Generalized Quasi-Variational Inequalities for Pseudo-Monotone Type III and Strongly Pseudo-Monotone Type III Operators on Non-Compact Sets

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Abstract: In this paper, the authors prove some existence results of solutions for a new class of generalized quasi-variational inequalities (GQVI) for pseudo-monotone type III operators and strongly pseudo-monotone type III operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In obtaining these results on GQVI for pseudo-monotone type III operators, we shall use Chowdhury and Tan’s generalized version [1] of Ky Fan’s minimax inequality [2] as the main tool.

Keywords: Generalized Quasi-Variational Inequalities, Pseudo-Monotone Type III Operators, Locally Convex Topological Vector Spaces

1. Introduction

Let \( X \) be a non-empty set, and \( 2^X \) be the family of all non-empty subsets of \( X \). Let \( E \) be a topological vector space. We shall denote by \( E^* \) the continuous dual of \( E \), by \( \langle w, x \rangle \) the pairing between \( E^* \) and \( E \) for \( w \in E^* \) and \( x \in E \) and by \( Re(w, x) \) the real part of \( \langle w, x \rangle \). Given the maps \( S: X \to 2^X \) and \( T: X \to 2^{E^*} \), the generalized quasi-variational inequality problem (GQVI) is to find a point \( y \in X \) and a point \( \bar{y} \in T(y) \) such that \( Re(\bar{w}, \bar{y} - x) \leq 0 \) for all \( x \in S(y) \). The GQVI was introduced by Chan and Pang [3] in 1982 when \( E \) is finite dimensional and by Shih and Tan [4] in 1985 when \( E \) is infinite dimensional.

In [5] we established some existence theorems of generalized variational inequalities and generalized complementarity problems in topological vector spaces for pseudo-monotone type III operators defined as follows:

Definition 1.1. Let \( E \) be a topological vector space, \( X \) a non-empty subset of \( E \) and \( T: X \to 2^{E^*} \) a map. If \( h: X \to \mathbb{R} \), then \( T \) is said to be an \( h \)-pseudo-monotone (respectively, a strongly \( h \)-pseudo-monotone) type III operator if for each \( x, y \in X \) and every net \( \{ y_\alpha \}_{\alpha \in \Gamma} \) in \( X \) converging to \( y \) (respectively, weakly to \( y \)) with

\[
\limsup_{\alpha \in \Gamma} [\inf_{u \in F(y_\alpha)} Re(u, y_u - x) + h(y_u) - h(x)] \leq 0,
\]

we have

\[
\limsup_{\alpha \in \Gamma} [\inf_{u \in F(y_\alpha)} Re(u, y_u - y) + h(y_u) - h(y)]
\geq \inf_{w \in \overline{F}(y)} Re(w, y - x) + h(y) - h(x).
\]

\( T \) is said to be a pseudo-monotone (respectively, a strongly pseudo-monotone) type III operator if \( T \) is an \( h \)-pseudo-monotone type III (respectively, a strongly \( h \)-pseudo-monotone type III) operator with \( h \equiv 0 \).

The above operators were originally named \( h \)-hemi-continuous (respectively, strong \( h \)-hemi-continuous) operators in [5]. Later, in [6], we re-named these operators pseudo-monotone type III operators.

The following result in [5] justified the validity of a set-valued pseudo-monotone (respectively, strongly pseudo-monotone) type III operator.
Proposition 1.1. Let $X$ be a non-empty compact subset of a topological vector space $E$ and $T: X \to 2^E$ an upper semi-
 continuous mapping from the relative weak topology on $X$ to the strong topology on $E^*$. Then $T$ is both a pseudo-monotone and a strongly pseudo-monotone type III operator.

If $T$ is single-valued and continuous, the compactness of $X$ is not required and the following result was obtained in [5]:

Proposition 1.2. Let $X$ be a non-empty bounded subset of a topological vector space $E$ and $T: X \to E^*$ a continuous mapping from the relative weak topology on $X$ to the strong topology on $E^*$. Then $T$ is both a pseudo-monotone and a strongly pseudo-monotone type III operator.

In this paper, we shall first obtain some general theorems on solutions for a new class of generalized quasi-variational inequalities for pseudo-monotone type III operators and strongly pseudo-monotone type III operators defined on non-compact sets in topological vector spaces. In obtaining these results, we shall mainly use the following generalized version of Ky Fan’s minimax inequality [2] due to M.S.R. Chowdhury and K.-K Tan [1].

Theorem 1.3. Let $E$ be a topological vector space, $X$ be a non-empty convex subset of $E$, $h: X \to \mathbb{R}$ be a lower semi-
 continuous on $co(A)$ for each $A \in \mathcal{F}(X)$, and $f: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that
(a) for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $co(A)$;
(b) for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\min_{x \in A} f(x, y) + h(y) - h(x) \leq 0$;
(c) for each $A \in \mathcal{F}(X)$ and each $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in $X$ converging to $y$ with $f(tx + (1 - t)y, y_\alpha) + h(y_\alpha) - h(tx + (1 - t)y) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$, we have $f(x, y) + h(y) - h(x) \leq 0$;
(d) there exist a non-empty closed and compact subset $K$ of $X$ and $x_0 \in K$ such that $f(x_0, y) + h(y) - h(x_0) > 0$ for all $y \not\in X \setminus K$.

Then there exists $\tilde{y} \in K$ such that $f(x, \tilde{y}) + h(\tilde{y}) - h(x) \leq 0$ for all $x \in X$.

2. Preliminaries

Let $E$ be a topological vector space over $\mathbb{F}$. Then, for each $x_0 \in E$, each non-empty subset $A$ of $E$ and each $\varepsilon > 0$, let $W(x_0, \varepsilon) := \{y \in E^* : |y(x_0)| < \varepsilon\}$ and $U(A; \varepsilon) := \{y \in \mathbb{F}^* : \sup_{x \in A} |y(x)| < \varepsilon\}$.

Let $\sigma(E^*, E)$ be the topology on $E^*$ generated by the family
\[\{W(x_0, \varepsilon) : x \in E, \varepsilon > 0\}\]
as a subbase for the neighborhood system at $0$ and $\delta(E^*, E)$ be the topology on $E^*$ generated by the family \[\{U(A; \varepsilon) : A \text{ is a non-empty bounded subset of } E, \varepsilon > 0\}\]as a base for the neighborhood system at $0$. We note that $E^*$, when equipped with the topology $\sigma(E^*, E)$ or the topology $\delta(E^*, E)$, becomes a locally convex Hausdorff topological vector space. Furthermore, for a net \[\{y_\alpha\}_{\alpha \in \Gamma} \in E^*\] and for $y \in E^*$, (i) $y_\alpha \to y$ in $\sigma(E^*, E)$ if and only if $(y_\alpha, x) \to (y, x)$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta(E^*, E)$ if and only if $(y_\alpha, x) \to (y, x)$ uniformly for $x \in A$ for each non-empty bounded subset $A$ of $E$. The topology $\sigma(E^*, E)$ (respectively, $\delta(E^*, E)$) is called the weak*-topology (respectively, the strong topology) on $E^*$.

If $X$ is a topological space and \[\{U_n : \alpha \in A\}\] is an open cover for $X$, then a partition of unity subordinate to the open cover \[\{U_n : \alpha \in A\}\] is a family \[\{\beta_n : \alpha \in A\}\] of continuous real-valued functions $\beta_n: X \to [0,1]$ such that
(a) $\beta_n(y) = 0$ for all $y \in X \setminus U_n$,
(b) \{support $\beta_n : \alpha \in A\}$ is locally finite and
(c) $\sum_{\alpha \in A} \beta_n(y) = 1$ for each $y \in X$.

We shall first state the following result which is Lemma 1 of Shih and Tan in [4, pp.334-335]:

Lemma 2.1. Let $X$ be a non-empty subset of a Hausdorff topological vector space $E$ and $S: X \to 2^E$ be an upper semi-
 continuous map such that $S(x)$ is a bounded subset of $E$ for each $x \in X$. Then for each continuous linear functional $p$ on $E$, the map $f_p : X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} Re(p, x)$ is upper semi-continuous; i.e. for each $\alpha \in \mathbb{R}$, the set \[\{y \in X : f_p(y) = \sup_{x \in S(y)} Re(p, x) < \lambda\}\] is open in $X$.

The following result is Lemma 3 of Takahashi in [7, p.177] (see also Lemma 3 in [8, pp.68-85]):

Lemma 2.2. Let $X$ and $Y$ be topological spaces, $f: X \to \mathbb{R}$ be non-negative and continuous and $g: Y \to \mathbb{R}$ be lower semi-continuous. Then the map $F: X \times Y \to \mathbb{R}$ defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semi-
 continuous.

We shall need the following Kneser’s minimax theorem in [9, pp.2418-2420] (see also [10, pp.40-41]):

Theorem 3.1. Let $E$ be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = co(A)$, and $T: X \to 2^E$ be upper semi-
 continuous from $X$ to the weak*-topology on $E^*$ such that $T(x)$ is weak*-compact. Let $f: X \times X \to \mathbb{R}$ be defined by $f(x, y) = \inf_{y \in T(x)} Re(w, y - x)$ for all $x, y \in X$. Then for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on $X$.

The following result is Lemma 3 in [1]:

Theorem 2.4. Let $E$ be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = co(A)$, and $T: X \to 2^E$ be upper semi-
 continuous from $X$ to the weak*-topology on $E^*$ such that $T(x)$ is weak*-compact. Let $f: X \times X \to \mathbb{R}$ be defined by $f(x, y) = \inf_{y \in T(x)} Re(w, y - x)$ for all $x, y \in X$. Then for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on $X$.

3. Generalized Quasi-Variational
Inequalities of Pseudo-Monotone Type
III and Strongly Pseudo-Monotone Type
III Operators

In this section, we shall obtain some general existence theorems for the solutions to the generalized quasi-variational inequalities for pseudo-monotone type III operators and strongly pseudo-monotone type III operators on non-compact sets.

We shall first establish the following result:

Theorem 3.1. Let $E$ be a locally convex Hausdorff

topological vector space, $X$ be a non-empty paracompact convex and bounded subset of $E$ and $h: E \rightarrow \mathbb{R}$ be convex with $h(X)$ bounded. Let $S:X \rightarrow 2^X$ be upper semi-continuous such that each $S(x)$ is compact convex and $T:X \rightarrow 2^X$ be an $h$-pseudo-monotone type III operator and be upper semi-continuous from $co(A)$ to the weak*-topology on $E$ for each $A \in F(X)$ and $T(X)$ is strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that the set

$$
\Sigma = \{y \in X: \sup_{x \in E(y)} [\inf_{x \in E(y)} Re(w, y - x) + h(y) - h(x)] > 0\}
$$

is open in $X$ and the following conditions are satisfied:

(a) for each $A \in F(X)$ and each $x, y \in co(A)$ and any net $(y_\alpha)_{\alpha \in \mathcal{A}}$ in $X$ converging to $y$, we have \(\lim \sup \alpha \inf_{x \in E(y_\alpha)} Re(u, y_\alpha - x) + h(y_\alpha) - h(x)\) \leq 0 whenever \(\lim \sup \alpha \inf_{x \in E(y_\alpha)} Re(u, y_\alpha - y) + h(y_\alpha) - h(y)\) \leq 0, and

(b) \(\lim \sup \alpha \inf_{x \in E(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x)\) \geq \(\inf \sup \alpha \inf_{x \in E(y_\alpha)} Re(w, y_\alpha - y) + h(y_\alpha) - h(y)\) \geq \(\inf \sup \alpha \inf_{x \in E(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(y)\).

Suppose further that there exists a non-empty compact subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0, k \in K \cap S(y)$ and $\inf_{x \in E(y)} Re(w, y - x_0) + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists a point $\tilde{y} \in K$ such that

(i) $\tilde{y} \in S(\tilde{y})$

(ii) there exists a point $\tilde{w} \in T(\tilde{y})$ with $Re(\tilde{w}, \tilde{y} - x) \leq h(x) - h(\tilde{y})$ for all $x \in \tilde{S}(\tilde{y})$.

Proof. We shall complete the proof in three steps as follows:

Step 1. There exists a point $\tilde{y} \in X$ such that $\tilde{y} \in S(\tilde{y})$ and

$$
\sup_{x \in E(\tilde{y})} [\inf_{x \in E(\tilde{y})} Re(w, y - x) + h(y) - h(x)] \leq 0.
$$

Suppose the contrary. Then for each $y \in X$, either $y \not\in S(y)$ or there exists $x \in E(y)$ such that $\inf_{x \in E(y)} Re(w, y - x) + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \not\in S(y)$ or $y \in \Sigma$. If $y \not\in S(y)$, then by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that $Re(p, y) > \sup_{x \in E(y)} Re(p, x) > 0$. For each $y \in X$, set

$$
y(y) := \sup_{x \in E(y)} [\inf_{x \in E(y)} Re(w, y - x) + h(y) - h(x)].
$$

Let $V_0 := \{y \in X: y(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$
V_p := \{y \in X: Re(p, y) > \sup_{x \in E(y)} Re(p, x) > 0\}.
$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each $V_p$ is open in $X$ by Lemma 2.1 and $V_0$ is open in $X$ by hypothesis, \(\{V_0, V_p: p \in E^*\}\) is an open covering for $X$. Since $X$ is paracompact, there is a continuous partition of unity \(\{\beta_0, \beta_p: p \in E^*\}\) for $X$ subordinated to the open cover \(\{V_0, V_p: p \in E^*\}\) (see, for example, Theorem VIII.4.2 of Dugundji in [11]), i.e., for each $p \in E^*$, $\beta_p: X \rightarrow [0, 1]$ and $\beta_0: X \rightarrow [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and \(\sup p\beta_p \sup p\beta_p: p \in E^*\) is locally finite and $\beta_0(y) + \sum_{p\in E^*} \beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in F(X)$, $h$ is continuous on $co(A)$ (see e.g. [12, Corollary 10.1.1, p.83]). Define $\phi: X \times X \rightarrow \mathbb{R}$ by

$$
\phi(x, y) = \beta_0(y) \left[ \min_{w \in E(y)} Re(w, y - x) + h(y) - h(x) \right]
$$

+ \sum_{p\in E^*} \beta_p(y) Re(p, y - x)

for each $x, y \in X$. Then we have the following:

(i) Since $E$ is Hausdorff, for each $A \in F(X)$ and each fixed $x \in co(A)$, the map

$$
y \mapsto \min_{w \in E(y)} Re(w, y - x) + h(y) - h(x)
$$

is continuous on $co(A)$ by Lemma 2.3 and the fact that $h$ is continuous on $co(A)$ and therefore the map

$$
y \mapsto \beta_0(y) \left[ \min_{w \in E(y)} Re(w, y - x) + h(y) - h(x) \right]
$$

is lower semi-continuous on $co(A)$ by Lemma 2.2. Also, for each fixed $x \in X$,

$$
y \mapsto \sum_{p\in E^*} \beta_p(y) Re(p, y - x)
$$

is continuous on $X$. Hence, for each $A \in F(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semi-continuous on $co(A)$.

(ii) For each $A \in F(X)$ and for each $y \in co(A)$, $\min_{x \in E} \phi(x, y) \leq 0$. If this were false, then there exists some $A = \{x_1, ..., x_n\} \in F(X)$ and some $y \in co(A)$, say $y = \sum_{i=1}^n \lambda_i x_i$ with $\sum_{i=1}^n \lambda_i = 1$, such that $\min_{x \in E} \phi(x, y) > 0$. Then for each $i = 1, ..., n$, $\beta_0(y) \left[ \min_{w \in E(y)} Re(w, y - x_i) + h(y) - h(x_i) \right] + \sum_{p\in E^*} \beta_p(y) Re(p, y - x_i) > 0$, so that

$$
0 = \phi(x, y) = \beta_0(y) \left[ \min_{w \in E(y)} Re(w, y - \sum_{i=1}^n \lambda_i x_i) + h(y) \right]
$$

$$
- \left( \sum_{i=1}^n \lambda_i x_i \right)
$$

+ \sum_{p\in E^*} \beta_p(y) Re(p, y - \sum_{i=1}^n \lambda_i x_i)

$$
\geq \sum_{i=1}^n \lambda_i \left[ \beta_0(y) \left[ \min_{w \in E(y)} Re(w, y - x_i) + h(y) - h(x_i) \right] \right]
$$

+ \sum_{p\in E^*} \beta_p(y) Re(p, y - x_i) > 0,

for each $x, y \in X$. Then we have the following:
which is a contradiction.

(iii) Suppose that $A \in \mathcal{F}(X)$, and $(y_\alpha)_{\alpha \in \Gamma}$ is a net in $X$ converging to $y$ with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$

Case 1: $\beta_0(y) = 0$.

Since $\beta_0$ is continuous and $y_\alpha \to y$, we have $\beta_0(y_\alpha) \to \beta_0(y) = 0$. Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $(y_\alpha)_{\alpha \in \Gamma}$ is a bounded set, it follows that

$$\limsup_{\alpha} \left( \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \right) = 0.$$  \hspace{1cm} (2.1)

Also, we have

$$\beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] = 0.$$  \hspace{1cm} (2.1)

Thus it follows that

$$\limsup_{\alpha} \left( \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \right) + \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) =$$

$$\beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] + \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.2)$$

When $t = 1$, we have $\phi(x, x_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.

$$\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] + \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.3)$$

for all $\alpha \in \Gamma$. Therefore, by (2.3), we have

$$\limsup_{\alpha} \left( \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \right) +$$

$$\liminf_{\alpha} \left( \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \right) \leq \lim_{\alpha} \sup \left[ \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \right]$$

$$+ \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.4)$$

and so

$$\limsup_{\alpha} \left( \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \right) + \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.4)$$

Hence, by (2.2) and (2.4), we have $\phi(x, y) \leq 0$.

Case 2: $\beta_0(y) > 0$.

Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$.

When $t = 0$, we have $\phi(y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.

$$\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(y) \right] +$$

$$\sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.4)$$

Thus

$$\limsup_{\alpha} \left[ \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(y) \right] \right] + \sum_{p \in \mathcal{E}} \beta_p(y_\alpha) Re(p, y_\alpha - x) \leq 0 \hspace{1cm} (2.5)$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$, it follows that

$$\beta_0(y) \limsup_{\alpha} \left[ \beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(y) \right] \right] \leq 0 \hspace{1cm} (2.6)$$

Since $\beta_0(y) > 0$, by (2.6) and (2.7) we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(y) \right] \leq 0.$$  \hspace{1cm} (2.7)

Then, by hypothesis (a), we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \leq 0.$$  \hspace{1cm} (2.8)

Since $T$ is a pseudo-monotone type III operator, we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \geq \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x).$$

Then, by hypothesis (b), we have

$$\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \geq \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x).$$

Since $\beta_0(y) > 0$, we have

$$\beta_0(y) \limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - x) + h(y_\alpha) - h(x) \right] \geq \beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] \geq 0.$$  \hspace{1cm} (2.8)

Thus
for all \( x \in X \).

If \( \beta_0(\tilde{y}) > 0 \), then \( \tilde{y} \in V_0 = \Sigma \), so that \( y(\tilde{y}) > 0 \). Choose \( \tilde{x} \in S(\tilde{y}) \subset X \) such that

\[
\inf_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - \tilde{x}) + h(\tilde{y}) - h(\tilde{x}) \geq \frac{y(\tilde{y})}{\varepsilon} > 0.
\]

Then it follows that

\[
\beta_p(\tilde{y}) \left[ \inf_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - \tilde{x}) + h(\tilde{y}) - h(\tilde{x}) \right] > 0.
\]

If \( \beta_p(\tilde{y}) > 0 \) for some \( p \in E^* \), then \( \tilde{y} \in V_p \) and hence

\[
\text{Re}(p, \tilde{y}) > \sup_{x \in S(\tilde{y})} \text{Re}(p, x) \geq \text{Re}(p, \tilde{x})
\]

and so \( \text{Re}(p, \tilde{y} - \tilde{x}) > 0 \). Then we see that \( \beta_p(\tilde{y}) \text{Re}(p, \tilde{y} - \tilde{x}) > 0 \) whenever \( \beta_p(\tilde{y}) > 0 \) for \( p \in E^* \). Since \( \beta_p(\tilde{y}) > 0 \) for some \( p \in E^* \), it follows that

\[
\phi(\tilde{x}, \tilde{y}) = \beta_0(\tilde{y}) \left[ \inf_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - \tilde{x}) + h(\tilde{y}) - h(\tilde{x}) \right] + \sum_{p \in E^*} \beta_p(\tilde{y}) \text{Re}(p, \tilde{y} - \tilde{x}) > 0,
\]

which contradicts (2.10). This contradiction proves Step 1. Hence we have shown that there exists a point \( \tilde{y} \in X \) such that \( \tilde{y} \in S(\tilde{y}) \) and

\[
\sup_{x \in S(\tilde{y})} \left[ \inf_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \right] \leq 0.
\]

Step 2. We need to show that there exists a point \( \tilde{w} \in T(\tilde{y}) \) such that \( \text{Re}(\tilde{w}, \tilde{y} - \tilde{x}) + h(\tilde{y}) - h(x) \leq 0 \) for all \( x \in S(\tilde{y}) \). From Step 1, we have

\[
\sup_{x \in S(\tilde{y})} \left[ \inf_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \right] \leq 0, \quad (2.11)
\]

where \( T(\tilde{y}) \) is a weak*-compact convex subset of the Hausdorff topological vector space \( E^* \) and \( S(\tilde{y}) \) is a convex subset of \( X \).

Now, we define \( f : S(\tilde{y}) \times T(\tilde{y}) \rightarrow \mathbb{R} \) by \( f(x, w) = \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \) for each \( x \in S(\tilde{y}) \) and \( w \in T(\tilde{y}) \). Then, for each fixed \( x \in S(\tilde{y}) \), the mapping \( w \mapsto f(x, w) \) is convex and continuous on \( T(\tilde{y}) \) and, for each fixed \( w \in T(\tilde{y}) \), the mapping \( x \mapsto f(x, w) \) is concave on \( S(\tilde{y}) \). So, we can apply Kneser’s Minimax Theorem (Theorem 2.3) and obtain the following:

\[
\min_{w \in T(\tilde{y})} \sup_{x \in S(\tilde{y})} \left[ \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \right] = \sup_{x \in S(\tilde{y})} \left[ \min_{w \in T(\tilde{y})} \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \right].
\]

Hence, by (2.11), we obtain

\[
\min_{w \in T(\tilde{y})} \sup_{x \in S(\tilde{y})} \left[ \text{Re}(w, \tilde{y} - x) + h(\tilde{y}) - h(x) \right] \leq 0.
\]

Since \( T(\tilde{y}) \) is compact, there exists \( \tilde{w} \in T(\tilde{y}) \) such that

\[
\text{Re}(\tilde{w}, \tilde{y} - x) + h(\tilde{y}) - h(x) \leq 0
\]

for all \( x \in S(\tilde{y}) \). This completes the proof. \( \blacksquare \)

When \( X \) is compact, we obtain the following immediate consequence of Theorem 3.1:
Theorem 3.2. Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty compact convex subset of $E$ and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be upper semi-continuous such that each $S(x)$ is closed convex and $T: X \to 2^E$ be an $h$-pseudo-monotone type III (respectively, a strongly $h$-pseudo-monotone type III) operator and be upper semi-continuous from $co(A)$ to the weak*-topology on $E^*$ for each $A \in \mathcal{F}(X)$ and $T(x)$ is strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that the set

$$
\Sigma = \{ y \in X : \sup_{x \in E(y)} \left[ \inf_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] > 0 \}
$$

is open in $X$ and the following conditions are satisfied:

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $(y_\lambda)_{\lambda \in \Lambda}$ in $X$ converging to $y$, we have

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(x) \right] \leq 0,
$$

whenever

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \leq 0,
$$

and

(b) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $(y_\lambda)_{\lambda \in \Lambda}$ in $X$ converging to $y$, we have

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(x) \right] \geq \lim \inf_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \geq 0,
$$

whenever

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \geq \lim \inf_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] > 0.
$$

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$, and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re(\hat{w}, \hat{y} - x) \leq h(\hat{y}) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S: X \to 2^X$ is, in addition, lower semi-continuous and for each $y \in \Sigma$, $T$ is upper semi-continuous at $y$ in $X$, then the set $\Sigma$ in Theorem 3.1 is always open in $X$ and we obtain the following theorem:

Theorem 3.3. Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty paracompact convex and bounded subset of $E$ and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be continuous such that each $S(x)$ is compact convex, $T: X \to 2^E$ be an $h$-pseudo-monotone type III (respectively, strongly $h$-pseudo-monotone type III) operator which is upper semi-continuous from $co(A)$ to the weak*-topology on $E^*$ for each $A \in \mathcal{F}(X)$, with $T(X)$ strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that for each $x \in \Sigma = \{ y \in X : \sup_{x \in E(y)} \left[ \inf_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] > 0 \}$, $T$ is upper semi-continuous at $y$ from the relative topology on $X$ to the strong topology on $E^*$ and the following conditions are satisfied:

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $(y_\lambda)_{\lambda \in \Lambda}$ in $X$ converging to $y$, we have

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(x) \right] \leq 0,
$$

whenever

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \leq 0,
$$

and

(b) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $(y_\lambda)_{\lambda \in \Lambda}$ in $X$ converging to $y$, we have

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(x) \right] \geq \lim \inf_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \geq 0,
$$

whenever

$$
\lim \sup_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] \geq \lim \inf_{\lambda \in \Lambda} \left[ \inf_{w \in T(y_\lambda)} Re(w, y_\lambda - x) + h(y_\lambda) - h(y) \right] > 0.
$$

Then $W$ is a strongly open neighborhood of $0$ in $E^*$ and so $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in $E^*$. Since $T$ is upper semi-continuous at $y_0$, there exists an open neighborhood $N_1$ of $y_0$ in $X$ such that $T(y) \subset U_1$ for all $y \in N_1$. Since the mapping $x \mapsto \inf_{w \in T(y_0)} Re(w, x - x) + h(x) - h(x)$ is continuous at $x_0$, there exists an open neighborhood $V_1$ of $x_0$ in $X$ such that

$$
\inf_{w \in T(y_0)} Re(w, x - x) + h(x) - h(x) < \alpha/6
$$

for all $x \in V_1$.

Since $x_0 \in V_1 \cap S(y_0) \neq \emptyset$ and $S$ is lower semi-continuous at $y_0$, there exists an open neighborhood $N_2$ of $y_0$ in $X$ such that $S(y) \cap V_1 \neq \emptyset$ for all $y \in N_2$. Since the mapping $y \mapsto \inf_{w \in T(y_0)} Re(w, y - y_0) + h(y) - h(y_0)$ is continuous at $y_0$, there exists an open neighborhood $N_3$ of $y_0$ in $X$ such that

$$
\inf_{w \in T(y_0)} Re(w, y - y_0) + h(y) - h(y_0) < \alpha/6
$$

for all $y \in N_3$.

Let $N_4 := N_1 \cap N_2 \cap N_3$. Then $N_4$ is an open neighborhood of $y_0$ in $X$ such that for each $y_1 \in N_4$, we have the following:
(a) $S(y_1) \cap V_1 \neq \emptyset$ as $y_1 \in N_2$; so, we can choose any $x_1 \in S(y_1) \cap V_1$.

(b) $\inf_{w \in T(y_0)} R(e, y_1 - y_0) + h(y_1) - h(y_0) < \alpha / 6$ as $y_1 \in N_2$.

(c) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$.

(d) $\inf_{w \in T(y_0)} R(e, x_0 - x_1) + h(x_0) - h(x_1) < \alpha / 6$ as $x_1 \in V_1$.

Hence, we can obtain the following by omitting the details:

\[
\inf_{w \in T(y_1)} R(e, y_1 - x_1) + h(y_1) - h(x_1) \\
\geq \inf_{w \in T(y_0) + W} R(e, y_1 - x_1) + h(y_1) - h(x_1) \\
\geq \inf_{w \in T(y_0)} R(e, y_1 - x_1) + h(y_1) - h(x_1) + \inf_{w \in T(y_0)} R(e, y_1 - y_0) + h(y_1) - h(y_0) + \inf_{w \in T(y_0)} R(e, w, x_0 - x_1) + h(x_0) - h(x_1)
\]

\[
\geq \inf_{w \in T(y_0)} R(e, w, x_0 - x_1) + h(x_0) - h(x_1) + \inf_{w \in T(y_0)} R(e, w, y_1 - x_1) \geq -\frac{\alpha}{6} - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{3} > 0.
\]

Consequently, we have

\[
\sup_{x \in S(y_1)} \inf_{w \in T(y_1)} R(e, w, x - x) + h(y_1) - h(x) > 0
\]

since $x_1 \in S(y_1)$. Hence, $y_1 \in \Sigma$ for all $y_1 \in N_0$. Therefore, $y_0 \in N_0 \subset \Sigma$. But $y_o$ was arbitrary. Consequently, $\Sigma$ is open in $X$. Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence, the conclusion follows from Theorem 3.1. This completes the proof. ■

When $X$ is compact, we obtain the following theorem:

Theorem 3.4. Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty compact convex subset of $E$ and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be continuous such that each $S(x)$ is closed convex, $T: X \to 2^{E^*}$ be an $h$-pseudo-monotone type III (respectively, strongly $h$-pseudo-monotone type III) operator which is upper semi-continuous from co$(A)$ to the weak* topology on $E^*$ for each $A \in \mathcal{F}(X)$, with $T(X)$ strongly bounded. Also, for each $x \in X$, $T(x)$ is weak* compact convex. Suppose that for each $y \in \Sigma = \{ y \in X : \sup_{x \in S(y)} \inf_{w \in T(y)} R(e, w, y - x) + h(y) - h(x) \geq 0 \}$, $T$ is upper semi-continuous at $y$ from the relative topology on $X$ to the strong topology on $E^*$ and the following conditions are satisfied:

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $(y_\alpha)_{\alpha < e}$ in $X$ converging to $y$, we have

\[
\limsup_{\alpha \to e} \left[ \inf_{w \in T(y_\alpha)} R(e, y_\alpha - x) + h(y_\alpha) - h(x) \right] \leq 0
\]

whenever $\limsup_{\alpha \to e} \left[ \inf_{w \in T(y_\alpha)} R(e, y_\alpha - y) + h(y_\alpha) - h(y) \right] \leq 0$, and

(b) $\limsup_{\alpha \to e} \left[ \inf_{w \in T(y_\alpha)} R(e, w, y - x) + h(y_\alpha) - h(x) \right] \geq \inf_{w \in T(y)} R(e, w, y - x) + h(y) - h(x)$ whenever

\[
\limsup_{\alpha \to e} \left[ \inf_{w \in T(y_\alpha)} R(e, y_\alpha - y) + h(y_\alpha) - h(y) \right] \geq \inf_{w \in T(y)} R(e, w, y - x) + h(y) - h(x).
\]

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $R(e, \hat{w}, \hat{y} - x) \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Remark 3.5. (1) Theorems 3.1, 3.2, 3.3 and 3.4 of this paper are further extensions of the results obtained in [4] on generalized quasi-variational inequalities of pseudo-monotone type III and strongly pseudo-monotone type III operators.

(2) In 1985, Shih and Tan ([4]) obtained results on generalized quasi-variational inequalities in locally convex topological vector spaces and their results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [4] using pseudo-monotone type III and strongly pseudo-monotone type III operators on non-compact sets.

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References


