, because, it is more realistic and more interesting in the research for a better dynamic interaction comprehension between a predator and its prey. Considering the following variable changes:

\[ t = a_0 S, x = \left(\frac{a_0}{\delta_1}\right) \tilde{x}, W_1(t, x) = \frac{b_0}{a_0} U_1(T, X), W_2(t, x) = \frac{v_1 b_0}{a_0} U_2(T, X), W_3(t, x) = \frac{v_2 v_3 b_0}{a_0} U_3(T, X), \]

\[ \tilde{r}_1 = \max \{\tilde{r}_1; \tilde{r}_2\} \]

and \( \tilde{r} \) respectively designate the rate of prey, intermediaries predators and that of super predators increase at instant \( S \) and the position \( X \) depending on the following ecological parameters:

\( \delta_1 \) is the prey diffusion coefficient \( U_1 \),

\( \delta_2 \) is the predator diffusion coefficient \( U_2 \),

\( \delta_3 \) is the super predator diffusion coefficient \( U_3 \),

\( \Delta \) is the operator of Laplace.

In two dimensions, this schema was subject to many studies. There is a lot of articles about it [1, 3]. However, in dimension three, fewer works have been done. Aziz [7] studied a similar model to (1).

Nindjin and al [2] included the term \(-\nu_2 U_3(S, X)\) in the dynamics of the predator \( U_2 \). They studied the impact of certain time delays on the dynamic of these three species. This insertion permits to take into account the internal competition between the members of this population especially in the research of food, procreation or occupation of the space.

Indeed, in the instantaneous case, most of them require the considered system to satisfy the so-called negative instantaneous diagonal feedback dominance condition. In the delayed system Lotka-Volterra-type, Kuang and Smith [8] showed that if, for every specy, the instantaneous intraspecific competition (instantaneous negative feedback) dominates the total competition due to delayed intraspecific competition and interspecific competition, then, the positive equilibrium point of the system remains globally and asymptotically stable. Most of the global stability or convergence results appearing so far for delayed ecological systems, require that the instantaneous negative feedbacks dominate both delayed feedback and interspecific interactions. Such requirement is rarely met in real systems when feedbacks are generally delayed. The model studied by Nindjin and al did not have any term of diffusion [2,7]. Therefore, the issue of mobility of the species in \( \Omega \) has not been tackled. This approach is not always consistent with both ecological and biological realities. We introduce a term of diffusion to take into account the mobility of species to complement their works. The resulting model seems to be more realistic.
\[ a = \frac{d_0h_0}{a_0}, \quad b = \frac{a_1}{a_0}, \quad \sigma_2 = \frac{\delta_2}{\delta_1}, \quad \sigma_3 = \frac{\delta_2}{\delta_1}, \quad c = v_1, \quad e = \frac{a_0b_1}{a_0}, \quad d = \frac{d_3b_0h_0}{a_0}, \quad p = \frac{c_3}{a_0}, \quad q = \frac{v_3}{v_2}, \quad s = \frac{d_3b_0h_0}{a_0}, \quad r_1 = a_0\bar{r}_1 \text{ and } r_2 = a_0\bar{r}_2 \]

we get the following completed model which is subject to our study:

\[
\begin{align*}
\frac{dW_1(t,x)}{dt} &= \Delta W_1(t,x) + (1 - W_1(t-r_1,x)) \\
&- \frac{qW_2(t,x)}{W_2(t,x)+a}W_1(t,x), \\
\frac{dW_2(t,x)}{dt} &= \sigma_2\Delta W_2(t,x) + (-b + \frac{cW_2(t,x)}{W_4(t,x)+a})W_2(t,x) \\
&- eW_2(t, -r_2, x) - \frac{w_3(t,x)}{w_2(t,x)+a}W_2(t,x), \\
\frac{dW_3(t,x)}{dt} &= \sigma_3\Delta W_3(t,x) \\
&+ (p - \frac{qW_3(t,x)}{W_3(t,x)+a})W_3(t,x), \\
W_1(0,x) &> 0, W_2(0,x) > 0, W_3(0,x) > 0, \\
\phi &= (\phi_1, \phi_2, \phi_3) \in C([-r; 0] \cap \Omega), \\
\frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_3}{\partial n} = 0 \quad \text{on } [0; +\infty) \cap \partial \Omega.
\end{align*}
\]

Where \( x \in \Omega, t > 0, \theta \in [-r; 0] \) with

\[ r = \max\{r_1; r_2\} \]

In this paper, our goal is to find out the natural, realistic and easily verifiable conditions under which, the global stability established in the instantaneous model remains the same. To achieve this aim, we set up an appropriated Lyapunov’s functional. But, long before that, we showed the solutions boundedness by using methods employed in article [9]. Then, locally we analysed the equilibrium points of the system (3).

### 3. Global Boundedness of Solutions

Let us determine the sufficient conditions that ensure the global boundedness of the solutions of model (3). For that,

\[
\begin{align*}
\frac{dW_2(t,x)}{dt} &\leq \sigma_2\Delta W_2(t,x) + (c - b - eW_2(t-r_2,x))W_2(t,x)
\end{align*}
\]

According to lemma 3.1,

\[ \lim_{t \to -\infty} \max_{x \in \Omega} W_1(t,x) \leq e^{r_1} \text{ and } \]

\[ \limsup_{t \to +\infty} \max_{x \in \Omega} W_2(t,x) \leq \frac{c-b}{e}e^{(c-b)r_2}. \]

Thus, \( \forall \epsilon > 0, \exists T_1 > 0, T_2 > 0/\forall t > \max(T_1, T_2), W_1(t,x) \leq e^{r_1} + \epsilon \) and

\[ W_2(t,x) \leq \frac{c-b}{e}e^{(c-b)r_2} + \epsilon, \quad \forall x \in \Omega. \]

In that case, one gets: \( W_1(t,x) \leq M_1 \) and \( W_2(t,x) \leq M_2 \) with \( M_2 > 0 \) because \( c > b \).

for smallest and positive \( \epsilon \), let us define the following numbers:

\[ M_1 = e^{r_1} + \epsilon, M_2 = \frac{c-b}{e}e^{(c-b)r_2} + \epsilon, M_3 = \frac{c-b}{q}M_2 + s + \epsilon, m_4 = \frac{c-b}{a} + \epsilon \]

\[ m_3 = \frac{c-b}{a} + \epsilon, m_3 = \frac{c-b}{q}M_2 + s + \epsilon \]

\[ K = 1 - \frac{m_3}{a} + \epsilon, m_2 = \frac{c-b}{e}e^{(c-b)r_2} - \epsilon, \quad \text{with } B = b - \frac{m_3}{a} + \epsilon. \]

Then, considering the domain

\[ D = [m_1, M_1] \times [m_2, M_2] \times [m_3, M_3] \text{ of } \mathbb{R}^3. \]

Theorem 3.1 : If \( c > b \), \( \frac{e^{(c-b)r_2}}{c-b} < \frac{\epsilon}{c-b} \) and \( \frac{cm_1}{a+\epsilon r_1} > \frac{\epsilon}{a+\epsilon r_1} \)

\[ \text{so, the system (3) is globally bounded and any solution of this system stays in the domain } D. \]

In order to prove the theorem (3.1), let us state the following lemma.

Lemma 3.1 : Let us consider the system

\[ \begin{align*}
\frac{dv(t,x)}{dt} &= \sigma\Delta v(t,x) + v(t,x)g(v(t-r,x)) \\
\frac{\partial v}{\partial n} &= 0 \quad \text{on } [0; +\infty) \cap \partial \Omega \\
v(s,x) &\geq 0 \quad \forall \theta \in [-r; 0]
\end{align*} \]

Where \( t > 0, x \in \Omega \)

If \( v(0,x) \neq 0 \), then, we have:

if \( g(\nu) \leq \alpha(1 - \frac{1}{a}v) \) so, any solution \( \nu \) of (5) verifies \( \limsup_{t \to +\infty} \max_{x \in \Omega} \nu(t,x) \leq de^{at} \),

if \( g(\nu) \geq \alpha(1 - \frac{1}{a}v) \) then, any solution \( \nu \) of (5) verifies \( \liminf_{t \to +\infty} \min_{x \in \Omega} \nu(t,x) \geq de^{at} \) with \( A = 1 - e^{at} \).

Proof: See the reference [9].

Now, let us prove theorem (3.1).

Proof : The two first equations of model (3) permit to lead to the following inequalities:

\[ \frac{dW_2(t,x)}{dt} \leq \Delta W_1(t,x) + (1 - W_{12}(t-r_{12},x))W_2(t,x) \]

\[ \frac{dW_3(t,x)}{dt} \leq \sigma_3\Delta W_3(t,x) + (p - \frac{qW_3(t,x)}{M_4+t})W_3(t,x) \]

By proceeding like previously: \( \exists T_2 > 0/\forall t > T_3 \)

\[ W_2(t,x) \leq M_3 \]. Consequently, \( W_3(t,x) \leq M_3 \text{ and } W_3(t,x) \leq M_3/\forall t > \max(T_1, T_2, T_3), \forall x \in \Omega. \]

Let us seek the lowest values.

The first equation of (3) gives:
\[ \frac{dW_i(t,x)}{dt} \geq \Delta W_i(t,x) + \left(1 - \frac{M_i}{a} - W_i(t-r, x)\right)W_i(t,x) \]

\[ x \in \Omega, \ t > \max(T_1 + r_1T_2) \]

By posing \( K = 1 - \frac{M_i}{a} \) and applying the lemma 3.1 to (7), we show that

\[ \exists \ T_6 > 0 / \forall t > T_6, W_i(t,x) \geq Ke^{1-Kr_1}e^{-\varepsilon t} \forall x \in \Omega. \]

It is clear that \( K > 0 \). So, for the smallest \( \varepsilon \), \( m_1 > 0 \) and \( W_i(t,x) \geq m_1 \).

A similar approach permits to conclude that

\[ \exists \ T_5 > 0 / \forall t > T_5, m_3 \leq W_i(t,x), \forall x \in \Omega. \]

Taking into account the lowest values of \( W_3 \) and \( W_1 \), the second equation of (3) is:

\[ \frac{dW_2(t,x)}{dt} \geq \sigma_2 \Delta W_2(t,x) + \left(B - eW_2(t-r_2x)\right)W_2(t,x) \]

Thus, \( \exists \ T_6 > 0 / \forall t > T_6, W_2(t,x) \geq \frac{\beta}{\varepsilon} e^{(1-\varepsilon r_2) t} - \varepsilon. \) Then, \( W_2(t,x) \geq m_2 \).

Let us demonstrate that \( m_2 > 0 \). For that, we only have to prove that \( B > 0 \).

We have, \( B = -b - \frac{\sigma_2}{d} + \frac{cm_1}{\alpha + e^r_1} \). Considering the two first conditions of theorem 3.1 and by decaying \( \varepsilon \) toward zero, one gets:

\[ B > -c - \frac{pa}{d} - \frac{ps}{d} + \frac{cm_1}{d} (a + s) \]

Whereas

\[ \frac{cm_1}{d} > \frac{p}{d} (a + s), \] so, \( B > 0 \)

We conclude that \( m_2 > 0 \). Hence the result.

Remark 3.1 : When \( r_1 = 0 \) and \( r_2 = 0 \), we get the following values:

\[ M_1 = 1 + \varepsilon, \ M_2 = \frac{c-b}{\varepsilon} + \varepsilon, \ M_3 = \frac{\beta}{\varepsilon} (M_2 + s) + \varepsilon, \]

\[ m_1 = 1 - \frac{M_2}{a} + \varepsilon \text{ and } m_2 = 1 + \varepsilon \text{ with } \]

\[ B = -b - \frac{M_2}{a} + \frac{cm_1}{a + M_1} \]

\[ (10) \]

In that case, the conditions to have the boundedness are:

\[ c > b, \ 1 < \frac{ar}{c-b} \text{ and } \frac{cm}{a} > \frac{p}{d} (a + s). \]

Remark 3.2 : In all the following work, we assume that the model is globally bounded.

4. The Equilibria Points

The system (3) has trivial equilibria points which are:

\[ S_0 = (0,0,0), \ S_1 = (1,0,0), \ S_2 = (0,0, \frac{ps}{q}) \]

\[ \gamma_0 = -\left( ea^3 + \left( de + \frac{p}{q} \right) a^2 + \frac{ps}{q} a + bda + ba^2 \right), \]

\[ \gamma_1 = -e(-2a^2 + 3a^2) - d^2 + c^d + ac - bd - b(2a - a^2) \]

On the plan \( W_1 = 0 \), there is no equilibrium point.

On the plan \( W_2 = 0 \), one has only a trivial equilibrium point which is \( S_3 = (1,0, \frac{ps}{q}) \).

On the plan \( W_3 = 0 \), one has only a non trivial equilibrium point which is \( S_4 = (W_1^*,(1-W_1^*)(W_2^* + a),0) \) where \( W_1^* \)

is eventually a positive root of

\[ P_4(x) = y_3x^3 + y_2x^2 + y_1x + y_0 \]

\[ \gamma_0 = -a(b + e), \gamma_1 = c - b - ac, \gamma_2 = e(2a - 1), \gamma_3 = e. \]

The polynomial \( P_4 \) admits at least a real root because it has an odd-degree.

Moreover, the product of these roots is \( \frac{ab}{a+b} > 0 \).

So, \( P_4 \) at least has a strictly positive root noted \( W_1^* \). This must verify the condition.

\[ \frac{ab}{a+b} < W_1^* < 1 \]

(11)

for \( S_4 \) to be a constant equilibrium point.

We have the following result relative of the interior equilibrium point.

Theorem 4.1 : If \( \frac{ps + bd}{8} > \frac{ab}{a+b} \) and \( c > max(b,3e + \frac{ps}{2q} + \frac{bd}{2}) \), then, the system (3) admits a positive constant interior equilibrium point.

Proof : The system (3) admits a constant interior equilibrium point

\[ S_5 = (W_1^{**}, W_2^{**}, W_3^{**}) \] if and only if \( (W_1^{**}, W_2^{**}, W_3^{**}) \) is the following system solution:

\[ \begin{cases} 
1 - W_1^{**} - \frac{W_1^*}{W_1^* + a} = 0, \\
-b + \frac{cw_{11}^*}{W_1^* + a} - eW_2^{**} - \frac{w_{12}^*}{W_2^* + d} = 0, \\
p - \frac{w_{22}^*}{W_2^* + s} = 0.
\end{cases} \] (12)

By setting \( W_2^{**} \) and \( W_3^{**} \) according to \( W_1^{**} \) in (12), \( W_1^{**} \) becomes a root of the polynomial \( P_2 \).

\[ P_2(x) = y_3x^5 + y_4x^4 + y_5x^3 + y_2x^2 + y_1x + y_0 \]

With
\[ \gamma_2 = -e(a^3 - 6a^2 + 3a) - \left( de + \frac{p}{q} - b \right) (1 - 2a) + c(1 - a) \]
\[ \gamma_3 = -e(3a^2 - 6a + 1) + \left( de + \frac{p}{q} \right) - c + b \]
\[ \gamma_4 = -e(3a - 2) \gamma_5 = -e \]

As the degree of \( P_2 \) is odd, the polynomial \( P_2 \) admits at least a real root.
We get \( P_2(0) = \gamma_0 < 0 \) and
\[ P_2(1) = \gamma_5 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 + \gamma_0 = a[-6e + 2c + 4(c - b)] - (a + 1) \left( \frac{ps}{q} + bd \right) \]
One has \( P_2(1) > 0 \) because \( c > 3e \) and
\[ \frac{a}{a + 1} > \frac{ps}{q} + bd \]
\[ -6e + 2c + 4(c - b) \]

As \( P_2 \) is continuous, under the conditions of theorem 4.1, at least, one of these roots which belong to \( \{0; 1\} \) noted \( W_1^{**} \).
Hence \( S_5 \) exists and showed under the following form:
\[ S_5 = (W_1^{**}, W_2^{**}, W_3^{**}) = (W_1^{**}, (1 - W_1^{**})(W_1^{**} + a), \frac{p}{q} (1 - W_1^{**})(W_1^{**} + a)) \]

5. Local Stability

In this subsection, we study the local stability in the neighborhood of \( S_k \) where \( k = 0, \ldots, 5 \). We have two cases. One of them is the instantaneous system and the other is the system with delays.

5.1. Local Stability of the Instantaneous System

The model with no time delays looks as follows:
\[ \frac{\partial W_1(t,x)}{\partial t} = \Delta W_1(t,x) + (1 - W_1(t,x)) - \frac{W_2(t,x)}{W_1(t,x) + a} W_1(t,x), \]
\[ \frac{\partial W_2(t,x)}{\partial t} = \sigma_3 \Delta W_2(t,x) + (-b + \frac{c W_1(t,x)}{W_2(t,x) + a}) - e W_2(t,x) - \frac{W_1(t,x)}{W_2(t,x)} W_2(t,x), \]
\[ \frac{\partial W_3(t,x)}{\partial t} = \sigma_3 \Delta W_3(t,x) + (p - \frac{q W_1(t,x)}{W_2(t,x) + a}) W_3(t,x), \]
\[ W_1(0,.) > 0, W_2(0,.) > 0, W_3(0,.) > 0, \]
\[ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_3}{\partial n} = 0, \text{ on } \partial \Omega; + \infty \times \partial \Omega. \]

For that, let us consider \( (\mu_i, \varphi_i)_{i=0}^{\infty} \) the set developed by the eigenvalue and eigenvector pairs of the operator \(-\Delta\) on \( \Omega \) with homogeneous Neumann boundary conditions such as \( 0 = \mu_0 < \mu_1 < \mu_2 < \ldots \).

\[ f_1(W_1(t,x), W_2(t,x), W_3(t,x)) = \left( 1 - W_1(t,x) - \frac{W_2(t,x)}{W_1(t,x) + a} \right) W_1(t,x) \]
\[ f_2(W_1(t,x), W_2(t,x), W_3(t,x)) = (-b + \frac{c W_1(t,x)}{W_2(t,x) + a}) - e W_2(t,x) - \frac{W_1(t,x)}{W_2(t,x) + a} W_2(t,x) \]
\[ f_3(W_1(t,x), W_2(t,x), W_3(t,x)) = \left( p - \frac{q W_1(t,x)}{W_2(t,x) + a} \right) W_3(t,x). \]
For any fixed \( i \geq 0 \), \( X_i \) is invariant under \( \mathbb{I} \). Thus, the \( \mathbb{I} \) matrix in \( X_i \) becomes:

\[
\begin{pmatrix}
-\mu_i + A^{(k)}_{11} & A^{(k)}_{12} & A^{(k)}_{13} \\
A^{(k)}_{21} & -\sigma \mu_i + A^{(k)}_{22} & A^{(k)}_{23} \\
A^{(k)}_{31} & A^{(k)}_{32} & -\sigma_3 \mu_i + A^{(k)}_{33}
\end{pmatrix}
\]  

(19)

Our aim is to study the eigenvalues signs of (19) in order to establish the stability of the equilibria points \( S_k \) where \( k = 1, \ldots, 5 \).

- a. Stability of \( S_0 = (0,0,0) \)
  - One has: \( A^{(1)}_{11} = 1, A^{(1)}_{12} = 0, A^{(1)}_{13} = 0, A^{(2)}_{21} = 0, A^{(2)}_{22} = -b, A^{(2)}_{23} = 0, A^{(3)}_{31} = 0, A^{(3)}_{32} = 0, A^{(3)}_{33} = p \).
  - The eigenvalues of (19) to the neighborhood of \( S_0 \) are \( 1 - \mu_i - \sigma_2 \mu_i - b \) and \(-\sigma_3 \mu_i + p \). We get:
    - If \( \max \left\{ \frac{\mu_i}{\sigma_2}, 1 \right\} < \mu_i \) then, \( S_0 \) is stable.
    - Otherwise, \( S_0 \) is unstable.

- b. Stability of \( S_1 = (1,0,0) \)
  - One has: \( A^{(1)}_{11} = -1, A^{(1)}_{12} = -\frac{1}{1+a}, A^{(1)}_{13} = 0, A^{(2)}_{21} = 0, A^{(2)}_{22} = -b + \frac{a}{1+a}, A^{(2)}_{23} = 0, A^{(3)}_{31} = 0, A^{(3)}_{32} = 0, A^{(3)}_{33} = p \).
  - The eigenvalues of (19) to the neighborhood of \( S_1 \) are \( -1 - \mu_i - \sigma_2 \mu_i - b + \frac{a}{1+a} \) and \(-\sigma_3 \mu_i + p \). We have:
    - If \( \mu_i > \max \left\{ \frac{1}{\sigma_2} (-b + \frac{a}{1+a}), \frac{1}{\sigma_3} \right\} \) then, \( S_1 \) is stable.
    - Otherwise, \( S_1 \) is unstable.

- c. Stability of \( S_2 = (0,0, \frac{q}{p}) \)
  - We have: \( A^{(2)}_{11} = 1, A^{(2)}_{12} = 0, A^{(2)}_{13} = 0, A^{(2)}_{21} = 0, A^{(2)}_{22} = -b - \frac{q}{p}, A^{(2)}_{23} = 0, A^{(3)}_{31} = 0, A^{(3)}_{32} = 0, A^{(3)}_{33} = -p \).
  - The eigenvalues of (19) to the neighborhood of \( S_2 \) are \( -1 - \mu_i - \sigma_2 \mu_i + -\frac{q}{p} \) and \(-\sigma_3 \mu_i - p \). We have:
    - If \( \mu_i > \frac{q}{\sigma_2} \) then, \( S_2 \) is a stable node.
    - Otherwise, \( S_2 \) is unstable.

- d. Stability of \( S_3 = (1,0, \frac{q}{p}) \)
  - we have: \( A^{(3)}_{11} = -1, A^{(3)}_{12} = -\frac{1}{1+a}, A^{(3)}_{13} = 0, A^{(3)}_{21} = 0, A^{(3)}_{22} = -b + \frac{a}{1+a} \frac{q}{p}, A^{(3)}_{23} = 0, A^{(3)}_{31} = 0, A^{(3)}_{32} = \frac{q^2}{p}, A^{(3)}_{33} = -p \).
  - The eigenvalues of (19) to the neighborhood of \( S_3 \) are \( -1 - \mu_i - \sigma_2 \mu_i + A^{(3)}_{22} \) and \(-\sigma_3 \mu_i - p \). We get:
    - If \( \frac{q}{\sigma_2} < \mu_i \) so, \( S_3 \) is a stable node.
    - Otherwise, \( S_3 \) is unstable.

- e. Stability of \( S_4 = (W_1^{*}, W_2^{*}, 0) \) and \( S_5 = (W_1^{**}, W_2^{**}, W_3^{**}) \)
  - The eigenvalues of (19) are the solutions of the following equation:
    \[
    \lambda^3 + \alpha_2^{(k)} \lambda^2 + \alpha_1^{(k)} \lambda + \alpha_0^{(k)} = 0
    \]
  - With
    \[
    \alpha_2^{(k)} = (1 + \sigma_2 + \sigma_3) \mu_i - (A_{22}^{(k)} + A_{11}^{(k)} + A_{33}^{(k)})
    \]
    \[
    \alpha_1^{(k)} = (\sigma_2 \sigma_3 + \sigma_2 + \sigma_3) \mu_i^2 - (\sigma_2 A_{11}^{(k)} + \sigma_3 A_{11}^{(k)} + A_{22}^{(k)} + A_{33}^{(k)} + \sigma_2 A_{33}^{(k)} + \sigma_3 A_{33}^{(k)}) \mu_i + (A_{11}^{(k)} A_{22}^{(k)} + A_{11}^{(k)} A_{33}^{(k)} + A_{22}^{(k)} A_{33}^{(k)})
    \]
    \[
    -A_{23}^{(k)} A_{32}^{(k)} - A_{12}^{(k)} A_{12}^{(k)}
    \]
    \[
    \alpha_0^{(k)} = \sigma_2 \sigma_3 \mu_i^3 - (A_{11}^{(k)} \sigma_2 A_{12}^{(k)} + A_{33}^{(k)} \sigma_2 A_{32}^{(k)}) \mu_i^2 + (A_{11}^{(k)} A_{22}^{(k)} + A_{11}^{(k)} A_{33}^{(k)} + A_{33}^{(k)} A_{22}^{(k)} - A_{23}^{(k)} A_{32}^{(k)} - A_{12}^{(k)} A_{12}^{(k)}) \mu_i
    \]
    \[
    -A_{11}^{(k)} A_{22}^{(k)} A_{33}^{(k)} + A_{11}^{(k)} A_{23}^{(k)} A_{32}^{(k)} + A_{21}^{(k)} A_{12}^{(k)} A_{33}^{(k)}
    \]
  - Theorem 5.1: \( S_k \) is stable if and only if
    \[
    \alpha_2^{(k)} > 0, \alpha_0^{(k)} > 0 \text{ and } \alpha_2^{(k)} \alpha_1^{(k)} > \alpha_0^{(k)} \quad (k = 4; 5)
    \]
  - Proof: The eigenvalues of (19) are the solutions of the following equation:
    \[
    \lambda^3 + \alpha_2^{(k)} \lambda^2 + \alpha_1^{(k)} \lambda + \alpha_0^{(k)} = 0
    \]
    Whereas according to Routh-Hurwitz criterion, this
equation admits some solutions with a real negative part only if:

\[ a_2^{(k)} > 0, \ a_0^{(k)} > 0 \text{ and } a_2^{(k)} \times a_1^{(k)} > a_0^{(k)} \quad (22) \]

Hence we have the following result.

Remark 5.1: To get \( a_2^{(5)} > 0, \ a_0^{(5)} > 0 \) and \( a_2^{(5)} \times a_1^{(5)} > a_0^{(5)} \), we only have \( A_{11}^{(5)} < 0 \) and \( A_{22}^{(5)} < 0 \).

We already know that

\[ A_{11}^{(5)} = 1 - 2W_{1}^{**} \frac{aW_{2}^{**}}{(W_{1}^{**} + a)^2} = \frac{W_{1}^{**}(-2W_{1}^{**} + 1 - a)}{W_{1}^{**} + a} \]

So, when \( W_{1}^{**} > \frac{1 - a}{2} \), one gets \( A_{11}^{(5)} < 0 \).

In a similar way, when \( W_{1}^{**} < \frac{ab}{c - b} A_{22}^{(5)} < 0 \).

Consequently a sufficient condition to have the local stability of \( S_{k}^{(5)} \) is:

\[ \frac{c - b}{a} < W_{1}^{**} < \frac{ab}{c - b} \text{ provided that } \max\left(\frac{c - b}{a}, -\frac{ab}{c - b}\right) < a. \]

5.2. Local Stability of the System with Time Delays

Let us note \( (W_{1}^{(k)}, W_{2}^{(k)}, W_{3}^{(k)}) \) the equilibria points \( S_{k} \) with \( k = 0, \ldots, 5 \) and let us pose \( V_{1}(t, x) = W_{1}(t, x) - W_{1}^{(k)}, \ V_{2}(t, x) = W_{2}(t, x) - W_{2}^{(k)}, \) and \( V_{3}(t, x) = W_{3}(t, x) - W_{3}^{(k)} \)

The linearized system to the neighborhood of the equilibria points \( S_{k} \) is for all \( x \in \Omega \) and \( t > 0 \):

\[
\begin{align*}
\frac{dV_{1}(t, x)}{dt} &= \Delta V_{1}(t, x) + A_{11}^{(k)}V_{1}(t, x) + A_{12}^{(k)}V_{2}(t, x) + A_{13}^{(k)}V_{3}(t, x), \\
\frac{dV_{2}(t, x)}{dt} &= \sigma_{2}\Delta V_{2}(t, x) + A_{21}^{(k)}V_{1}(t, x) + A_{22}^{(k)}V_{2}(t, x) + A_{23}^{(k)}V_{3}(t, x), \\
\frac{dV_{3}(t, x)}{dt} &= \sigma_{3}\Delta V_{3}(t, x) + A_{31}^{(k)}V_{1}(t, x) + A_{32}^{(k)}V_{2}(t, x) + A_{33}^{(k)}V_{3}(t, x),
\end{align*}
\]

with \( A_{11}^{(k)} = 1 - W_{1}^{(k)} - \frac{aW_{2}^{(k)}}{(W_{1}^{(k)} + a)^2}, \ A_{12}^{(k)} = \frac{W_{1}^{(k)}}{W_{1}^{(k)} + a}, \ A_{14}^{(k)} = -W_{1}^{(k)}, \ A_{21}^{(k)} = \frac{caw_{2}^{(k)}}{(W_{1}^{(k)} + a)^2}, \ A_{22}^{(k)} = -b + \frac{caw_{2}^{(k)}}{W_{1}^{(k)} + a} - eW_{2}^{(k)} - \frac{dw_{2}^{(k)}}{(W_{1}^{(k)} + a)^2}, \)

\[ A_{23}^{(k)} = -\frac{W_{1}^{(k)}}{W_{1}^{(k)} + a}, \ A_{24}^{(k)} = -eW_{2}^{(k)}, \ A_{32}^{(k)} = \frac{g(w_{2}^{(k)})^2}{(W_{1}^{(k)} + a)^2}, \ A_{33}^{(k)} = p - \frac{2g(w_{2}^{(k)})}{W_{1}^{(k)} + a}. \]

5.2.1. Local Stability of the System with \( r_{1} \neq 0 \) and \( r_{2} = 0 \)

In this subsection, \( A_{22}^{(k)} = 0 \). The characteristic equation of the linearized model (23) is:

\[ \Delta^{(k)}(\lambda, \mu_{1}, r_{1}) = P^{(k)}(\lambda) + Q^{(k)}(\lambda)e^{-r_{1}\lambda} = 0 \]

Where

\[ P^{(k)}(\lambda) = (A_{11}^{(k)} - \lambda - \mu_{1})[(A_{22}^{(k)} - \lambda - \sigma_{2}\mu_{1})(A_{33}^{(k)} - \lambda - \sigma_{3}\mu_{1}) - A_{23}^{(k)}(A_{33}^{(k)} - \lambda - \sigma_{3}\mu_{1})] - A_{12}^{(k)}A_{21}^{(k)}(A_{33}^{(k)} - \lambda - \sigma_{3}\mu_{1}) \]

And

\[ Q^{(k)}(\lambda) = A_{14}^{(k)}[(A_{22}^{(k)} - \lambda - \sigma_{2}\mu_{1})(A_{33}^{(k)} - \lambda - \sigma_{3}\mu_{1}) - A_{23}^{(k)}A_{33}^{(k)}] \]

For each cases, we explain the conditions to obtain

\[ P^{(k)}(0) + Q^{(k)}(0) \neq 0 \]

It is clear that \( P^{(k)}(-j\gamma) = P^{(k)}(j\gamma) \) and \( Q^{(k)}(-j\gamma) = Q^{(k)}(j\gamma) \), with \( t^{2} = -1 \).

(iv). we have \( \limsup\{Q^{(k)}(\lambda)/|\lambda| \to +\infty \text{ and } Re(\lambda) \geq 0\} = 0 \). So, \( \limsup\{|P^{(k)}(\lambda)/|\lambda| \to +\infty \text{ and } Re(\lambda) \geq 0\} < 1 \).

(v). Let pose \( F^{(k)}(\gamma) = |P^{(k)}(j\gamma)| - |Q^{(k)}(j\gamma)|^{2} \). The function \( F^{(k)} \) could be under the following form
\[
F^{(k)}(y) = y^6 + \eta_2^{(k)} y^4 + \eta_1^{(k)} y^2 + \eta_0^{(k)}
\]  

Where:

\[
\eta_0^{(k)} = [(A_{11}^{(k)} - \mu_i)(A_{22}^{(k)} - \mu_i \sigma_2)(A_{33}^{(k)} - \mu_i \sigma_3) - A_{12}^{(k)} A_{23}^{(k)}(A_{11}^{(k)} - \mu_i) - A_{12}^{(k)} A_{23}^{(k)}(A_{33}^{(k)} - \mu_i \sigma_3)]^2 - (A_{12}^{(k)})^2[A_{22}^{(k)} - \mu_i \sigma_2 - \mu_i \sigma_3] - A_{32}^{(k)} A_{23}^{(k)}
\]

\[
\eta_1^{(k)} = -2(A_{11}^{(k)} - \mu_i + A_{22}^{(k)} - \mu_i \sigma_2 + A_{33}^{(k)} - \mu_i \sigma_3)[(A_{11}^{(k)} - \mu_i)(A_{22}^{(k)} - \mu_i \sigma_2)(A_{33}^{(k)} - \mu_i \sigma_3) - A_{32}^{(k)} A_{23}^{(k)}(A_{11}^{(k)} - \mu_i)]
\]

The analysis of the local stability of (3) to the neighborhood of \( S_k \) is based on the existence of a positive root of \( F^{(k)} \). It is apparent that:

- \( F^{(k)} \) admits a positive root if \( \eta_0^{(k)} < 0 \).
- If \( \eta_1^{(k)} > 0 \), then, \( F^{(k)} \) does not admit any positive root.

Remark 5.2:

- If \( \mu_i < \frac{1}{\sigma_2} \) and \( \mu_i \neq \frac{A_{11}^{(k)}}{\sigma_2} \), so,

\[
P^{(1)}(0) + Q^{(1)}(0) \neq 0 \quad \text{and} \quad F^{(1)} \text{ admits at least a positive root}.
\]

\[
\left(A_{11}^{(k)} - \mu_i \right) \left(A_{22}^{(k)} - \sigma_2 \mu_i \right) \left(A_{33}^{(k)} - \mu_i \right) - A_{32}^{(k)} A_{23}^{(k)} \left(A_{11}^{(k)} - \mu_i \right) - A_{12}^{(k)} A_{23}^{(k)} \left(A_{11}^{(k)} - \mu_i \right) - A_{12}^{(k)} A_{23}^{(k)} \left(A_{33}^{(k)} - \sigma_2 \mu_i \right) - A_{14}^{(k)} \left(A_{11}^{(k)} - \mu_i \right) - A_{14}^{(k)} \left(A_{33}^{(k)} - \sigma_2 \mu_i \right) - A_{14}^{(k)} \left(A_{23}^{(k)} - \mu_i \right) > 0
\]

So, \( P^{(5)}(0) + Q^{(5)}(0) \neq 0 \) and \( F^{(5)} \) admits at least a positive root.

Let us find out \( \delta(r_1^{(k)}) = \text{sign}(\frac{d \Re(F^{(k)}(y))}{dr_1}|_{y=y_1}) \) the real part sign of a solution \( \lambda \) from the characteristic of the equation \( \Delta^{(k)}(\lambda, \mu_i, r_1^{(k)}) = 0 \).

Lemma 5.1: Let us take account \( \lambda \) as a positive solution of the characteristic of the equation \( \Delta^{(k)}(\lambda, \mu_i, r_1^{(k)}) = 0 \).

Let us name \( y_1^{(k)} = y(r_1^{(k)}) \) the positive root of \( F^{(k)} \) and \( r_1^{(k)} \) the associated time delay verifying, for all \( n \in \mathbb{N} \), the

\[
\delta(r_1^{(k)}) = \text{sign}(\frac{d \Re(F^{(k)}(y))}{dr_1}|_{y=y_1}) = \text{Sign}(\Re(F^{(k)}(y_1)))
\]

Whereas \( \frac{d \delta_n}{dr_1} = 1 \), so,

\[
\delta(r_1^{(k)}) = \text{sign}(3y_1^{(k)} + 2\eta_2^{(k)}y_1^{(k)} + \eta_1^{(k)}).
\]

Remark 5.3:

1. If \( \eta_0^{(k)} > 0 \) and \( \eta_1^{(k)} > 0 \) where

\[
\eta_0^{(k)} < 0 \quad \text{or} \quad \eta_1^{(k)} < 0
\]

2. If \( \eta_1^{(k)} < 0 \), then, if \( y_1^{(k)} \in \mathbb{R} \), then, \( \delta(r_1^{(k)}) > 0 \).

3. If \( \eta_1^{(k)} > 0 \), then, if \( y_1^{(k)} \in \mathbb{R} \), then, \( \delta(r_1^{(k)}) > 0 \).

\[
\text{If} \quad y_1^{(k)} \in [0, \sqrt{X_1}] \quad \text{then}, \quad \delta(r_1^{(k)}) < 0
\]

with \( X_0 \) the positive solution of the equation \( 3X^2 + 2\eta_2^{(k)}X + \eta_1^{(k)} = 0 \).

Let us state out the following theorem.

Theorem 5.2: Let us assume that the stability hypothesis in the instantaneous case are verified. Thus, for \( k = 1, 3, 4, 5 \):

1. there is no change in the stability of the equilibrium points \( S_k \) in the following case: \( n_1^{(k)} > 0 \), \( l = 0, 1, 2 \).

2. We notice the following stability changes:

In the case where \( \delta(r_1^{(k)}) > 0 \).

If for \( r_1^{(k)} = 0 \), \( S_k \) was stable, it remains stable when

\[
0 \leq r_1^{(k)} < r_1^{(k)}
\]

and becomes unstable if \( r_1^{(k)} > r_1^{(k)} \).
If for \( r_1 = 0 \), \( S_k \) was unstable, it remains for \( r_1 \geq 0 \). In the case where \( \delta (r_1) < 0 \).

If for \( r_1 = 0 \), \( S_k \) was stable, it stays stable when \( r_1 \geq 0 \).

If for \( r_1 = 0 \), \( S_k \) was unstable, it remains unstable when \( 0 \leq r_1 < r_1^{(k)} \) and becomes stable if \( r_1 > r_1^{(k)} \).

Proof : 1. If \( \eta > 0, l = 0, 1, 2 \) so, \( F(k) \) defined by (25) does not admit any real root.

2. If \( \eta < 0 \) then, \( F(k) \) admits at least a positive root \( \eta_1 \) associated to the time delay \( r_1^{(k)} \). From the reference [5], and by taking into account the remarks (5.2) and (5.3), we have the stability of \( S_k \).

\[
\lambda_k (\lambda) = R_1^{(k)} (\lambda) + R_2^{(k)} (\lambda) e^{-r_1 \lambda} + (R_3^{(k)} (\lambda) + R_4^{(k)} (\lambda) e^{-r_2 \lambda}) e^{-r_2 \lambda} = 0
\]  

(26)

Where

\[
R_1^{(k)} (\lambda) = (A_{11}^{(k)} - \lambda - \mu_1) (A_{22}^{(k)} - \lambda - \sigma_2 \mu_2) - A_{12}^{(k)} A_{21}^{(k)} (A_{33}^{(k)} - \lambda - \sigma_3 \mu_3) - A_{32}^{(k)} A_{23}^{(k)} (A_{44}^{(k)} - \lambda - \sigma_4 \mu_4)
\]

\[
R_2^{(k)} (\lambda) = A_{14}^{(k)} [(A_{22}^{(k)} - \lambda - \sigma_2 \mu_2) (A_{33}^{(k)} - \lambda - \sigma_3 \mu_3) - A_{32}^{(k)} A_{23}^{(k)}] R_3^{(k)} (\lambda) = A_{24}^{(k)} (A_{11}^{(k)} - \lambda - \mu_1) (A_{33}^{(k)} - \lambda - \sigma_3 \mu_3)
\]

\[
R_4^{(k)} (\lambda) = A_{44}^{(k)} [(A_{11}^{(k)} - \lambda - \mu_1) (A_{33}^{(k)} - \lambda - \sigma_3 \mu_3)]
\]

When \( r_1 \) is the parameter and \( r_2 \) the variable, (26) becomes:

\[
\lambda_k (\lambda, \mu_1, r_1, r_2) = p(k) (\lambda) + Q(k) (\lambda) e^{-r_2 \lambda} = 0
\]  

(27)

With \( P(k) (\lambda) = R_1^{(k)} (\lambda) + R_2^{(k)} (\lambda) e^{-r_2 \lambda} \) and \( Q(k) (\lambda) = R_3^{(k)} (\lambda) + R_4^{(k)} (\lambda) e^{-r_2 \lambda} \). However, if \( r_2 \) becomes the parameter and \( r_1 \) the variable, (26) is:

\[
\lambda_k (\lambda, \mu_1, r_1, r_2) = p(k) (\lambda) + Q(k) (\lambda) e^{-r_1 \lambda} = 0
\]  

(28)

with \( P(k) (\lambda) = R_1^{(k)} (\lambda) + R_3^{(k)} (\lambda) e^{-r_2 \lambda} \) and \( Q(k) (\lambda) = R_2^{(k)} (\lambda) + R_4^{(k)} (\lambda) e^{-r_2 \lambda} \). It should be noticed that, for the equilibria points \( S_k \), with \( k = 0, 1, 2, 3 \), \( A_{24}^{(k)} = 0 \) or \( A_{14}^{(k)} = 0 \). So, the stability study to the neighborhood of these equilibria points refers to the previous case.

This is why, we only study the system stability to the neighborhood of the equilibria points \( S_k \) for \( k = 4, 5 \).

We are interesting in the impact of the time delay \( r_2 \) by keeping \( r_1 \) as a parameter. Considering the characteristic of the equation (27) we have:

1. \( P(k) (-j \omega) = P(k) (\omega) \) and \( Q(k) (-j \omega) = Q(k) (\omega) \) with \( \omega^2 = -1 \).

2. If \( \frac{\mu_1 - A_{11}^{(k)}}{A_{14}^{(k)}} > 1 \) so, \( P(k) (\lambda) \) and \( Q(k) (\lambda) \) have no common imaginary roots.

Remark 5.4 : When \( r_2 \neq 0 \) and \( r_1 = 0 \) then, \( A_{14}^{(k)} = 0 \) and \( A_{44}^{(k)} \neq 0 \). In that case, by using a similar process to the case \( r_1 \neq 0 \) and \( r_2 = 0 \), we study the stability of the equilibria points \( S_k \).

5.2.2. Local Stability of the System with \( r_1 \neq 0 \) and \( r_2 \neq 0 \)

In this part, the two time delays are considered as non null.

We focus on the impact of one of them meanwhile the other one is viewed as a parameter.

The characteristic equation of (23) to the neighborhood of \( S_k \), for all \( k = 0, \ldots, 5 \), is:

\[
\lambda_k (\lambda) = R_1^{(k)} (\lambda) + R_2^{(k)} (\lambda) e^{-r_1 \lambda} + (R_3^{(k)} (\lambda) + R_4^{(k)} (\lambda) e^{-r_2 \lambda}) e^{-r_2 \lambda} = 0
\]  

Indeed, if \( P(k) (\lambda) \) and \( Q(k) (\lambda) \) have common imaginary roots \( j \omega, \) \( \omega \) verify the relation \( \cos \omega t = \frac{p(k)}{\sqrt{\rho}} \frac{A_{11}^{(k)}}{A_{14}^{(k)}} \).

Which is impossible because \( \frac{\mu_1 - A_{11}^{(k)}}{A_{14}^{(k)}} > 1 \).

3. \( \limsup \{ |\frac{Q(k)}{P(k)}|/|\lambda| \to +\infty \) and \( Re (\lambda) \geq 0 \) = 0. So, \( \limsup \{ |\frac{Q(k)}{P(k)}|/|\lambda| \to +\infty \) and \( Re (\lambda) \geq 0 \} < 1 \).

4. If \( A_{11}^{(k)} - \mu_1 < 0, A_{22}^{(k)} - \sigma_2 \mu_2 < 0 \) and \( A_{33}^{(k)} - \sigma_3 \mu_3 < 0 \) so, \( P(k) (0) \) and \( Q(k) (0) \) ≠ 0.

Let us consider the function defined on \( \mathbb{R} \) by

\[
F(k) (\omega) = |P(k) (\omega)|^2 - |Q(k) (\omega)|^2
\]

The function \( F(k) \) could be under the following form:

\[
H(k) (\omega) = 2H_1^{(k)} (\omega) \cos (r_1 \omega) + 2H_2^{(k)} (\omega) \sin (r_1 \omega),
\]

\[
G' (\omega) = y^6 + \eta_2 y^4 + \eta_1 y^2 + \eta_0, \]

\[
H_2^{(k)} (\omega) = y^5 + \xi_2 y^3 + \xi_1 y + \xi_0 \]

Where

\[
\eta_2 (\omega) = (E_1^{(k)})^2 + (E_2^{(k)})^2 + (E_3^{(k)})^2 + 2D_1^{(k)} + 2D_2^{(k)} + (D_3^{(k)})^2 - (D_4^{(k)})^2
\]

\[
\eta_1 (\omega) = 2D_1^{(k)} (D_3^{(k)})^2 + 2D_2^{(k)} (E_1^{(k)})^2 + (D_1^{(k)})^2 + (E_2^{(k)})^2 + (E_1^{(k)} E_2^{(k)})^2 + (E_3^{(k)})^2 + (E_2^{(k)} E_3^{(k)})^2 + 2D_1^{(k)} D_2^{(k)}
\]

\[
-2D_1^{(k)} E_1^{(k)} E_2^{(k)} - 2D_2^{(k)} E_2^{(k)} E_3^{(k)} + (D_3^{(k)})^2 + (D_4^{(k)} E_4^{(k)})^2 + 2D_3^{(k)} D_4^{(k)} - (D_4^{(k)} E_4^{(k)})^2 - (D_4^{(k)} E_4^{(k)})^2
\]
\[ \eta_0^{(k)} = [E_1^{(k)}(E_2^{(k)}E_3^{(k)} - D_2^{(k)}) - E_2^{(k)}E_1^{(k)}] + [D_1^{(k)}(E_2^{(k)}E_3^{(k)} - D_2^{(k)})] - (D_4^{(k)}E_1^{(k)}E_3^{(k)})^2 - (D_4^{(k)}D_3^{(k)}E_3^{(k)})^2 \]

\[ \xi_2^{(k)} = D_3^{(k)}E_1^{(k)} \]

\[ \xi_1^{(k)} = 2D_2^{(k)}D_3^{(k)}E_1^{(k)} - D_3^{(k)}D_3^{(k)}E_2^{(k)} + D_3^{(k)}E_1^{(k)}(E_2^{(k)})^2 + D_4^{(k)}E_1^{(k)}(E_2^{(k)})^2 - D_3^{(k)}E_1^{(k)}(D_4^{(k)})^2 \]

\[ \xi_0^{(k)} = D_3^{(k)}E_1^{(k)}(E_2^{(k)})^2 - D_3^{(k)}D_1^{(k)}E_2^{(k)}E_3^{(k)} - D_3^{(k)}D_1^{(k)}E_2^{(k)}(E_3^{(k)})^2 + D_3^{(k)}D_1^{(k)}D_2^{(k)}E_3^{(k)} - D_3^{(k)}D_2^{(k)}E_1^{(k)}E_2^{(k)}E_3^{(k)} \]

\[ + D_3^{(k)}(D_1^{(k)})^2E_1^{(k)} - (D_4^{(k)})^2D_3^{(k)}E_1^{(k)}(E_3^{(k)})^2 \]

\[ \gamma_2^{(k)} = D_3^{(k)} \]

\[ \gamma_1^{(k)} = 2D_2^{(k)}D_2^{(k)} + D_3^{(k)}D_1^{(k)} + D_4^{(k)}(E_2^{(k)})^2 + D_3^{(k)}(E_3^{(k)})^2 - (D_4^{(k)})^2D_2^{(k)} \]

\[ \gamma_0^{(k)} = D_3^{(k)}D_2^{(k)}D_1^{(k)} - 2D_2^{(k)}D_3^{(k)}E_2^{(k)}E_3^{(k)} + D_3^{(k)}(D_2^{(k)})^2 + D_3^{(k)}(E_3^{(k)})^2 + D_3^{(k)}D_1^{(k)}(E_2^{(k)})^2 - (D_4^{(k)})^2D_3^{(k)}(E_3^{(k)})^2 \]

with

\[ E_1^{(k)} = A_{11}^{(k)} - \mu_1E_2^{(k)} = A_{22}^{(k)} - \mu_1E_3^{(k)} = A_{33}^{(k)} - \mu_1 \]

\[ D_1^{(k)} = A_{12}^{(k)}A_{21}^{(k)}D_2^{(k)} = A_{23}^{(k)}A_{32}^{(k)}D_3^{(k)} = A_{43}^{(k)} \] and \[ D_4^{(k)} = A_{24}^{(k)} \]

The following result reveals the existence conditions of a positive root from \( F^{(k)} \) to the neighborhood of \( S_k \).

Proposition 5.1: Let us note

\[ Z^{(k)} = \sqrt{(H_1^{(k)})^2 + (H_2^{(k)})^2} \] (33)

\[ \beta_1 = \left( \frac{\xi_1^{(k)}}{\xi_2^{(k)}} \right)^2 + \eta_1^{(k)} - \frac{\xi_1^{(k)}}{\xi_2^{(k)}} \quad \text{and} \]

\[ \beta_0 = \frac{\xi_0^{(k)}\xi_1^{(k)}}{(\xi_2^{(k)})^2} + \eta_0^{(k)} - \frac{\eta_0^{(k)}}{\xi_2^{(k)}} \]

1. If \( Y_2^{(k)}\xi_1^{(k)} = Y_2^{(k)}\xi_1^{(k)} \), \( Y_0^{(k)}\xi_2^{(k)} = Y_0^{(k)}\xi_2^{(k)} \) so,

\( F^{(k)} \) admits a positive root \( \sqrt{\frac{-\beta_1^{(k)}}{\beta_0^{(k)}}} \) if \( \beta_0\beta_1 < 0 \).

\( F^{(k)} \) does not admit any root if not.

2. Let us assume that \( |G^{(k)}(y)| < |Z^{(k)}(y)| \).

Let us pose \( \forall n \in \mathbb{N} \),

\[ \psi_n(y) = y - \frac{1}{r_1}\arctan\left(\frac{H_1^{(k)}(y)}{H_2^{(k)}(y)}\right) - \frac{1}{r_1}\arccos(-\frac{G^{(k)}(y)}{Z^{(k)}(y)}) + \frac{2n\pi}{r_1} \]

Whereas \( Y_1^{(k)}\xi_1^{(k)} = Y_2^{(k)}\xi_1^{(k)} \), \( Y_0^{(k)}\xi_2^{(k)} = Y_2^{(k)}\xi_2^{(k)} \) so,

\( H_2^{(k)}(y) = 0 \).

In that case, \( F^{(k)} \) is reduced to \( G^{(k)} \).

Let us replace again \( y^4 \) by \(-\frac{1}{\xi_2^{(k)}}(\xi_1^{(k)}y^2 + \xi_0^{(k)}) \) in \( G^{(k)} \),

we have: \( G^{(k)}(y) = \beta_1^{(k)}y^2 + \beta_0^{(k)} \). So, when \( \beta_1^{(k)} \) and \( \beta_0^{(k)} \) are all non null and have the same sign then, \( G^{(k)} \) does not admit any positive root.

If \( \beta_1^{(k)} \) and \( \beta_0^{(k)} \) are opposite signs so, \( \sqrt{\frac{-\beta_1^{(k)}}{\beta_0^{(k)}}} \) becomes a unique positive root of \( G^{(k)} \) and \( F^{(k)} \).

Let us suppose that \( |G^{(k)}(y)| < |Z^{(k)}(y)| \) so, \( \cos(r_1y - \varphi_1) = \cos(\theta_1) \) with \( \theta_1 = \arccos(-\frac{G^{(k)}(y)}{Z^{(k)}(y)}) \).

Hence \( r_1y = \varphi_1 + \theta_1 + 2n\pi \).
Let us pose 
\[ \psi_n(y) = y - \frac{1}{r_1} \arctan(\frac{\mu_2^{(k)}(y)}{H_1^{(k)}(y)}) - \frac{1}{r_1} \arccos(\frac{G^{(k)}(y)}{Z^{(k)}(y)}) + \frac{2\pi}{r_1} \]

If \( n \) exists and the equation \( \psi_n(y) = 0 \) admits a positive solution \( y_0 \), then, \( F^{(k)}(y) = 0 \) admits \( y_0 \) as a positive root for all time delay \( r_1 \).

If \( y_1 > 0 \) and \( |G^{(k)}(y)| > |Z^{(k)}(y)| \), so, \( F^{(k)} \) does not admit any positive roots. □

Remark 5.5: If \( A^{(k)}_1 + A^{(k)}_2 > 0 \), we have 
\[ \gamma_1^{(k)} \zeta_1^{(k)} = \gamma_2^{(k)} \zeta_2^{(k)}, \gamma_0 = \gamma_2^{(k)} \zeta_2^{(k)} \]

\[ S_1^{(k)} = E_3^{(k)} = -S_2^{(k)} \]

\[ (A^{(k)}_1 + A^{(k)}_2)(1 - \sigma_3) = (A^{(k)}_1 - A^{(k)}_3)(1 + \sigma_2). \]

We know that \( A^{(k)}_3 = -p \), \( A^{(k)}_1 > 0 \) and \( A^{(k)}_2 > 0 \), so, in order to have 
\[ (A^{(k)}_1 + A^{(k)}_2)(1 - \sigma_3) = (A^{(k)}_1 - A^{(k)}_3)(1 + \sigma_2) \]

\[ \delta_2^{(k)} = Sgn(\frac{d\Re(\lambda)}{dr_2}) = Sgn(F^{(k)})Y Sgn(\frac{dS_2^{(k)}}{dr_2}) \; r_2 = r_2^{(k)} \]

However, \( \frac{dS_2^{(k)}}{dr_2} = 1 \), so, if \( Sgn(F^{(k)})Y > 0 \) and that \( r_2 = 0 \) the equilibrium point \( S_2^{(k)} \) is stable so, when \( r_2 \in [0; r_2^{(k)}] \), it stays stable. If \( r_2 > r_2^{(k)} \), then, it becomes stable.

Contrarily, if it is unstable, it remains.

If \( Sgn(F^{(k)})Y < 0 \) and that, \( r_2 = 0 \), \( S_2^{(k)} \) was stable, it stays. But if it was unstable, it remains until \( r_2^{(k)} \). Then, for \( r_2 > r_2^{(k)} \), it becomes stable.

If \( F^{(k)} \) does not admit any positive roots, there is no stability change.

Proof: See the reference [5].


Let's pose 
\[ l_1(W_1, W_2, W_3)(t, x) = [W_1(t, x) - W_1^{**} - W_1^{**} \ln \frac{W_1(t, x)}{W_2^{**}}] + [W_2(t, x) - W_2^{**} - W_2^{**} \ln \frac{W_2(t, x)}{W_3^{**}}] + W_3^{**}[W_3(t, x) - W_3^{**} - W_3^{**} \ln \frac{W_3(t, x)}{W_3^{**}}] \]

The function \( l_1 \) admits zero for the global minimum reached in \( (W_1^{**}, W_2^{**}, W_3^{**}) \).

Let us pose 
\[ L_1(W_1, W_2, W_3)(t, x) = \int_\Omega l_1(W_1, W_2, W_3)(t, x) \, dx \]

Let us demonstrate that the function \( L \) as developed is a Lyapunov's function for the system (14).

1. We have: \( L_1(W_1^{**}, W_2^{**}, W_3^{**}) = 0 \).

2. For any solution \((W_1, W_2, W_3)\) positive of (14), \( L_1(W_1, W_2, W_3) \) is positive.

3. Let us prove the following inequality: \( \frac{dl_1}{dt} < 0 \).

We have:
\[ \frac{dl_1}{dt} = \int_\Omega \frac{\partial l_1(W_1, W_2, W_3)(t, x)}{\partial t} \, dx \]
Therefore,
\[
\frac{dL_1}{dt} = \int_{\Omega} \left[ \frac{\partial l_1(W_1, W_2, W_3)(t, x) \partial W_1(t, x)}{\partial W_1(t, x)} \frac{\partial W_1(t, x)}{\partial t} + \frac{\partial l_1(W_1, W_2, W_3)(t, x) \partial W_2(t, x)}{\partial W_2(t, x)} \frac{\partial W_2(t, x)}{\partial t} + \frac{\partial l_1(W_1, W_2, W_3)(t, x) \partial W_3(t, x)}{\partial W_3(t, x)} \frac{\partial W_3(t, x)}{\partial t} \right] dx
\]
\[
= \int_{\Omega} \left[ \frac{W_3(t, x) - W_3^{**} \frac{\partial W_3(t, x)}{\partial t}}{W_3(t, x)} \right] dx + W_3^{**} \int_{\Omega} \left[ \frac{W_3(t, x) - W_3^{**} \frac{\partial W_3(t, x)}{\partial t}}{W_3(t, x)} \right] dx + W_3^{**} \int_{\Omega} \left[ \frac{W_3(t, x) - W_3^{**} \frac{\partial W_3(t, x)}{\partial t}}{W_3(t, x)} \right] dx
\]
(37)

By using (12), (14) turns into :
\[
\begin{align*}
\frac{\partial W_1(t, x)}{\partial t} &= \frac{\Delta W_1(t, x)}{W_1(t, x)} - (W_1(t, x) - W_1^{**}) \\
&\quad - \frac{1}{W_1^{**} + a}(W_2(t, x) - W_2^{**}) \\
&\quad + \frac{W_2(t, x)}{(W_1^{**} + a)(W_1(t, x) + a)}(W_1(t, x) - W_1^{**}), \\
\frac{\partial W_2(t, x)}{\partial t} &= \frac{\sigma_2 \Delta W_2(t, x)}{W_2(t, x)} - e(W_2(t, x) - W_2^{**}) \\
&\quad + \frac{ca \sigma_2 \Delta W_2(t, x)}{W_2(t, x)} - e(W_2(t, x) - W_2^{**}) \\
&\quad + \frac{W_2(t, x)}{(W_1^{**} + a)(W_1(t, x) + a)}(W_1(t, x) - W_1^{**}) \\
&\quad - \frac{1}{W_2^{**} + d}(W_3(t, x) - W_3^{**}), \\
\frac{\partial W_3(t, x)}{\partial t} &= \frac{\sigma_3 \Delta W_3(t, x)}{W_3(t, x)} - \frac{q}{W_2^{**} + s}(W_3(t, x) - W_3^{**}) \\
&\quad + \frac{q W_3(t, x)}{(W_1^{**} + s)(W_1(t, x) + s)}(W_1(t, x) - W_1^{**})
\end{align*}
\]
(38)

So,
\[
\begin{align*}
\int_{\Omega} \frac{\partial l_1(W_1, W_2, W_3)}{\partial t} dx &= \int_{\Omega} \left[ W_1(t, x) - W_1^{**} \right] \frac{\Delta W_1(t, x)}{W_1(t, x)} - (W_1(t, x) - W_1^{**}) - \frac{1}{W_1^{**} + a}(W_2(t, x) - W_2^{**}) \\
&\quad + \frac{W_2(t, x)}{(W_1^{**} + a)(W_1(t, x) + a)}(W_1(t, x) - W_1^{**}) + \int_{\Omega} \left[ W_2(t, x) - W_2^{**} \right] \frac{\sigma_2 \Delta W_2(t, x)}{W_2(t, x)} - e(W_2(t, x) - W_2^{**}) \\
&\quad + \frac{ca \sigma_2 \Delta W_2(t, x)}{W_2(t, x)} - e(W_2(t, x) - W_2^{**}) - \frac{1}{W_2^{**} + d}(W_3(t, x) - W_3^{**}) + W_3^{**} \int_{\Omega} \left[ W_3(t, x) - W_3^{**} \right] \frac{\sigma_3 \Delta W_3(t, x)}{W_3(t, x)} - \frac{q}{W_2^{**} + s}(W_3(t, x) - W_3^{**}) \\
&\quad + \frac{q W_3(t, x)}{(W_1^{**} + s)(W_1(t, x) + s)}(W_1(t, x) - W_1^{**}) dx
\end{align*}
\]
(39)

By posing
\[
T_1 = \int_{\Omega} \left[ W_1(t, x) - W_1^{**} \right] (-W_1(t, x) - W_1^{**}) - \frac{1}{W_1^{**} + a}(W_2(t, x) - W_2^{**}) + \frac{W_2(t, x)}{(W_1^{**} + a)(W_1(t, x) + a)}(W_1(t, x) - W_1^{**}) \\
- \frac{W_3(t, x)}{(W_2^{**} + d)(W_2(t, x) + d)}(W_2(t, x) - W_2^{**}) - \frac{1}{W_2^{**} + d}(W_3(t, x) - W_3^{**}) dx
\]
(40)

And
\[
T_2 = \int_{\Omega} \left[ W_1(t, x) - W_1^{**} \right] \Delta W_1(t, x) + \frac{\sigma_2(W_2(t, x) - W_2^{**})}{W_2(t, x)} \Delta W_2(t, x) + W_3^{**} \frac{\sigma_3(W_3(t, x) - W_3^{**})}{W_3(t, x)} \Delta W_3(t, x) dx
\]
(41)
We have \( T_1 + T_2 = \int_{\Omega} \frac{\partial W_3}{\partial n}(W_1(t,x),W_2(t,x))\) dx.

Let us transform \( T_2\). Per Green’s formula and considering Neumann’s condition
\[
\frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_3}{\partial n} = 0,
\]
we get:
\[
T_2 = -W_1^{**} \int_{\Omega} \frac{|\nabla W_1(x)|^2}{W_1(t,x)^2} dx - \sigma_2 W_2^{**} \int_{\Omega} \frac{|\nabla W_2(x)|^2}{W_2(t,x)^2} dx - \sigma_3 (W_3^{**})^2 \int_{\Omega} \frac{|\nabla W_3(x)|^2}{W_3(t,x)^2} dx
\]
(42)

Let us compute the supremum value \( T_1,\)
\[
T_1 = \int_{\Omega} \left[-1 + \frac{W_2(t,x)}{(W_2^{**} + a)(W_1(t,x) + a)}\right] (W_1(t,x) - W_1^{**})^2 + \left[- e + \frac{W_3}{(W_2^{**} + d)(W_2(t,x) + d)}\right] (W_2(t,x) - W_2^{**})^2
\]
\[
- \frac{q W_3^{**}}{(W_2^{**} + s)} (W_3(t,x) - W_3^{**})^2 + \left[- \frac{1}{W_2^{**} + a} + \frac{c a}{(W_1^{**} + a)(W_1(t,x) + a)}\right] (W_2(t,x) - W_2^{**}) (W_1(t,x) - W_1^{**})
\]
\[
+ \frac{1}{W_2^{**} + d} + \frac{q W_3^{**}}{(W_2^{**} + s)(W_2(t,x) - W_2^{**})(W_3(t,x) - W_3^{**})}\right] dx
\]
\[
\leq \int_{\Omega} \left[-e + \frac{M_3}{2a} + \frac{1}{2a} + \frac{c}{2a} + \frac{p M_3}{2s}\right] (W_2(t,x) - W_2^{**})^2 + \left[-1 + \frac{M_2}{a^2} + \frac{c}{2a}\right] (W_1(t,x) - W_1^{**})^2
\]
\[
+ \frac{-p + \frac{p M_3}{2s} + \frac{1}{2a}}(W_3(t,x) - W_3^{**})^2 dx.
\]
(43)

So,
\[
\frac{dL_1}{dt} \leq - \int_{\Omega} \frac{W_1^{**}}{M_1^2} |\nabla W_1(t,x)|^2 dx - \int_{\Omega} \sigma_2 \left(\frac{W_2^{**}}{M_2}\right) |\nabla W_2(t,x)|^2 dx - \int_{\Omega} \sigma_3 \frac{W_3^{**}}{M_3} |\nabla W_3(t,x)|^2 dx + \int_{\Omega} C_1 (W_1(t,x) - W_1^{**})^2 dx
\]
\[
+ \int_{\Omega} C_2 (W_2(t,x) - W_2^{**})^2 dx + \int_{\Omega} C_3 (W_3(t,x) - W_3^{**})^2 dx
\]
(44)

So, under the conditions of theorem 6.1, \( \frac{dL_1}{dt} < 0.\)

Consequently, the equilibrium point \( S_5 = (W_1^{**}, W_2^{**}, W_3^{**})\) of the system is globally and asymptotically stable.

7. Global Stability of the System with Time Delays

Theorem 7.1: Let us assume the hypothesis of the theorems 4.1 and 6.1. Then, \( r_{01} \) and \( r_{02} \) exist such as, for all \( (r_1, r_2) \in [0; r_{01}] \times [0; r_{02}],\) the interior equilibrium point \( S_5 \) is globally and asymptotically stable in \( \mathbb{R}^3.\)

Proof: Let us assume that the theorem 4.1 is verified. Then, the model (3) admits a unique interior point \( S_5 = (W_1^{**}, W_2^{**}, W_3^{**})\) and it is bounded. Let us set
\[
G(W_1, W_2, W_3)(t,x) = (l_2(W_1, W_2, W_3) + \Sigma)(t,x)
\]
(45)

With,
\[
\Sigma(t,x) = \frac{1}{2} \int_{t-r_1}^{t} \int_{y} \left[|\nabla W_1(s,x)|^2 ds dy + \frac{\sigma_2}{2} \int_{t-r_1}^{t} \int_{y} |\nabla W_2(s,x)|^2 ds dy + \frac{r_3 M_2 e^2}{2} \int_{t-r_2}^{t} (W_2(s,x) - W_2^{**})^2 ds dy
\]
\[
+ \frac{M_1}{2} \int_{t-r_1}^{t} \int_{y} (W_1(s, r_1, x) - W_1^{**})^2 ds dy + \frac{M_2 e^2}{2} \int_{t-r_2}^{t} (W_2(s, r_2, x) - W_2^{**})^2 ds dy
\]
\[
+ \frac{M_4 e}{2} \int_{t-r_1}^{t} \int_{y} (W_1(s, x) - W_1^{**})^2 ds dy + \frac{M_1 M_2}{2a^2} \int_{t-r_1}^{t} \int_{y} (W_2(s, x) - W_2^{**})^2 ds dy
\]
\[
+ \frac{M_4 e}{2a} \int_{t-r_1}^{t} \int_{y} (W_1(s, x) - W_1^{**})^2 ds dy + \frac{M_1 M_4 e}{2d^2} \int_{t-r_2}^{t} \int_{y} (W_2(s, x) - W_2^{**})^2 ds dy
\]
\[
+ \frac{M_4 e}{2a} \int_{t-r_1}^{t} \int_{y} (W_1(s, x) - W_1^{**})^2 ds dy + \frac{r_3 M_1}{2} \int_{t-r_1}^{t} (W_1(s, x) - W_1^{**})^2 ds dy
\]
(46)

And
The function \( L_2 \) admits zero for the global minimum reached in \((W_1^*, W_2^*, W_3^*)\).

So, \( G(W_1, W_2, W_3)(t, x) \geq 0 \) with \( G(W_1^*, W_2^*, W_3^*) = 0 \)

Let us pose

\[
L_2(W_1, W_2, W_3)(t, x) = \int_\Omega G(W_1, W_2, W_3)(t, x) \, dx
\]

Let us show that the function \( L_2 \) as developed is a Lyapunov's functional for the system (3).

1. We have: \[ \frac{dL_2}{dt} < 0. \] We have:

\[
\frac{dL_2}{dt} = \int_\Omega \frac{\partial L_2(W_1, W_2, W_3)}{\partial t}(t, x) \, dx + \int_\Omega \frac{\partial L_2(W_1, W_2, W_3)}{\partial t}(t, x) \, dx
\]

One gets:

\[
\int_\Omega \frac{\partial L_2(W_1, W_2, W_3)}{\partial t}(t, x) \, dx = \int_\Omega \frac{\partial L_2(W_1, W_2, W_3)}{\partial t}(t, x) \, dx
\]

By using (12), (3) becomes:

\[
\frac{\partial W_1(t, x)}{\partial t} = \frac{\partial W_1(t, x)}{W_1(t, x)} - (W_1(t - r_1, x) - W_1^*)
- \frac{W_2(t, x)}{W_1(t, x)} - W_2^*
+ \frac{(W_1^* + a)(W_1(t, x) + a)}{W_1(t, x)}(W_1(t, x) - W_1^*)
\]

So, by setting

\[
T_3 = \int_\Omega [W_1(t, x) - W_1^*] - (W_3(t - r_2, x) - W_3^*) - \frac{1}{W_1^* + a} (W_2(t, x) - W_2^*)
\]

\[
+ \frac{W_2(t, x)}{(W_1^* + a)(W_1(t, x) + a)} (W_1(t, x) - W_1^*)\]
\[-\frac{1}{W_2^{**} + d} (W_3(t,x) - W_3^{**})] dx + W_3^{**} \int_{\Omega} [W_3(t,x) - W_3^{**}][-\frac{q}{W_2^{**} + s} (W_3(t,x) - W_3^{**}) + \frac{qW_3(t,x)}{(W_2^{**} + s)(W_3(t,x) + s)} (W_2(t,x) - W_2^{**})] dx \]

And

\[T_3 = \int_{\Omega} \left\{ \frac{W_1(t,x) - W_1^{**}}{W_1(t,x)} \Delta W_1(t,x) + \frac{W_2(t,x) - W_2^{**}}{W_2(t,x)} \sigma_2 \Delta W_2(t,x) + W_3^{**} \frac{W_2(t,x) - W_2^{**}}{W_3(t,x)} \sigma_3 \Delta W_3(t,x) \right\} dx \]

We have:

\[T_3 + T_4 = \int_{\Omega} \partial_{t^2} (W_1, W_2, W_3) dx \]

Let us change \(T_4\). Based on Green’s formula and taking into account Neumann’s condition

\[\frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_3}{\partial n} = 0 \]

We have:

\[T_4 = -W_1^{**} \int_{\Omega} \frac{|\nabla W_1(t,x)|^2}{W_1(t,x)^2} dx - \sigma_2 W_2^{**} \int_{\Omega} \frac{|\nabla W_2(t,x)|^2}{W_2(t,x)^2} dx - \sigma_3 (W_3^{**})^2 \int_{\Omega} \frac{|\nabla W_3(t,x)|^2}{W_3(t,x)^2} dx \]

Let us transform \(T_3\).

In view of the relation: for all \(i = 1, 2, W_i(t - r_i, x) = W_i(t, x) - \int_{t-r_i}^{t} \frac{\partial W_i(s,x)}{\partial s} ds, \)

\(T_3\) changes into:

\[T_3 = \int_{\Omega} [W_1(t,x) - W_1^{**}](-(W_1(t,x) - W_1^{**}) - \frac{1}{W_1^{**}} + a) (W_2(t,x) - W_2^{**}) + \frac{W_2(t,x)}{(W_1^{**} + a)(W_1(t,x) + a)} (W_1(t,x) - W_1^{**})] dx + \int_{\Omega} [W_2(t,x) - W_2^{**}](-e(W_2(t,x) - W_2^{**}) + \frac{ca}{W_1^{**} + a} (W_1(t,x) - W_1^{**}) + \frac{W_3(t,x)}{(W_1^{**} + a)(W_1(t,x) + a)} (W_1(t,x) - W_1^{**})] dx + \int_{\Omega} [W_3(t,x) - W_3^{**}](-\frac{q}{W_2^{**} + s} (W_3(t,x) - W_3^{**}) + \frac{qW_3(t,x)}{(W_2^{**} + s)(W_3(t,x) + s)} (W_2(t,x) - W_2^{**})] dx + \int_{t-r_1}^{t} (W_1(t,x) - W_1^{**}) \frac{\partial W_1(s,x)}{\partial s} ds \]

Let us pose

\[T_3 = T_{31} + T_{32}, \]

where

\[T_{31} = \int_{\Omega} [W_1(t,x) - W_1^{**}](-(W_1(t,x) - W_1^{**}) - \frac{1}{W_1^{**}} + a) (W_2(t,x) - W_2^{**}) + \frac{W_2(t,x)}{(W_1^{**} + a)(W_1(t,x) + a)} (W_1(t,x) - W_1^{**})] dx + \int_{\Omega} [W_2(t,x) - W_2^{**}](-e(W_2(t,x) - W_2^{**}) + \frac{ca}{W_1^{**} + a} (W_1(t,x) - W_1^{**}) + \frac{W_3(t,x)}{(W_1^{**} + a)(W_1(t,x) + a)} (W_1(t,x) - W_1^{**})] dx + \int_{\Omega} [W_3(t,x) - W_3^{**}](-\frac{q}{W_2^{**} + s} (W_3(t,x) - W_3^{**}) + \frac{qW_3(t,x)}{(W_2^{**} + s)(W_3(t,x) + s)} (W_2(t,x) - W_2^{**})] dx \]

And
\[ T_{32} = \int_0^t \int_{t-r_1}^t (W_1(t,x) - W_1^{**}) \frac{\partial W_1(x,t)}{\partial s} ds + \int_{t-r_2}^t e(W_2(t,x) - W_2^{**}) \frac{\partial W_2(x,t)}{\partial s} ds \]  
\[dx \]  

(60)

Let us compute the supremum value of \( T_3 \). For that, first, let us compute the supremum value of \( T_{31} \) and \( T_{32} \). Thus,

\[ T_{32} = \int_0^t \int_{t-r_1}^t (W_1(t,x) - W_1^{**}) \Delta W_1(s,x) ds + \sigma_2 e \int_0^t \int_{t-r_2}^t (W_2(t,x) - W_2^{**}) \Delta W_2(s,x) ds \]

\[dx \]

(61)

Finally, we have:

\[ T_{32} \leq \int_0^{r_2} \frac{1}{2} |\nabla W_1(t,x)|^2 + \frac{e \sigma_2 r_2}{2} |\nabla W_2(t,x)|^2 + \int_{t-r_1}^t \int_0^t \frac{1}{2} |\nabla W_1(s,x)|^2 ds + \frac{e \sigma_2}{2} \int_{t-r_2}^t |\nabla W_2(s,x)|^2 ds \]  

(62)

So,

\[ T_{32} \leq \frac{r_2}{2} \int_0^t |\nabla W_1(t,x)|^2 dx + \frac{e \sigma_2 r_2}{2} \int_0^t |\nabla W_2(t,x)|^2 dx + \int_0^{r_2} \frac{M_1}{2} (1 + \frac{1}{a} + \frac{M_2}{a^2})(W_1(t,x) - W_1^{**})^2 dx \]

(63)
\[ T_{31} = \int_\Omega \left[ -1 + \frac{W_2(t,x)}{(W_1^{**} + a)(W_1(t,x) + a)} \right] \left( W_1(t,x) - W_1^{**} \right)^2 + \left[-e + \frac{W_3}{(W_2^{**} + d)(W_2(t,x) + d)} \right] \left( W_2(t,x) - W_2^{**} \right)^2 \]
\[ - \frac{qW_3^{**}}{(W_2^{**} + s)} \left( W_2(t,x) - W_2^{**} \right)^2 + \left[ -e + \frac{1}{W_1^{**} + a} + \frac{ca}{(W_1^{**} + a)(W_1(t,x) + a)} \right] \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \]
\[ + \left[ -e + \frac{1}{W_1^{**} + a} + \frac{qW_3^{**}W_2(t,x)}{(W_2^{**} + s)(W_2 + s)} \right] \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \right] \right] dx 
\leq \int_\Omega \left[ -1 + \frac{1}{2a} + \frac{M_2}{2a} + \frac{c}{2a} \right] \left( W_1(t,x) - W_1^{**} \right)^2 + \left[ -e + \frac{M_3}{d^2} + \frac{1}{2a} + \frac{c}{2a} + \frac{1}{2a} + \frac{pM_3}{2s} \right] \left( W_2(t,x) - W_2^{**} \right)^2
\[ + \left[ -e + \frac{M_3}{d^2} + \frac{1}{2a} + \frac{pM_3}{2s} \right] \right] \left( W_3(t,x) - W_3^{**} \right)^2 \right] dx \quad (64) \]

Let us pose \( \Gamma = \int_\Omega \frac{\partial \mathcal{E}(W_1(t,x),W_2(t,x),W_3(t,x))}{\partial t} \) dx.

The relations (54) and (58) allow us to conclude that

\[ \Gamma \leq \int_\Omega \left( \frac{W_3}{W_1} \right) \left| \nabla W_1(t,x) \right|^2 dx + \sigma_2 \left( - \frac{W_3}{W_2} + \frac{er_2}{2} \right) \left| \nabla W_2(t,x) \right|^2 dx - \int_\Omega \sigma_3 \frac{W_3^{**}}{W_3} \left| \nabla W_3(t,x) \right|^2 dx
\]
\[ + \int_\Omega \left[ -1 + \frac{1}{2a} + \frac{M_2}{2a} + \frac{c}{2a} + \frac{r_1M_1}{2a^2} + \frac{r_1M_1M_2}{2a^2} \left( W_1(t,x) - W_1^{**} \right) \right] \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx
\[ + \int_\Omega \left[ -e + \frac{M_3}{d^2} + \frac{1}{2a} + \frac{c}{2a} + \frac{1}{2a} + \frac{pM_3}{2s} \right] \left( W_3(t,x) - W_3^{**} \right)^2 \left( W_3(t,x) - W_3^{**} \right) \right] dx \quad (65) \]

So, by using the lemma 5.1 of the article [1], \( \int_\Omega \frac{\partial \mathcal{E}(t)}{\partial t} dx \) becomes:

\[ \int_\Omega \frac{\partial \mathcal{E}(t)}{\partial t} dx = \int_\Omega \left[ \frac{1}{2a} \right] \left| \nabla W_1(t,x) \right|^2 dx - \int_\Omega \left[ \frac{1}{2a} \right] \left| \nabla W_2(t,x) \right|^2 dx - \int_\Omega \left[ \frac{1}{2a} \right] \left| \nabla W_3(t,x) \right|^2 dx
\]
\[ + \int_\Omega \left[ \frac{r_1M_1}{2a} \left( W_1(t,x) - W_1^{**} \right) \right] \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx
\[ - \frac{M_1e^2}{2a} \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx
\[ + \int_\Omega \left[ \frac{r_1M_1M_2}{2a^2} \left( W_1(t,x) - W_1^{**} \right) \right] \left( W_1(t,x) - W_1^{**} \right) \left( W_1(t,x) - W_1^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx
\[ - \frac{M_2e^2}{2a} \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_2(t,x) - W_2^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx
\[ + \int_\Omega \left[ \frac{r_2M_2}{2d^2} \left( W_3(t,x) - W_3^{**} \right) \right] \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \left( W_3(t,x) - W_3^{**} \right) \right] \right] dx \quad (66) \]
To sum up,
\[
\frac{dg}{dt} \leq \int_{\Omega} C_4 |\nabla W(t,x)|^2 dx + \int_{\Omega} C_5 |\nabla W(t,x)|^2 dx + \int_{\Omega} C_6 (W(t,x) - W_1^*)^2 dx + \int_{\Omega} C_8 (W(t,x) - W_2^*)^2 dx + \int_{\Omega} C_9 (W(t,x) - W_3^*)^2 dx.
\]  
(67)

Where
\[
C_4 = - \frac{W_{11}^*}{M_1^2} + r_1, C_5 = \sigma_2 \left( - \frac{W_{22}^*}{M_2^2} + e r_2 \right),
\]
(68)
\[
C_6 = - \sigma_3 \left( \frac{(W_{11}^*)^2}{M_1^2} \right),
\]
(69)
\[
C_7 = C_4 + \frac{r_r M_2}{2a} + r_1 M_1 + \frac{r_r M_2 M_1}{a^2} + 2 \int_0^1 \frac{r_r C_2 M_2}{2a} \]
(70)
\[
C_8 = C_2 + r_2 e^2 M_2 + \frac{r_r M_2 C_2}{a^2} + \frac{r_r C_2 M_2}{2a} + \frac{r_r C_2 M_2}{2a} + \frac{r_1 M_1}{a^2}
\]
(71)
\[
C_9 = C_3 + \frac{r_r e M_2}{2a^2}
\]
(72)

Under the hypothesis of the theorem 6.1, \( C_i < 0 \) for \( i = 1, 2, 3 \). Then, \( r_{11} \) and \( r_{12} \) exist such as, for all \( (r_1, r_2) \in [0; r_{11}] \times [0; r_{12}], C_i \) for \( i = 4, \ldots, 9 \) are all inferior to zero. In that case,
\[
\frac{dg}{dt} < 0
\]  
(73)

Consequently, the equilibrium \( S_6 = (W_1^*; W_2^*; W_3^*) \) of the system is globally and asymptotically stable. 

Remark 7.1: The global stability analysis shows that the stability established in the model with no time delays remains until the value of \( r_1 \) and \( r_2 \). At the neighborhood of these threshold values, there is a stability change. To get these values, it suffices to resolve the system \( C_i < 0 \) for \( i = 4, \ldots, 9 \). So, we may conclude that the time delays have a real impact on the stability study.

8. Conclusion

In this paper, we studied a food chain model with diffusion and time delays which implies three species whose corresponding densities are globally bounded. We demonstrated that, these delays inserted in order to heed the internal competition between preys and that of intermediary predators, lead up to a change of the local stability of some equilibria points under certain conditions. We are ending this study with the establishment of the local stability of some equilibrium points under certain conditions. Indeed, the stability established in the instantaneous model remains up to a threshold value of delays beyond which a change of global stability is observed. This conclusion remains valid even if, we consider the internal competition between preys for any species’ number in presence and in interaction.

Conflicts of Interest

The authors declare that they have no competing interests.

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