

Solution of FPDE in Fluid Mechanics by ADM, VIM and NIM

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Abstract: By handling the one dimensional partial differential equation with three methods i.e. Adomain decomposition method(ADM), Variation iteration method(VIM) and the New iterative method(NIM) and applied logarithmic and exponential functions as initial condition. A general framework of these methods is presented for analytical treatment of fractional partial differential equation arises in fluid mechanics. The fractional derivatives are described in the Caputo sense. The equation used in this paper is fractional wave equation, fractional burgers equation and fractional Klein-Gordon equation. After comparison of the results, the series of solution are found which is very helpful. The basic idea described in this paper is accepted to be further in use to solve other similar linear problems in fractional calculus.

Keywords: Adomain Decomposition Method (ADM), Variation Iteration Method (VIM), New Iterative Method (NIM), Fractional Wave Equation, Fractional Burgers Equation, Fractional Klein-Gordon Equation

1. Introduction

Now a day's fractional differential equations are motivated by new example of applications in fluid mechanics, mathematics, mathematical biology, physics, electrochemistry and viscoelasticity. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [1], and differential equation with fractional order have currently proved to be valuable tools to the modeling of many physical phenomena [2]. The NIM, planned by Daftardar-Gejji and Jafari in 2006 [3] and improved by Hemeda [4], was effectively applied to a variety of linear and nonlinear equations such as algebraic equations, integral equations, integrodifferential equations, ordinary and partial differential equations of integer and fractional order, and system of equations as well. NIM is simple to understand and easy to implement using computer packages and yields better result [5] than the existing ADM [6], Homotopy perturbation method(HPM) [7], or VIM [8]. Henderson [9] investigates the existence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with coupled integral boundary conditions. Bekri [10] used the fractional complex transformation method to convert fractional order partial differential equation to ordinary differential equation.

Yang [11] extended the classical HPM to local fractional HPM; Bhrawy [12] solve the second and fourth order fractional diffusion-wave equations and fractional wave equations with damping.

The objective of this work is to extend the application of the ADM, VIM and the NIM to obtain analytical solutions with initial conditions like logarithmic and exponential function to some fractional partial differential equations in fluid mechanics. These equations include Wave equation, Burgers equation and Klein-Gordon equation.

2. Preliminaries and Notations

Some basic definitions and properties of the fractional calculus theory which are used in this paper are given in this section.

Definition (1): A real function $f(t)$, $t > 0$, is said to be in the space C_{μ} , $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_{μ}^m iff $f^{(m)} \in C_{\mu}$, $m \in N$.

Definition (2): The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (1)$$

$$\alpha > 0, t > 0,$$

$$I^0 f(t) = f(t).$$

Properties of the operators I^α can be found in [13]; we mention only the following: for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$,

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t),$$

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t),$$

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\alpha+\gamma}.$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified functional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [13].

Definition(3): The fractional derivative of $f(t)$ in caputo sense is defined as

$$\begin{aligned} D^\alpha f(t) &= I^{(m-\alpha)} D^{(m)} f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \end{aligned} \quad (2)$$

for $m-1 < \alpha \leq m, m \in N, t > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma(1): If $m-1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then $D^\alpha I^\alpha f(t) = f(t)$,

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, t > 0$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of problem [14]. In this paper, we consider the one-dimensional linear inhomogeneous functional partial differential equations in fluid mechanics, where the unknown function $u(x, t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The fractional derivative is taken in caputo sense as follows.

Definition(4): For m to be the smallest integer that exceeds α , the caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$\begin{cases} D_t^\alpha u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \\ \left\{ \begin{array}{l} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m}{\partial \tau^m} u(x, \tau) d\tau, \\ \text{for } m-1 < \alpha < m \\ \frac{\partial^m}{\partial t^m} u(x, t), \text{ for } \alpha = m, m \in N. \end{array} \right. \end{cases} \quad (3)$$

3. Linear Equation

To include ADM [15], VIM [15] and NIM [16], the three linear fractional partial differential equations will be studied. The three methods are used to construct the solution of the given examples.

Example(1): Consider the following One-dimensional linear inhomogeneous fraction wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t \cos(x), \quad (4)$$

$$t > 0, x \in R, 0 < \alpha \leq 1,$$

subject to initial condition

$$u(x, 0) = \log x \quad (5)$$

Problem “(4)” and “(5)” can be obtain by using ADM [15] the recurrence relation is given by

$$u_0(x, t) = u(x, 0) + I^\alpha \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t \cos(x) \right),$$

$$u_{j+1}(x, t) = -I^\alpha \left(\frac{\partial}{\partial x} u_j(x, t) \right), j \geq 0. \quad (6)$$

In view of “(6)” equation $u_0, u_1, u_2 \dots$ are as follows:

$$\begin{aligned} u_0(x, t) &= \log x + t \sin(x) \\ &+ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x), \end{aligned} \quad (7)$$

$$\begin{aligned} u_1(x, t) &= -\frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\ &+ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x), \end{aligned} \quad (8)$$

$$\begin{aligned} u_2(x, t) &= -\frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) \\ &- \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x), \end{aligned} \quad (9)$$

⋮

and so on, in this way the remaining equations of ADM in series can be obtain. The solution in series form is given by

$$\begin{aligned} u(x, t) &= \log x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\ &- \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\ &+ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ &\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x) + \dots \end{aligned} \quad (10)$$

Cancelling the noise terms and keeping the non-noise terms we yield,

$$u(x, t) = \log x + t \sin(x) - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \dots \tag{11}$$

Where in Mittage-Leffer form it is written as

$$u(x, t) = \log x + t \sin(x) - \left[\sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right] = \log x + t \sin(x) - E_\alpha \frac{1}{x} t^\alpha$$

where E_α is the Mittage-Leffer function.

Also the problem “(4)” and “(5)” is solved in [15] by using the VIM. By using iteration formula as

$$I_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x, t) + \frac{\partial}{\partial x} u_k(x, t) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) - t \cos(x) \right] = u_{k+1}(x, t) - u_k(x, t) \tag{12}$$

In view of “(12)” $u_1, u_2 \dots$ and by benign with $u_0 = \log x$, we can obtain,

$$u_1(x, t) = \log x - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x), \tag{13}$$

$$u_2(x, t) = \log x - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x) - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) \vdots \tag{14}$$

Cancelling the noise terms and keeping the non-noise term we yield,

$$u(x, t) = \log x + t \sin(x) - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots \tag{15}$$

Where in Mittage-Leffer form

$$u(x, t) = \log x + t \sin(x) - \left[\sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right] = \log x + t \sin(x) - E_\alpha \frac{1}{x} t^\alpha$$

According to the NIM [16], and by using the formula,

$$u(x, t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A$$

$$= f + N(u), \tag{16}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

$$\text{and } N(u) = I_t^\alpha A \text{ and } u_0 = f, u_{n+1} = N(u), n = 0, 1, 2, \dots$$

Therefore in view of “(16)” we obtain

$$u_0(x, t) = \log x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x, \tag{17}$$

$$u_1(x, t) = N(u_0(x, t)) = -I_t^\alpha \left[\frac{\partial}{\partial x} u_0(x, t) \right] = -\frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x), \tag{18}$$

$$u_2(x, t) = N(u_1(x, t)) = -I_t^\alpha \left[\frac{\partial}{\partial x} u_1(x, t) \right] = -\frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x) \vdots \tag{19}$$

The NIM in series form is given by

$$u(x, t) = \log x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x) - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \sin(x) - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x) - \dots \tag{20}$$

Cancelling the noise terms and keeping the non-noise terms we yield,

$$u(x, t) = \log x + t \sin(x) - \frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{1}{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots \tag{21}$$

Where in Mittage-Leffer form

$$u(x, t) = \log x + t \sin(x) - \left[\sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right]$$

$$= \log x + t \sin(x) - E_{\alpha} \frac{1}{x} t^{\alpha}$$

From “(11)”, “(15)” and “(21)”, it is clear that the three methods have the same results.

Example(2): Consider the following One-dimensional linear inhomogeneous fraction wave equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t \cos(x), \quad (22)$$

$$t > 0, x \in R, 0 < \alpha \leq 1,$$

subject to initial condition

$$u(x, 0) = e^x \quad (23)$$

Problem “(22)” and “(23)” can be obtain by using ADM [15] the recurrence relation is given by

$$u_0(x, t) = u(x, 0) + I^{\alpha} \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t \cos(x) \right) \quad (24)$$

$$u_{j+1}(x, t) = -I^{\alpha} \left(\frac{\partial}{\partial x} u_j(x, t) \right), j \geq 0.$$

In View of “(24)” $u_0, u_1, u_2 \dots$ are as follows:

$$u_0(x, t) = e^x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x), \quad (25)$$

$$u_1(x, t) = -e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x)$$

$$+ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x), \quad (26)$$

$$u_2(x, t) = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x)$$

$$- \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x) \quad (27)$$

⋮

The solution in series form is given by

$$u(x, t) = e^x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x)$$

$$- e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x)$$

$$+ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$- \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x)$$

$$- \dots \quad (28)$$

Cancelling the noise terms and keeping the non-noise terms we yield,

$$u(x, t) = e^x + t \sin(x) - e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots \quad (29)$$

Where in Mittage-Leffer form

$$u(x, t) = t \sin(x) + e^x \left[1 - \sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right]$$

$$= t \sin(x) + e^x - e^x E_{\alpha}(t^{\alpha})$$

Also the problem “(22)” and “(23)” is solved in [15] by using the VIM. By using iteration formula as

$$u_{k+1}(x, t) = u_k(x, t) - I_t^{\alpha} \left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} u_k(x, t) \right.$$

$$\left. + \frac{\partial}{\partial x} u_k(x, t) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) \right.$$

$$\left. - t \cos(x) \right] \quad (30)$$

In view of “(30)” $u_1, u_2 \dots$ and by benign with $u_0 = e^x$ we can obtain,

$$u_1(x, t) = e^x - e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\cos(x), \quad (31)$$

$$u_2(x, t) = e^x - e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\cos(x) + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x$$

$$+ \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin x, \quad (32)$$

$$\vdots$$

Cancelling the noise terms and keeping the non-noise term we yield,

$$u(x, t) = e^x + t \sin(x) - e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x$$

$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots \quad (33)$$

Where in Mittage-Leffer form

$$u(x, t) = t \sin(x) + e^x \left[1 - \sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right]$$

$$= t \sin(x) + e^x - e^x E_{\alpha}(t^{\alpha})$$

According to the NIM [16], and by using the formula,

$$\begin{aligned}
 u(x, t) &= \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A \\
 &= f + N(u),
 \end{aligned}
 \tag{34}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

and $N(u) = I_t^\alpha A$ and $u_0 = f, u_{n+1} = N(u), n = 0, 1, 2, \dots$

Therefore in view of “(34)” we obtain

$$u_0(x, t) = e^x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos x,
 \tag{35}$$

$$\begin{aligned}
 u_1(x, t) &= -e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\
 &\quad + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x),
 \end{aligned}
 \tag{36}$$

$$\begin{aligned}
 u_2(x, t) &= e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) \\
 &\quad - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x),
 \end{aligned}
 \tag{37}$$

⋮

The NIM in series form is given by

$$\begin{aligned}
 u(x, t) &= e^x + t \sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\
 &\quad - e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \cos(x) \\
 &\quad + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
 &\quad - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \cos(x) \\
 &\quad - \dots
 \end{aligned}
 \tag{38}$$

Cancelling the noise terms and keeping the non-noise terms we yield,

$$\begin{aligned}
 u(x, t) &= e^x + t \sin(x) - e^x \frac{t^\alpha}{\Gamma(\alpha+1)} \\
 &\quad + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots
 \end{aligned}
 \tag{39}$$

Where in Mittag-Leffer form

$$u(x, t) = t \sin(x) + e^x \left[1 - \sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha i + 1)} \right]$$

$$= t \sin(x) + e^x - e^x E_\alpha(t^\alpha)$$

From “(29)”, “(33)” and “(39)”, it is clear that the three methods have the same results.

Example(3): Consider the following One-dimensional linear inhomogeneous fractional Burger’s equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2,
 \tag{40}$$

$$t < 0, x \in R, 0 < \alpha \leq 1,$$

subject to initial condition

$$u(x, 0) = \log x
 \tag{41}$$

Problem “(40)” and “(41)” can be obtain by using ADM [15] the recurrence relation is given by

$$u_0(x, t) = u(x, 0) + I^\alpha \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} 2x - 2 \right),$$

$$\begin{aligned}
 u_{j+1}(x, t) &= -I^\alpha \left(\frac{\partial}{\partial x} u_j(x, t) - \frac{\partial^2}{\partial x^2} u_j(x, t) \right), \\
 j &\geq 0.
 \end{aligned}
 \tag{42}$$

In View of “(42)” equations $u_0, u_1, u_2 \dots$ are as follows:

$$u_0(x, t) = \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)},
 \tag{43}$$

$$u_1(x, t) = -\left(\frac{1}{x} + \frac{1}{x^2}\right) \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)},
 \tag{44}$$

$$\begin{aligned}
 u_2(x, t) &= \left(-\frac{1}{x^2} - \frac{2}{x^3}\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4}\right) \\
 &\quad \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},
 \end{aligned}
 \tag{45}$$

$$\begin{aligned}
 u_3(x, t) &= -\left(\frac{2}{x^3} + \frac{6}{x^4}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5}\right) \\
 &\quad \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
 &\quad + \left(\frac{24}{x^5} + \frac{120}{x^6}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},
 \end{aligned}
 \tag{46}$$

The solution in series form is given by

$$\begin{aligned}
 u(x, t) &= \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)} \\
 &\quad - \left(\frac{1}{x} + \frac{1}{x^2}\right) \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} \\
 &\quad + \left(-\frac{1}{x^2} - \frac{2}{x^3}\right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4}\right) \\
 &\quad \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}
 \end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{6}{x^4} - \frac{24}{x^5} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5} \right) \\
& \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(\frac{24}{x^5} + \frac{120}{x^6} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \quad (47)
\end{aligned}$$

The exact solution of ADM is

$$\begin{aligned}
u(x, t) &= \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)} \\
& - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} \\
& + \left(-2 - \frac{1}{x^2} - \frac{4}{x^3} - \frac{6}{x^2} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& + \left(-\frac{2}{x^3} - \frac{18}{x^4} - \frac{72}{x^5} - \frac{120}{x^6} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
& + \dots \quad (48)
\end{aligned}$$

Also the problem “(40)” and “(41)” is solved in [15] by using the VIM. By using iteration formula as

$$\begin{aligned}
u_{k+1}(x, t) &= u_k(x, t) - I_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x, t) \right. \\
& + \frac{\partial}{\partial x} u_k(x, t) - \frac{\partial^2}{\partial x^2} u_k(x, t) \\
& \left. - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2x + 2 \right] \quad (49)
\end{aligned}$$

In view of “(49)” $u_1, u_2 \dots$ and by benign with

$u_0 = \log x$ we can obtain,

$$\begin{aligned}
u_1(x, t) &= \log x - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + t^2 \\
& + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (50)
\end{aligned}$$

$$\begin{aligned}
u_2(x, t) &= \log x - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + t^2 \\
& + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(-\frac{1}{x^2} - \frac{2}{x^3} \right) \\
& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4} \right) \\
& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (51)
\end{aligned}$$

$$\begin{aligned}
u_3(x, t) &= \log x - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + t^2 \\
& + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(-\frac{1}{x^2} - \frac{2}{x^3} \right) \\
& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4} \right)
\end{aligned}$$

$$\begin{aligned}
& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
& + \left(\frac{6}{x^4} + \frac{24}{x^5} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5} \right) \\
& \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \left(\frac{24}{x^5} + \frac{120}{x^6} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (52) \\
& \vdots
\end{aligned}$$

The exact solution is

$$\begin{aligned}
u(x, t) &= \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)} \\
& - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(-2 - \frac{1}{x^2} - \frac{4}{x^3} - \frac{6}{x^2} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \\
& \left(-\frac{2}{x^3} - \frac{18}{x^4} - \frac{72}{x^5} - \frac{120}{x^6} \right) + \dots \quad (53)
\end{aligned}$$

According to the NIM [16] and using the formula,

$$\begin{aligned}
u(x, t) &= \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A \\
& = f + N(u), \quad (54)
\end{aligned}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

and $N(u) = I_t^\alpha A$ and $u_0 = f, u_{n+1} = N(u),$

$$n = 0, 1, 2, \dots$$

Therefore in view of “(54)” we obtain

$$u_0(x, t) = \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (55)$$

$$u_1(x, t) = - \left(\frac{1}{x} + \frac{1}{x^2} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (56)$$

$$\begin{aligned}
u_2(x, t) &= \left(-\frac{1}{x^2} - \frac{2}{x^3} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{2}{x^3} + \frac{6}{x^4} \right) \\
& \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (57)
\end{aligned}$$

$$\begin{aligned}
u_3(x, t) &= - \left(\frac{2}{x^3} + \frac{6}{x^4} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5} \right) \\
& \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
& - \left(\frac{24}{x^5} + \frac{120}{x^6} \right) \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (58) \\
& \vdots
\end{aligned}$$

The NIM in series form is given by

$$\begin{aligned}
 u(x, t) = & \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - \left(\frac{1}{x} + \frac{1}{x^2}\right) \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
 & + \left(-\frac{1}{x^2} - \frac{2}{x^3}\right) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \left(\frac{2}{x^3} + \frac{6}{x^4}\right) \\
 & \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \left(\frac{2}{x^3} + \frac{6}{x^4}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} \\
 & + \left(-\frac{6}{x^4} - \frac{24}{x^5}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \left(-\frac{6}{x^4} - \frac{24}{x^5}\right) \\
 & \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} - \left(\frac{24}{x^5} + \frac{120}{x^6}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \quad (59)
 \end{aligned}$$

The exact solution is

$$\begin{aligned}
 u(x, t) = & \log x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - \left(\frac{1}{x} + \frac{1}{x^2}\right) \frac{t^\alpha}{\Gamma(\alpha + 1)} + \left(-2 - \frac{1}{x^2} - \frac{4}{x^3} - \frac{6}{x^4}\right) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \\
 & \left(-\frac{2}{x^3} - \frac{18}{x^4} - \frac{72}{x^5} + \frac{120}{x^6}\right) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \quad (60)
 \end{aligned}$$

From “(48)”, “(53)” and “(60)”, it is clear that the three methods have the same results.

Example(4): Consider the following One-dimensional linear inhomogeneous fractional Burger’s equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad (61)$$

$$t < 0, x \in R, 0 < \alpha \leq 1,$$

subject to initial condition

$$u(x, 0) = e^x. \quad (62)$$

Problem “(60)” and “(61)” can be obtain by using ADM [15] the recurrence relation is given by

$$u_0(x, t) = u(x, 0) + I^\alpha \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} 2x - 2 \right),$$

$$\begin{aligned}
 u_{j+1}(x, t) = & -I^\alpha \left(\frac{\partial}{\partial x} u_j(x, t) - \frac{\partial^2}{\partial x^2} u_j(x, t) \right), \\
 & j \geq 0. \quad (63)
 \end{aligned}$$

In View of “(63)” equation $u_0, u_1, u_2 \dots$ are as follows:

$$u_0(x, t) = e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad (64)$$

$$u_1(x, t) = -\frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (65)$$

$$u_2(x, t) = 0. \quad (66)$$

The exact solution of ADM is

$$\begin{aligned}
 u(x, t) = & e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (67)
 \end{aligned}$$

Also the problem is solved by using VIM [15] with $u_0 = e^x$ the following approximation can be obtain by using the formula

$$\begin{aligned}
 u_{k+1}(x, t) = & u_k(x, t) - I_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x, t) \right. \\
 & + \frac{\partial}{\partial x} u_k(x, t) - \frac{\partial^2}{\partial x^2} u_k(x, t) \\
 & \left. - \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2x + 2 \right] \quad (68)
 \end{aligned}$$

In view of “(68)” $u_1, u_2 \dots$ and by benign with $u_0 = e^x$ we can obtain,

$$u_1(x, t) = e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad (69)$$

$$\begin{aligned}
 u_2(x, t) = & e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (70)
 \end{aligned}$$

The exact solution of ADM is

$$\begin{aligned}
 u(x, t) = & e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)} \quad (71)
 \end{aligned}$$

According to the NIM [16] and by using the formula,

$$\begin{aligned}
 u(x, t) = & \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A \\
 = & f + N(u), \quad (72)
 \end{aligned}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

and $N(u) = I_t^\alpha A$ and $u_0 = f, u_{n+1} = N(u),$

$$n = 0, 1, 2, \dots$$

Therefore in view of “(72)” we obtain

$$u_0(x, t) = e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad (73)$$

$$u_1(x, t) = -\frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (74)$$

$$u_2(x, t) = 0. \quad (75)$$

The NIM in series form and the exact solution is

$$u(x, t) = e^x + t^2 + (2x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)} \tag{76}$$

From “(67)”, “(71)” and “(76)”, it is clear that the three methods have the same results.

Example(5): Consider the following One-dimensional linear inhomogeneous fractional Klein-Gordon equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \tag{77}$$

$$t < 0, x \in R, 1 < \alpha \leq 2,$$

subject to initial condition

$$u(x, 0) = \log x, u_t(x, 0) = 0 \tag{78}$$

Problem “(77)” and “(78)” can be obtain by using ADM [15] the recurrence relation is given by

$$u_0(x, t) = u(x, t) + tu_t(x, 0) + I^\alpha(6x^3 t + (x^3 - 6x)t^3),$$

$$u_{j+1}(x, t) = I^\alpha \left(\frac{\partial^2}{\partial x^2} u_j(x, t) - u_j(x, t) \right),$$

$$j \geq 0. \tag{79}$$

In view of “(79)” the first few components are derived as follows

$$u_0(x, t) = \log x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \tag{80}$$

$$u_1(x, t) = -\frac{1}{x^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} - \log x \frac{t^\alpha}{\Gamma(\alpha + 1)} - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \tag{81}$$

$$\vdots$$

The solution in series form is

$$u(x, t) = \log x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)} - \frac{1}{x^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} - \log x \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

$$-6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha + 4)} + \dots \tag{82}$$

According to VIM [15], the iteration formula for the problem is given by

$$u_{k+1}(x, t) = u_k(x, t) - (\alpha - 1) I_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x, t) - \frac{\partial^2}{\partial x^2} u_k(x, t) + u_k(x, t) - 6x^3 t - (x^3 - 6x)t^3 \right] \tag{83}$$

By above Variation iteration formula, if we begin with $u_0(x, t) = \log x$, we can obtain the following approximations

$$u_1(x, t) = \log x + (\alpha - 1) \left[-\frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha+1)} + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \right], \tag{84}$$

$$u_2(x, t) = \log x + (\alpha - 1) \left[-\frac{1}{x} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)} \right] + ((\alpha - 1)^2 - (\alpha - 1)) \frac{1}{x^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + (\alpha - 1)^2 \left(-\frac{6}{x^4} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right), \tag{85}$$

$$\vdots$$

Now according to NIM [16] and by using the formula,

$$u(x, t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A = f + N(u), \tag{86}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

and $N(u) = I_t^\alpha A$ and $u_0 = f, u_{n+1} = N(u),$

$$n = 0, 1, 2, \dots$$

Therefore in view of “(86)” we obtain

$$u_0(x, t) = \log x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \tag{87}$$

$$\begin{aligned}
 u_1(x, t) = & -\frac{1}{x^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \\
 & + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} - \log x \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \quad (88) \\
 & \vdots
 \end{aligned}$$

The NIM in series form and the exact solution is

$$\begin{aligned}
 u(x, t) = & \log x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \\
 & \frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)} - \frac{1}{x^2} \frac{t^\alpha}{\Gamma(\alpha + 1)} + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \\
 & + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} - \log x \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \\
 & + (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \dots \quad (89)
 \end{aligned}$$

The ADM, the VIM and the NIM gives the same solution for Classical Klein-Gordon equation(when $\alpha = 2$) which is given by

$$\begin{aligned}
 u(x, t) = & \log x + x^3 t^3 + (x^3 - 6x) 6 \frac{t^5}{\Gamma(6)} \\
 & - \frac{1}{x^2} \frac{t^2}{\Gamma(3)} + 36x \frac{t^5}{\Gamma(6)} + 36x \frac{t^7}{\Gamma(8)} \\
 & - \log x \frac{t^2}{\Gamma(3)} - 6x^3 \frac{t^5}{\Gamma(6)} \\
 & + (x^3 - 6x) 6 \frac{t^7}{\Gamma(8)} + \dots \quad (90)
 \end{aligned}$$

Cancelling the noise terms and keeping the non-noise terms in above equation yield the exact solution of the problem, for the case when $\alpha = 2$

$$u(x, t) = \log x + x^3 t^3. \quad (91)$$

Example(6): Consider the following One-dimensional linear inhomogeneous fractional Klein-Gordon equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad (92)$$

$$t < 0, x \in R, 1 < \alpha \leq 2,$$

subject to initial condition

$$u(x, 0) = e^x, u_t(x, 0) = 0 \quad (93)$$

Problem “(92)” and “(93)” can be obtain by using ADM [15] the recurrence relation is given by

$$\begin{aligned}
 u_0(x, t) = & u(x, 0) + tu_t(x, 0) + I^\alpha (6x^3 \\
 & + (x^3 - 6x)t^3) \\
 u_{j+1}(x, t) = & -I^\alpha \left(\frac{\partial^2}{\partial x^2} u_j(x, t) - u_j(x, t) \right), \\
 & j \geq 0. \quad (94)
 \end{aligned}$$

In view of “(94)” the first few components are derived as follows

$$\begin{aligned}
 u_0(x, t) = & e^x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \\
 & \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \quad (95) \\
 u_1(x, t) = & 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + (x^3 - 6x) \\
 & \frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \quad (96) \\
 & \vdots
 \end{aligned}$$

The solution in series form is

$$\begin{aligned}
 u(x, t) = & e^x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \\
 & + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & + \dots \quad (97)
 \end{aligned}$$

According to VIM [15], the iteration formula for the problem is given by

$$\begin{aligned}
 u_{k+1}(x, t) = & u_k(x, t) - (\alpha - 1) I_t^\alpha \left[\frac{\partial^\alpha}{\partial t^\alpha} u_k(x, t) \right. \\
 & - \frac{\partial^2}{\partial x^2} u_k(x, t) + u_k(x, t) - 6x^3 t \\
 & \left. - (x^3 - 6x)t^3 \right] \quad (98)
 \end{aligned}$$

By above Variation iteration formula, if we begin with $u_0(x, t) = e^x$, we can obtain the following approximations

$$\begin{aligned}
 u_1(x, t) = & e^x + (\alpha - 1) \left[e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - \right. \\
 & \left. 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \right], \quad (99)
 \end{aligned}$$

$$\begin{aligned}
 u_2(x, t) = e^x + (\alpha - 1) & \left[e^x \frac{t^\alpha}{\Gamma(\alpha + 1)} + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right. \\
 & \left. + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)} \right] \\
 & + (\alpha - 1)^2 \left(-e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right), \quad (100) \\
 & \vdots
 \end{aligned}$$

Now according to NIM [16] and by using the formula,

$$\begin{aligned}
 u(x, t) &= \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A \\
 &= f + N(u), \quad (101)
 \end{aligned}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B$$

and $N(u) = I_t^\alpha A$ and $u_0 = f, u_{n+1} = N(u), n = 0, 1, 2, \dots$

Therefore in view of “(101)” we obtain

$$\begin{aligned}
 u_0(x, t) &= e^x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (x^3 - 6x) \\
 & \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \quad (102)
 \end{aligned}$$

$$\begin{aligned}
 u_1(x, t) &= 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - (x^3 - 6x) \\
 & \frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \quad (103) \\
 & \vdots
 \end{aligned}$$

The solution in series form is

$$\begin{aligned}
 u(x, t) &= e^x + 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x) \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \\
 & + 36x \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + 36x \frac{t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & - 6x^3 \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - (x^3 - 6x) \frac{6t^{2\alpha+3}}{\Gamma(2\alpha + 4)} \\
 & + \dots \quad (104)
 \end{aligned}$$

The ADM, the VIM and the NIM gives the same solution for Classical Klein-Gordon equation (when $\alpha = 2$) which is given by

$$u(x, t) = e^x + x^3 t^3 + (x^3 - 6x) 6 \frac{t^5}{\Gamma(6)}$$

Cancelling the nose terms and keeping the non-noise terms in above equation yield the exact solution of the problem, for the case when $\alpha = 2$

$$u(x, t) = e^x + x^3 t^3. \quad (106)$$

4. Conclusion

These three types of method, i.e., ADM, VIM and the NIM are many useful for solving partial differential equations, ordinary differential equations and other equations. The present paper shows great potential for solving linear fraction partial differential equations. And using logarithmic and exponential function as initial condition in the given problem, the series of solution is obtained. This work can also be expected for further solving similar type of linear as well as nonlinear problems in fraction calculus.

References

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