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Control of Cauchy Problem for a Laplacian Operator

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Abstract: In this paper we study the control of an ill-posed system relating to the Cauchy problem for an elliptical operator. The control of Cauchy systems for an elliptical operator has already been studied by many authors. But it still seems to be globally an open problem. Of all the studies that have been done on this problem, it is assumed that the set of admissible couple-state must be nonempty to make sense of the problem. This is the case of J. L. Lions in [6] who gave various examples of the admissible set to make a sense of the problem. O. Nakoulima in [9] uses the regularization-penalization method to approach the problem by a sequence of well-posed control problems, he obtains the convergence of the processus in a particular case of the admissible set. G. Mophou and O. Nakoulima in [10] do the same study and obtain the convergence of the processus when the interior of the admissible set is non empty. In this work, we give an approximate solution without an additional condition on the set of admissible couple-state.We propose a method which consists in associating with the singular control problem a "family" of controls of well posed problems. We propose as an alternative the stackelberg control which is a multiple-objective optimization approach proposed by H. Von Stackelberg in [12].

Keywords: Systems Governed by PDEs, Stackelberg Control, Cauchy Problem, Cost Function

1. Statement of the Problem

Let Ω be an open bounded subset of R^n, with a boundary Γ of class C^2.

$\Gamma = \Gamma_e \cup \Gamma_i$ with $\Gamma_e \cap \Gamma_i = \emptyset$. The boundaries Γ_e and Γ_i are non-empty and

of positive measure. Let us consider in Ω the state z and the control $(v_1; v_2)$
linked by the relations:

$$\begin{cases} \Delta z = 0 \text{ in } \Omega \\ z = v_1, \frac{\partial z}{\partial v} = v_2 \text{ on } \Gamma_e \end{cases} \quad (1)$$

With $z \in L^2(\Omega)$ and $(v_1; v_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e)$.

Problem (1) is a Cauchy problem for an elliptic operator (here the Laplacian operator). In general, problem (1) does not admit a solution and there is the instability of the solution when it exists, see for instance [6], [11]. It is so an ill-posed problem. But, it is important to control the Cauchy data, considering the fact that, such problems come from many concrete situations. It is the case in gravimetry for instance. In the evolution case we have enzymatic reactions, see [4], and the bibliography of this work, the control of the transmission of electrical energy, the control of the form of plasmas. Let U_{ad}^1 and U_{ad}^2 be two no empty closed convex subsets of $L^2(\Gamma_e)$. We denote by

$$A = \left\{ \left((v_1, v_2); z\right) \in U_{ad}^1 \times U_{ad}^2 \times L^2(\Omega), \Delta z = 0 \text{ in } \Omega, z\Big|_{\Gamma_e} = v_1, \frac{\partial z}{\partial v}\Big|_{\Gamma_e} = v_2 \right\}$$

a subset of $L^2(\Gamma_e) \times L^2(\Gamma_e) \times L^2(\Omega)$.

We assume that

$$A \neq \emptyset \quad (2)$$

It is obvious that A is a closed convex subset of $L^2(\Gamma_e) \times L^2(\Gamma_e) \times L^2(\Omega)$.

A couple of control-state $\left((v_1, v_2); z\right)$ will be called "admissible couple" if it belongs in A. To simplify the notation, we will write $(v_1, v_2; z)$ instead of $\left((v_1, v_2); z\right)$.

$$\begin{cases} U_{ad}^1 = L^2(\Gamma_e) \\ U_{ad}^2 = no\ empty\ closed\ convex\ subset\ of\ L^2\left(\Gamma_e\right). \end{cases} \quad (3)$$

Then $A \neq \emptyset$ if $z\big|_{\Gamma_i} \in L^2(\Gamma_i)$, we build ς solution of

$$\begin{cases} \Delta\varsigma = 0 \text{ in } \Omega \\ \varsigma = 0\ on\ \Gamma_e \ ;\ \frac{\partial\varsigma}{\partial v} = v_1 \text{ on } \Gamma_e \end{cases} \quad (4)$$

The system (4) defines a unique $\varsigma \in H^{\frac{3}{2}}(\Omega)$. Therefore $\varsigma\big|_{\Gamma_e} \in H^1\left(\Gamma_e\right) \subset L^2\left(\Gamma_e\right)$. The triplet $\left(\varsigma\big|_{\Gamma_e}, v_2, \varsigma\right) \in A$. See [6] p. 416. Let us consider now, a strictly convex cost functional J, defined for all admissible couple control-state $(v_1, v_2;\ z)$ by

$$J(v_1, v_2, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Omega)} + \frac{N_2}{2}\|v_1\|^2_{L^2(\Omega)} \quad (5)$$

Where $(N_1, N_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $z_d \in L^2(\Omega)$ the desired state.

We are then interested in the problem:

$$\inf J(v_1, v_2;\ z), (v_1, v_2;\ z) \in A. \quad (6)$$

The assumption (2) and the structure of J, show easily that problem (6) has a unique solution *(u1, u2; y)* which we are going to characterize. The function J being differentiable, if *(u1, u2; y)* is the optimal control-state the first order Euler-Lagrange conditions gives.

$$\begin{cases} for\ all\ (v_1, v_2;\ z) \in A, \\ \int_\Omega (y - z_d)(z - y)dx + \int_{\Gamma_e} N_1 u_1(v_1 - u_1)d\Gamma_e + \int_{\Gamma_e} N_2 u_2(v_2 - u_2)d\Gamma_e \end{cases} \quad (7)$$

Many authors have already studied Cauchy systems.

In the parabolic and hyperbolic case we can cite M. Barry and O. Nakoulima in [1]; J. P. Kernevez [4]; M. Barry, O. Nakoulima and G. B. Ndiaye [2].

In the elliptic case we can cite J. L. Lions in [6]; O. Nakoulima in [9]; G.

Mophou and O. Nakoulima in [10].

To get a system where state and control are independent, J. L. Lions proposed in [5] a method of approximation by penalization. He obtained the convergence of the process when $U_{ad}^1 = L^2(\Gamma_e)$ and $U_{ad}^2 = L^2(\Gamma_e)$.

O. Nakoulima in [9] uses the regularization-penalization method. That means, he considered the control problem as a "singular" limit of a sequence of well-posed control problems. The convergence of the process is also obtained by the author in a particular case: $U_{ad}^1 = U_{ad}^2 = (L^2(\Gamma_e))^+$.

O. Nakoulima and G. Mophou in [10] use a regularization method that consists of viewing a singular problem as a limit of a family of well-posed problems. Following this method and assuming that the interior of considered convex is -non-empty, they obtain a singular optimality system (S. O. S.) for the considered control problem.

In this paper, we propose to give an approximate solution without an additional condition on the set of admissible couple-state.

We associate with the control problem of the ill-posed problem (1) -(2), (5) - (7) a family of hierarchical controls of well-posed problems.

The rest of this paper is organized as follows. In section 2.1 and section 2.2, we establish respectively the optimality conditions of the follower and of the leader. In section 2.3, we study the equivalence of the problem studied in sections 2.1 and 2.2, and the problem (1), (5) - (7).

2. Stackelberg Control of the Cauchy Problem for a Laplacian Operator

In this part, we study two well-posed systems associated with the ill-posed problem (1). We propose as an alternative the Stackelberg control.

The Stackelberg leadership model is a multiple-objective optimization approach proposed by H. Von Stackelberg in [12]. This model involves two companies (controls) that compete on the market of the same product. The first (leader) to act must integrate the reaction of the other firm (followers) in the choices it makes in the amount of product that decides to put on the market. In this case, the control of Stackelberg consists of the hierarchical control of two objective functions, one by the follower and the other by the leader.

More precisely, we consider the following systems:

$$\begin{cases} \Delta z_1 = 0 \text{ in } \Omega \\ z_1 = v_1\ on\ \Gamma_e\ ;\ \frac{\partial z_1}{\partial v} = \omega_1 \text{ on } \Gamma_i \end{cases} \quad (8)$$

$$\begin{cases} \Delta z_2 = 0 \text{ in } \Omega \\ \frac{\partial z_2}{\partial v} = v_2\ on\ \Gamma_e\ ;\ z_2 = \omega_2 \text{ on } \Gamma_i \end{cases} \quad (9)$$

Where $(z_1, z_2) \in L^2(\Omega) \times L^2(\Omega)$, $(\omega_1, \omega_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i)$ with (v_1, v_2) defined as in (7).

Remark 2 Problems (8) and (9) are well-posed.

$$If\left(z_1\big|_{\Gamma_e}; z_2\big|_{\Gamma_i}\right) = (v;\ \omega_2) \in L^2\left(\Gamma_e\right) \times L^2\left(\Gamma_i\right)$$

and

$$\left(\frac{\partial z_1}{\partial \nu}\Big|_{\Gamma_i}; \frac{\partial z_2}{\partial \nu}\Big|_{\Gamma_e}\right) = (\omega_1, v_2) \in L^2(\Gamma_e) \times L^2(\Gamma_i)$$

for the trace theorem, see [5], [7] $(z_1, z_2) \in H^{\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega)$. (8) and (9) admit some unique solutions z_1 *and* z_2 in $H^{\frac{1}{2}}(\Omega)$.

We consider now the cost function define by

$$J_2(v_1, v_2, w_1, w_2, z_1(w_1), z_2(w_2)) = \frac{1}{2}\left\|\frac{\partial z_1}{\partial \nu}(w_1) - \frac{\partial z_2}{\partial \nu}(w_2)\right\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\|z_1 - z_2\|^2_{L^2(\Gamma_e)} \quad (10)$$

Where $(z_1, z_2) \in L^2(\Omega) \times L^2(\Omega)$ is a solution of (8)-(9). And the minimization follow problem

$$\min J_2(v_1, v_2, w_1, w_2, z_1(w_1), z_2(w_2));\ (\omega_1, \omega_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i) \quad (11)$$

Remark 3 In a symmetrical way it is possible to introduce the functional cost

$$J_2(v_1, v_2, w_1, w_2, z_1(w_1), z_2(w_2)) = \frac{1}{2}\left\|\frac{\partial z_1}{\partial \nu}(w_1) - \frac{\partial z_2}{\partial \nu}(w_2)\right\|^2_{L^2(\Gamma_e)} + \frac{1}{2}\|z_1 - z_2\|^2_{L^2(\Gamma_i)} \quad (12)$$

The set A being non-empty, we now show that the resolution of the problem (1) is equivalent to that of the problem (8) - (11).

Proposition 4 Assume that the problem (1) admits a solution. Then, solving problem (1) is equivalent to solve the problem (8) - (11) and

$$\min J_2(v_1, v_2, \omega_1, \omega_2, z_1(\omega_1), z_2(\omega_2)) = 0,\ (\omega_1, \omega_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i) \quad (13)$$

Proof Let (v_1, v_2, z) be a solution of (1), such that $w_1 = \frac{\partial z}{\partial \nu}$ *on* Γ_i *and* $w_2 = z$ *on* Γ_i Then, we have

$$J_2(v_1, v_2, w_1, w_2, z, z) = \frac{1}{2}\left\|w_1 - \frac{\partial z}{\partial \nu}\right\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\|v_1 - z\|^2_{L^2(\Gamma_e)} = 0.$$

Conversely, let $(v_1, v_2, w_1^*, w_2^*, z_1^*, z_2^*)$ be a solution of (8)-(11), we have

$$0 \le J_2(v_1, v_2, w_1^*, w_2^*, z_1^*, z_2^*) \le J_1(v_1, v_2, w_1, w_2, z, z) = 0 \quad (14)$$

(14) *implies that* $z_1^* = z_2^*$ *on* Γ_e *and* $\frac{\partial z_1^*}{\partial \nu} = \frac{\partial z_2^*}{\partial \nu}$ *on* Γ_i. *We deduce that,*

$z_1^* - z_2^*$ *is the solution of the following problem*

$$\begin{cases} \Delta(z_1^* - z_2^*) = 0 \text{ in } \Omega \\ z_1^* - z_2^* = 0 \text{ on } \Gamma_e;\ \frac{\partial z_1^*}{\partial \nu} - \frac{\partial z_2^*}{\partial \nu} = 0 \text{ on } \Gamma_i \end{cases} \quad (15)$$

from (15), *we obtain* $z_1^* = z_2^* = z$ *in* Ω. ■

2.1. Optimality System of the Follower

Proposition 5 *The functional* J_2 *is twice Gâteaux differentiable and strictly convex.*

Proof. We first show that J_2 *is Gâteaux differentiable.*

Let $(\varphi_1, \varphi_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i)$, $\lambda \in \mathbb{R}$ *and denote by* $(.,.)$ *the scalar product in* L^2.

Let us consider the applications

$$\varphi_1 \to z_1(\varphi_1) - z_1(0) \text{ and } \varphi_2 \to z_2(\varphi_2) - z_2(0) \quad (16)$$

solution of

$$\begin{cases} \Delta\Phi_1 = 0 \text{ in } \Omega \\ \Phi_1 = 0 \text{ on } \Gamma_e;\ \frac{\partial \Phi_1}{\partial \nu} = \Phi_1 \text{ on } \Gamma_i \end{cases} \quad (17)$$

$$\begin{cases} \Delta\Phi_2 = 0 \text{ in } \Omega \\ \frac{\partial \Phi_2}{\partial \nu} = 0 \text{ on } \Gamma_e;\ \Phi_2 = \varphi_2 \text{ on } \Gamma_i \end{cases} \quad (18)$$

The applications defined in (16) are linear.

The following calculations are performed

$$J_2(v_1, v_2, \omega_1 + \lambda\varphi_1, \omega_2 + \lambda\varphi_2, z_1, z_2)$$

$$= \frac{1}{2}\left\|\frac{\partial z_1}{\partial \nu}(\omega_1 + \lambda\varphi_1) - \frac{\partial z_2}{\partial \nu}(\omega_2 + \lambda\varphi_2)\right\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\|v_1 - z_2(\omega_2 + \lambda\varphi_2)\|^2_{L^2(\Gamma_e)}$$

$$=\frac{1}{2}\left\|\omega_1-\frac{\partial z_2}{\partial v}(w_2)\right\|^2_{L^2(\Gamma_i)}+\frac{\lambda^2}{2}\left\|\varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right\|^2_{L^2(\Gamma_i)}+\frac{1}{2}\|v_1-z_2(\omega_2)\|^2_{L^2(\Gamma_e)}+$$

$$\frac{\lambda^2}{2}\|-z_2(\varphi_2)+z_2(0)\|^2_{L^2(\Gamma_e)}+\lambda\left(\omega_1-\frac{\partial z_2}{\partial v}(\omega_2),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+\lambda(v_1-z_2\,(\omega_2),-z_2(\varphi_2)+z_2(0))_{L^2(\Gamma_e)}$$

Hence

$$\lim_{\lambda\to 0}\frac{J_2(\omega_1+\lambda\,\varphi_1,\ \varphi_2+\ \lambda\,\varphi_2,v_1,v_2,z_1,z_2)-J_2(\omega_1,\ \varphi_2,v_1,v_2,z_1,z_2)}{\lambda}$$

$$=\left(\omega_1-\frac{\partial z_2}{\partial v}(\omega_2),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}+(v_1-z_2\,(\omega_2),-z_2(\varphi_2)+z_2(0))_{L^2(\Gamma_e)}$$

This last result shows that J_2 is Gâteaux differentiable and

$$dJ_2(v_1,v_2,\omega_1,\omega_2,z_1,z_2)\cdot(\varphi_1,\varphi_2)=\left(\omega_1-\frac{\partial z_2}{\partial v}(\omega_2),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}+$$

$$(v_1-z_2\,(\omega_2),-z_2(\varphi_2)+z_2(0))_{L^2(\Gamma_e)} \tag{19}$$

For $(\xi_1,\xi_2)\in\ L^2(\Gamma_i)\times L^2(\Gamma_i)$ and $\lambda\in\mathbb{R}$ *we have*

$$dJ_2(v_1,v_2,\omega_1+\lambda\xi_1,\omega_2+\lambda\xi_2,z_1,z_2)\cdot(\varphi_1,\varphi_2)$$

$$=\left(\omega_1+\lambda\xi_1-\frac{\partial z_2}{\partial v}(\omega_2+\lambda\xi_2),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+(v_1-z_2\,(\omega_2+\lambda\xi_2),-z_2(\varphi_2)+z_2(0))_{L^2(\Gamma_e)}$$

$$=\left(\omega_1-\frac{\partial z_2}{\partial v}(\omega_2),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+\left(\lambda\xi_1-\frac{\partial z_2}{\partial v}(\xi_2)+\frac{\partial z_2}{\partial v}(0),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+(v_1-z_2\,(\omega_2),-z_2(\varphi_2)+z_2(0))_{L^2(\Gamma_e)}$$

$$+\lambda\,(-z_2\,(\xi_2)\ +z_2(0),-z_2(\omega_2)+z_2(0))_{L^2(\Gamma_e)}$$

Which give

$$\lim_{\lambda\to 0}\left(\frac{J_2(v_1,v_2,\omega_1+\lambda\xi_1,\omega_2+\lambda\xi_2,z_1,z_2)\cdot(\varphi_1,\varphi_2)}{\lambda}-\frac{J_2(\omega_1,\omega_2,v_1,v_2,z_1,z_2)\cdot(\varphi_1,\varphi_2)}{\lambda}\right)$$

$$=\left(\frac{\partial z_1}{\partial v}(\xi_1)-\frac{\partial z_1}{\partial v}(0)-\frac{\partial z_2}{\partial v}(\xi_2)+\frac{\partial z_2}{\partial v}(0),\frac{\partial z_1}{\partial v}(\varphi_1)-\frac{\partial z_1}{\partial v}(0)-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+(-z_2\,(\xi_2)\ +z_2(0),-z_2(\omega_2)+z_2(0))_{L^2(\Gamma_e)}$$

and so

$$dJ_2^2(v_1,v_2,\omega_1,\omega_2,z_1,z_2)\cdot(\xi_1,\xi_2)$$

$$=\left(\xi_1-\frac{\partial z_2}{\partial v}(\xi_2)+\frac{\partial z_2}{\partial v}(0),\ \varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$

$$+(-z_2\,(\xi_2)\ +z_2(0),-z_2(\omega_2)+z_2(0))_{L^2(\Gamma_e)}.$$

This last result shows that J_2 is twice Gâteaux differentiable.

We have

$$dJ_2^2(v_1,v_2,\omega_1,\omega_2,z_1,z_2)\cdot(\varphi_1,\varphi_2)\cdot(\varphi_1,\varphi_2)$$

$$=\left\|\varphi_1-\frac{\partial z_2}{\partial v}(\varphi_2)+\frac{\partial z_2}{\partial v}(0)\right\|^2_{L^2(\Gamma_i)}$$

$$+\|-z_2(\varphi_2)+z_2(0)\|^2_{L^2(\Gamma_i)}$$

As $dJ_2^2(v_1, v_2, \omega_1, \omega_2, z_1, z_2) \cdot (\varphi_1, \varphi_2) \cdot (\varphi_1, \varphi_2) \geq 0$, then J_2 is convex.

In addition, we have $dJ_2^2(v_1, v_2, \omega_1, \omega_2, z_1, z_2) \cdot (\varphi_1, \varphi_2) \cdot (\varphi_1, \varphi_2) = 0$

Implies that $\varphi_1 - \frac{\partial z_2}{\partial v}(\varphi_2) + \frac{\partial z_2}{\partial v}(0) = 0 \; on \; \Gamma_i \; and \; z_2(\varphi_2) - z_2(0) = 0 \; on \; \Gamma_e$.

From (18) and the identity $z_2(\varphi_2) - z_2(0) = \; 0 \; on \; \Gamma_e$, the uniqueness of the Cauchy problem gives

$$z_2(\varphi_2) - z_2(0) = 0 \; in \; \bar{\Omega} \; and \; thus \; \varphi_2 = 0. \text{ F}$$

From the identity

$$\varphi_1 - \frac{\partial z_2}{\partial v}(\varphi_2) + \frac{\partial z_2}{\partial v}(0) = 0 \; on \; \Gamma_i \; we \; have \; \varphi_1 = 0.$$

Finally J_2 is strictly convex. ■

The strict convexity of J_2 leads to the uniqueness of the solution of prob-lem (11).

We introduce now some adjoint systems of (8) and (9) define by

$$\begin{cases} \Delta p_1 = 0 \text{ in } \Omega \\ p_1 = v_1 - z_2 \; on \; \Gamma_e; \; \frac{\partial p_1}{\partial v} = -\omega_1 + \frac{\partial p_2}{\partial v} \text{ on } \Gamma_i \end{cases} \quad (20)$$

$$\begin{cases} \Delta p_2 = 0 \text{ in } \Omega \\ \frac{\partial p_2}{\partial v} = v_1 - z_2 \; on \; \Gamma_e; \; p_2 = -\omega_1 + \frac{\partial p_2}{\partial v} \text{ on } \Gamma_i \end{cases} \quad (21)$$

We obtain the optimality system below

Proposition 6 *(ω_1^*, ω_2^*) is an optimal solution of (13) if and only if it exists*

(z_1^*, z_2^*) satisfying (8)- (9) and (p_1^*, p_2^*) satisfying (20)-(21) such as the

triplet $\{(\omega_1^*, \omega_2^*), (z_1^*, z_2^*), (p_1^*, p_2^*)\}$ *is the solution of the optimality systems:*

$$\begin{cases} \Delta p_1^* = 0 \text{ in } \Omega \\ z_1^* = v_1 \; on \; \Gamma_e; \; \frac{\partial z_1^*}{\partial v} = -\omega_1^* \text{ on } \Gamma_i \end{cases} \quad (22)$$

$$\begin{cases} \Delta p_2 = 0 \text{ in } \Omega \\ \frac{\partial z_2}{\partial v} = v_2 \; on \; \Gamma_e; \; z_2^* = \omega_2^* \text{on } \Gamma_i \end{cases} \quad (23)$$

$$\begin{cases} \Delta p_1^* = 0 \text{ in } \Omega \\ p_1^* = v_1 - z_2^* \; on \; \Gamma_e; \; \frac{\partial p_1^*}{\partial v} = -\omega_1^* + \frac{\partial z_2^*}{\partial v} \text{ on } \Gamma_i \end{cases} \quad (24)$$

$$\begin{cases} \Delta p_2^* = 0 \text{ in } \Omega \\ \frac{\partial p_2^*}{\partial v} = v_1 - z_2^* \; on \; \Gamma_e; \; p_2^* = -\omega_1^* + \frac{\partial z_2^*}{\partial v} \text{ on } \Gamma_i \end{cases} \quad (25)$$

and

$$\frac{\partial p_1^*}{\partial v} = 0 \; on \; \Gamma_i; \; \frac{\partial p_2^*}{\partial v} = 0 \; on \; \Gamma_i.$$

Proof. *Let* $(\omega_1, \omega_2,), (\varphi_1, \varphi_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i), \lambda \in \mathbb{R}$. *From* (19) *we have*

$$dJ_2(v_1, v_{2,} \omega_1, \omega_2, z_1, z_2) \cdot (\varphi_1, \varphi_2)$$
$$= \left(\omega_1 - \frac{\partial z_2}{\partial v}(\omega_2), \; \varphi_1 - \frac{\partial z_2}{\partial v}(\varphi_2) + \frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)} + (v_1 - z_2(\omega_2), -z_2(\varphi_2) + z_2(0))_{L^2(\Gamma_e)}.$$

According to the Euler optimality conditions, (ω_1^*, ω_2^*) is the optimal solution of (13) if and only if,

$$(\varphi_1, \varphi_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i), dJ_2(v_1, v_{2,} \omega_1^*, \omega_2^*, z_1^*, z_2^*) \cdot (\varphi_1, \varphi_2) \geq 0.$$

that is,

$$\forall (\varphi_1, \varphi_2) \in L^2(\Gamma_i) \times L^2(\Gamma_i), \left(\omega_1^* - \frac{\partial z_2^*}{\partial v}, \varphi_1 - \frac{\partial z_2}{\partial v}(\varphi_2) + \frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$
$$+ (v_1 - z_2^*, -z_2(\varphi_2) + z_2(0))_{L^2(\Gamma_e)} \geq 0.$$

Hence

$$\left(\omega_1^* - \frac{\partial z_2^*}{\partial v}, \varphi_1\right)_{L^2(\Gamma_i)} + \left(\omega_1^* - \frac{\partial z_2^*}{\partial v} - \frac{\partial z_2}{\partial v}(\varphi_2) + \frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)}$$
$$+ (v_1 - z_2^*, -z_2(\varphi_2) + z_2(0))_{L^2(\Gamma_e)} \geq 0,$$

As a result, talking into account (24) and (25) we have

$$\left(-\frac{\partial p_1^*}{\partial v}, \varphi_1\right)_{L^2(\Gamma_i)} + \left(p_2^*, \frac{\partial z_2}{\partial v}(\varphi_2) - \frac{\partial z_2}{\partial v}(0)\right)_{L^2(\Gamma_i)} \quad (26)$$
$$-- \left(\frac{\partial p_2^*}{\partial v}, z_2(\varphi_2) - z_2(0)\right)_{L^2(\Gamma_e)} \geq 0.$$

Multiplying the first equation of (25) by Φ_2 the solution of (18)

And integrating by parts we have

$$-\int_{\Gamma_e} \frac{\partial p_2^*}{\partial v} \Phi_2 ds + \int_{\Gamma_i} p_2^* \frac{\partial \Phi_2}{\partial v} ds = \int_{\Gamma_i} \frac{\partial p_2^*}{\partial v} \Phi_2 ds.$$

(26) become

$$\forall (\varphi_1, \varphi_2) \in \mathrm{L}^2(\Gamma_i) \mathrm{x} \, \mathrm{L}^2(\Gamma_i), \left(-\frac{\partial p_1^*}{\partial v}, \varphi_1\right)_{L^2(\Gamma_i)} + \left(\frac{\partial p_2^*}{\partial v}, \varphi_2\right)_{L^2(\Gamma_i)} = 0.$$

Finally we have $\frac{\partial p_1^*}{\partial v} = 0$ *on* Γ_i *and* $\frac{\partial p_2^*}{\partial v} = 0$ *on* Γ_i. ■

$$J_1(v_1, v_2, \omega_1^*, \omega_2^*, z_1^*, z_2^*) = \frac{\theta_1}{2}\|z_1^* - z_d\|^2_{L^2(\Omega)} + \frac{\theta_2}{2}\|z_2^* - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Gamma_e)} + \frac{N_2}{2}\|v_2\|^2_{L^2(\Gamma_e)} \quad (27)$$

where (N_1, N_2) *and* z_d are defined in (5), θ_1 *and* θ_2 *are some reals such that* $\theta_1 + \theta_2 = 1$, *and the problem*

$$\inf J_1(v_1, v_2, \omega_1^*, \omega_2^*, z_1^*, z_2^*), (v_1, v_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e) \quad (28)$$

Proposition 7 There existe $(u_1^*, u_2^*) \in L^2(\Gamma_e) \times L^2(\Gamma_e)$ *unique such that*

$$\forall (v_1, v_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e), (.)$$

$$J_1(u_1^*, u_2^*, \omega_1^*, \omega_2^*, z_1^*, z_2^*) \le J_1(v_1, v_2, \omega_1^*, \omega_2^*, z_1^*, z_2^*) \quad (29)$$

Proof. $L^2(\Gamma_e) \times L^2(\Gamma_e)$ is closed convex.
J_1 is coercive and strictly convex, then the problem (29) holds true. ■

Consider again one adjoint problem of (8) and (9) define by

$$\begin{cases} \Delta p_1 = z_1^* - z_d \text{ in } \Omega \\ p_1 = 0 \text{ on } \Gamma_e;\ \frac{\partial p_1}{\partial v} = 0 \text{ on } \Gamma_i \end{cases} \quad (30)$$

$$\begin{cases} \Delta p_2 = z_2^* - z_d \text{ in } \Omega \\ \frac{\partial p_2}{\partial v} = 0 \text{ on } \Gamma_e;\ p_2 = 0 \text{ on } \Gamma_i \end{cases} \quad (31)$$

We obtain the optimality system below

Proposition 8 (u_1^*, u_2^*) is an optimal solution of (28) if and only if it exists $(z_1^*; z_2^*)$ satisfying (8) - (9) and $(p_1^*; p_2^*)$ satisfying (30)-(31) such as the triplet $\{(u_1^*; u_2^*), (z_1^*; z_2^*), (p_1^*; p_2^*)\}$ is the solution of the optimality systems:

$$\begin{cases} \Delta z_1^* = 0 \text{ in } \Omega \\ z_1^* = u_1^* \text{ on } \Gamma_e;\ \frac{\partial z_1^*}{\partial v} = \omega_1^* \text{ on } \Gamma_i \end{cases} \quad (32)$$

$$\begin{cases} \Delta z_2^* = 0 \text{ in } \Omega \\ \frac{\partial z_2^*}{\partial v} = u_2^* \text{ on } \Gamma_e;\ z_2^* = \omega_2^* \text{ on } \Gamma_i \end{cases} \quad (33)$$

$$\begin{cases} \Delta p_2^* = z_1^* - z_d \text{ in } \Omega \\ p_1^* = 0 \text{ on } \Gamma_e;\ \frac{\partial p_1^*}{\partial v} = 0 \text{ on } \Gamma_i \end{cases} \quad (34)$$

$$\begin{cases} \Delta p_2^* = z_2^* - z_d \text{ in } \Omega \\ \frac{\partial p_2}{\partial v} = 0 \text{ on } \Gamma_e;\ p_2^* = 0 \text{ on } \Gamma_i \end{cases} \quad (35)$$

And

$$-\theta_1 \frac{\partial p_1^*}{\partial v} + N_1 u_1^* = 0 \text{ on } \Gamma_e;\ \theta_2\, p_2^* + N_2 u_2^* = 0 \text{ on } \Gamma_e \quad (36)$$

Proof. Let $(\varphi_1, \varphi_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e)$, $\lambda \in \mathbb{R}$ and denote by (.,.) the scalar

Product in L^2.

Let us consider the applications

$$\varphi_1 \to z_1(\varphi_1) - z_1(0) \text{ and } \varphi_2 \to z_2(\varphi_2) - z_2(0) \quad (37)$$

solutions of

$$\begin{cases} \Delta \Phi_1 = 0 \text{ in } \Omega \\ \Phi_1 = \varphi_1 \text{ on } \Gamma_e;\ \frac{\partial \Phi_1}{\partial v} = 0 \text{ on } \Gamma_i \end{cases} \quad (38)$$

$$\begin{cases} \Delta \Phi_2 = 0 \text{ in } \Omega \\ \frac{\partial \Phi_2}{\partial v} = \varphi_2 \text{ on } \Gamma_e;\ \Phi_2 = 0 \text{ on } \Gamma_i \end{cases} \quad (39)$$

The applications define in (37) are linear.
We have

$$\forall (\varphi_1, \varphi_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e),$$

$$dJ_1(v_1, v_2) \cdot (\varphi_1, \varphi_2) = \theta_1 (z_1^*(v_1) - z_d, z_1^*(\varphi_1) - z_1^*(0))_{L^2(\Omega)}$$

$$+\theta_2 (z_2^*(v_2) - z_d, z_2^*(\varphi_1) - z_2^*(0))_{L^2(\Omega)}$$

$$+N_1(v_1, \varphi_2)_{L^2(\Gamma_e)} + N_2(v_2, \varphi_2)_{L^2(\Gamma_e)}$$

According to the Euler optimality conditions, $(u_1^*; u_2^*)$ is an optimal solution
of (28) if and only if,

$$\forall (\varphi_1, \varphi_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e), dJ_1(u_1^*; u_2^*) \cdot (\varphi_1, \varphi_2) \ge 0.$$

That gives

$$\forall (\varphi_1, \varphi_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e), \quad (40)$$

$$\theta_1 (z_1^*(u_1^*) - z_d, z_1^*(\varphi_1) - z_1^*(0))_{L^2(\Omega)}$$

$$+\theta_2 (z_2^*(u_2^*) - z_d, z_2^*(\varphi_1) - z_2^*(0))_{L^2(\Omega)}$$

$$+N_1(u_1^*, \varphi_1)_{L^2(\Gamma_e)} + N_2(u_2^*, \varphi_2)_{L^2(\Gamma_e)} \ge 0.$$

Multiplying the firsts equations of (30) and (31) respectively by Φ_1 and Φ_2 (solutions of (38) and (39) and integrating by parts we obtain

$$(z_1^*(u_1^*) - z_d, z_1^*(\varphi_1) - z_1^*(0))_{L^2(\Omega)} = \left(-\frac{\partial p_1^*}{\partial v}, \varphi_1\right)_{L^2(\Gamma_e)} \quad (41)$$

And

$$(z_2^*(u_2^*) - z_d, z_2^*(\varphi_2) - z_2^*(0))_{L^2(\Omega)} = (p_2^*, \varphi_2)_{L^2(\Gamma_e)} \quad (42)$$

From (40)-(42) we obtain

$$\forall (\varphi_1, \varphi_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e),$$

$$\left(-\theta_1 \frac{\partial p_1^*}{\partial v} + N_1 u_1^*, \varphi_1\right)_{L^2(\Gamma_e)} + (\theta_2 p_2^* + N_2 u_2^*, \varphi_2)_{L^2(\Gamma_e)} = 0.$$

2.2. Optimality System of the Leader

Consider the cost function

and finally, we have

$$-\theta_1 \frac{\partial p_1^*}{\partial v} + N_1 u_1^* = 0 \; on \; \Gamma_e \; and \; \theta_2 p_2^* + N_2 u_2^* = 0 \; on \; \Gamma_e.$$

2.3. Equivalence to the Problem (1); (5)-(7)

In this section, we will show that the optimality system (32)-(36) makes it possible to calculate the solution of problem (1), (5) - (7).

Let (u_1, u_2, y) be the unique solution of problem (1), (5) - (7) and $(v_1, v_2) \in L^2(\Gamma_e) \times L^2(\Gamma_e)$.

J_2 being the cost function defined in (10) and (u_1^*, u_2^*); (ω_2^*, ω_2^*); (z_1^*, z_2^*) defines respectively in proposition 6 and proposition 8, we have

$$J_2(u_1^*, u_2^*, \omega_1^*, \omega_2^*, z_1^*, z_2^*) \leq J_2(u_1, u_2, \omega_1^*, \omega_2^*, z_1^*, z_2^*)$$
$$\leq J_2(u_1, u_2, \omega_1^*, \omega_2^*, y, y) = 0$$

From proposition 4 we conclude that

$$u_1^* = u_1, u_2^* = u_2; \text{ and } z_1^* = z_2^* = y$$

Therefore we have

$$J_1(u_1^*, u_2^*, \omega_1^*, \omega_2^*, z_1^*, z_2^*)$$
$$= \frac{\theta_1}{2} \|z_1^* - z_d\|^2_{L^2(\Omega)} + \frac{\theta_2}{2} \|z_2^* - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2} \|u_1^*\|^2_{L^2(\Gamma_e)} + \frac{N_2}{2} \|u_2^*\|^2_{L^2(\Gamma_e)}$$
$$= \frac{\theta_1}{2} \|y - z_d\|^2_{L^2(\Omega)} + \frac{\theta_2}{2} \|y - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2} \|u_1\|^2_{L^2(\Gamma_e)} + \frac{N_2}{2} \|u_2\|^2_{L^2(\Gamma_e)}$$
$$= (\frac{\theta_1}{2} + \frac{\theta_2}{2}) \|y - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2} \|u_1\|^2_{L^2(\Gamma_e)} + \frac{N_2}{2} \|u_2\|^2_{L^2(\Gamma_e)}$$
$$= \|y - z_d\|^2_{L^2(\Omega)} + \frac{N_1}{2} \|u_1\|^2_{L^2(\Gamma_e)} + \frac{N_2}{2} \|u_2\|^2_{L^2(\Gamma_e)}$$
$$= J(u_1, u_2, y)$$

And finally,

$$\inf J(u_1, u_2, y) \; \inf \; J_1(v_1, v_2, \omega_1^*, \omega_2^*, z_1^*, z_2^*).$$
$$(v_1, v_2, z) \in A \; (v_1, v_2,) \in L^2(\Gamma_e) \times L^2(\Gamma_e).$$

3. Conclusion

The present article presents an alternative possibility to study the optimal control problem (6) and to obtain an optimality condition where state and control are independent, different from the regularization-penalization method introduced by O. Nakoulima in [9] or the regularization method due to O. Nakoulima and G. Mophou in [10].

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