



Numerical Approach for Solving Stiff Differential Equations Through the Extended Trapezoidal Rule Formulae

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Abstract: The most popular methods for the solution of stiff initial value problems for ordinary differential equations are the backward differentiation formulae (BDF). In this paper, we focus on the derivation of the fourth, sixth and eighth order extended trapezoidal rule of first kind (ETRs) formulae through Hermite polynomial as basis function which we named FETR, SETR and EETR respectively. We then interpolate and collocate at some points of interest to generate the desire method. The stability analysis on our methods suggests that they are not only convergent but possess regions suitable for the solution of stiff ordinary differential equations (ODEs). The methods were very efficient when implemented in block form, they tend to perform better over existing methods.

Keywords: Stiffness, Hermite Polynomial, ETRs, A-Stability, Ordinary Differential Equations

1. Introduction

A very important special class of differential equations taken up in the initial value problems termed as stiff differential equations result from the phenomenon with widely differing time scales [3, 7]. There is no universally acceptable definition of stiffness. Stiffness is a subtle, difficult and important concept in the numerical solution of ordinary differential equations. It depends on the differential equation, the initial condition and the interval under consideration. The initial value problems with stiff ordinary differential equation systems occur in many field of engineering science, particularly in the studies of electrical circuits, vibrations, chemical reactions and so on. Stiff differential equations are ubiquitous in astrochemical kinetics, many non-industrial areas like weather prediction and biology.

A set of differential equations is 'stiff' when an excessively small step is needed to obtain correct integration. In other words we can say a set of differential equations is 'stiff' when it contains at least two 'time constants' (where time is supposed to be the joint independent variable) that

differ by several orders of magnitude. A more rigorous definition of stiffness was also given by Shampine and Gear: "By a stiff problem we mean one for which no solution component is unstable (no Eigen-value of the Jacobian matrix has a real part which is at all large and positive) and at least some component is very stable (at least one Eigen-value has a real part which is large and negative). Further, we will not call a problem stiff unless its solution is slowly varying with respect to most negative part of the Eigen-values. Consequently a problem may be stiff for some intervals and not for others.

We seek to propose a new numerical integrator for the solution of first-order ordinary differential equation of the form:

$$y'(x) = f(x, y), \quad y(a) = y_0, \quad I = [x_0, x_N] \quad (1)$$

where, f is continuous within the interval of integration, we assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1).

A lot of work has been conducted by many scholaron finding the solution of (1). They have adopted several methods using different polynomials as basis functions, some

of these scholars include; [6, 7, 3, 4, 1, 2, 9, 10].

2. Derivation of the Method

The approach in this paper basically entails substituting into (1) a trail solution of the form

$$y : [x_0, x_N] \rightarrow R^m \tag{2}$$

where $f : [x_0, x_N] \times R^m \rightarrow R^m$ and $I_j = a + jh$, are the interpolation and collocation points, $j = 0, \dots, N-1$, is the Hermite polynomial generated by the formula:

$$h = \frac{b-a}{N-1} \tag{3}$$

For the sake of reporting, we present some few terms of the Hermite polynomial as

$$k\text{-step}, \sum_{j=0}^k \alpha_j y_{n+j} = h \left[\sum_{j=0}^k \beta_j(x) f_{n+j} \right], y_{n+j}, f_{n+j}, y(x_{n+j}), f(x_{n+j}, y(x_{n+j}))$$

From (2)

$$e^{-\lambda t} \tag{4}$$

Substituting (4) into (1) we obtained

$$\lambda \tag{5}$$

2.1. Derivation of the Block ETRs Method

2.1.1. Fourth Order Block ETRs (FETR)

Interpolating (2) at $Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$ and collocating (5) at $f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]})$ gives a system of equation which can be put in the form

$$Y_1^{[n-1]}, Y_2^{[n-1]}, \dots, Y_r^{[n-1]} \tag{6}$$

Solving (6) for the $Y_1^{[n]}, Y_2^{[n]}, \dots, Y_r^{[n]}$ yields

$$Y^{[n]}, f(Y^{[n]}),$$

$$Y^{[n-1]}$$

$$Y^{[n]}$$

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix}$$

$$f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \cdot \\ \cdot \\ f(Y_s^{[n]}) \end{bmatrix} \tag{7}$$

substituting (7) into (2) for $Y^{[n-1]} = \begin{bmatrix} Y_1^{[n-1]} \\ Y_2^{[n-1]} \\ \cdot \\ \cdot \\ Y_r^{[n-1]} \end{bmatrix}$ and $Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_r^{[n]} \end{bmatrix}$ give

an equation which is evaluated at some points of interest to generate the following set of main and additional equations.

Additional Equations

$$Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$$

$$Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) Y^{[n-1]} \tag{8}$$

2.1.2. Sixth Order Block ETRs (SETR)

Interpolating (2) at

$$Y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) Y^{[n-1]}$$
 and collocating (5)

at \oplus we obtained

$$Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]} \tag{9}$$

whose solution gives the coefficients of

$$Y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]}$$
 as

$$M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}$$

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h \left[\sum_{j=k-2}^2 \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q} \right]$$

$$\alpha_j(x)$$

$$\beta_j(x)$$

$$j = 0(1) \frac{3}{2}, 2$$

$$\beta_q(x)$$

$$\alpha_q(x)$$

$$\tag{10}$$

substituting (10) into (2) for $q = \frac{a}{b}$, $\frac{2r+1}{2}, r=1,2$ and

evaluating at points of interest to generate:

Additional Equations

$$\begin{aligned} \phi_0(x) &= 1 - \frac{1225 \xi}{303 h} + \frac{8035 \xi^2}{1212 h^2} - \frac{3419 \xi^3}{606 h^3} + \frac{3187 \xi^4}{1212 h^4} - \frac{193 \xi^5}{303 h^5} + \frac{19 \xi^6}{303 h^6} \\ \phi_1(x) &= -\frac{22016 \xi^3}{101 h^3} + \frac{17268 \xi^2}{101 h^2} - \frac{3876 \xi^5}{101 h^5} + \frac{13329 \xi^4}{101 h^4} - \frac{5040 \xi}{101 h} + \frac{436 \xi^6}{101 h^6} \\ \phi_2(x) &= -\frac{4017 \xi^5}{101 h^5} + \frac{60687 \xi^2}{404 h^2} - \frac{40631 \xi^3}{202 h^3} + \frac{475 \xi^6}{101 h^6} - \frac{4365 \xi}{101 h} + \frac{52203 \xi^4}{404 h^4} \\ \phi_{\frac{3}{2}}(x) &= \frac{29440 \xi}{303 h} - \frac{99328 \xi^2}{303 h^2} + \frac{23872 \xi^5}{303 h^5} - \frac{2752 \xi^6}{303 h^6} + \frac{128704 \xi^3}{303 h^3} - \frac{79936 \xi^4}{303 h^4} \end{aligned} \tag{11}$$

2.1.3. Eighth Order Block ETRs (EETR)

Following the same procedure as in fourth and sixth order ETRs, we obtained Additional Equations

$$\begin{aligned} \psi_1(x) &= \frac{11947 \xi^4}{303 h^3} - \frac{6953 \xi^3}{101 h^2} + \frac{17704 \xi^2}{303 h} - \frac{1940 \xi}{101} + \frac{364 \xi^6}{303 h^5} - \frac{1112 \xi^5}{101 h^4} \\ \psi_2(x) &= \frac{7359 \xi^4}{202 h^3} - \frac{8167 \xi^2}{202 h} + \frac{5584 \xi^3}{101 h^2} + \frac{1155 \xi}{101} - \frac{142 \xi^6}{101 h^5} + \frac{1166 \xi^5}{101 h^4} \\ \psi_{\frac{3}{2}}(x) &= \frac{704 \xi^2}{303 h} - \frac{64 \xi}{101} + \frac{704 \xi^4}{303 h^3} + \frac{32 \xi^6}{303 h^5} - \frac{336 \xi^3}{101 h^2} - \frac{80 \xi^5}{101 h^4} \end{aligned}$$

0	0	0	0	0	0	0	0	0	1
$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$
0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$
0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$-\frac{152}{211}$	$-\frac{257}{39879}$
0	$-\frac{75}{404}$	0	$\frac{675}{808}$	$\frac{15}{101}$	0	$\frac{675}{1616}$	$\frac{225}{101}$	$-\frac{325}{404}$	$-\frac{9}{1616}$
0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$
0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$
0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$-\frac{152}{211}$	$-\frac{257}{39879}$
0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$
$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(12)

$$I = [x_0, x_N] \tag{14}$$

Expanding (14) in Taylor Series gives

$$y : [x_0, x_N] \rightarrow R^m \tag{15}$$

Following (15), we obtained the order and error constants of (8), including its main equation in (13) as $f : [x_0, x_N] \times R^m \rightarrow R^m$ and $I_j = a + jh$, respectively.

3.1.2. Consider Equation (2.6)

Applying (5) on (11), including its main equation in (13), yields

$$j = 0, \dots, N-1, \tag{16}$$

Expanding (16) in Taylor Series gives

$$h = \frac{b-a}{N-1}$$

k - step

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \left[\sum_{j=0}^k \beta_j(x) f_{n+j} \right]$$

$$y_{n+j}$$

$$f_{n+j}$$

$$y(x_{n+j}) \tag{17}$$

Following (17), we obtained a uniform order $f(x_{n+j}, y(x_{n+j}))$ for (11) and its main equation in (13), whose error constants were calculated as $e^{-\lambda t}$.

3.1.3. Consider Equation (2.7)

Application of (5) on (12), including its main equation in (13) yields

$$\lambda Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]} \tag{18}$$

Expanding (18) in Taylor Series gives

$$f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]})$$

$$Y_1^{[n-1]}, Y_2^{[n-1]}, \dots, Y_r^{[n-1]}$$

$$Y_1^{[n]}, Y_2^{[n]}, \dots, Y_r^{[n]}$$

$$Y^{[n]}, f(Y^{[n]}),$$

2.2. Main Equation for FETR, SETR and EETR Respectively

$$B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$$

$$y'(x) = f(x, y), \tag{13}$$

The class of methods we have developed shall each be implemented in block form to generate approximate solutions $y(a) = y_0$ simultaneously without the need for starters.

3. Analysis of Basic Properties of the Newly Derived Block Methods

3.1. Order of the Block ETRs Method

3.1.1. Consider Equation (8) and Its Main Equation in (13)

Applying (5), we obtained

$$\begin{aligned}
 & y^{[n-1]} \\
 & y^{[n]} \\
 Y^{[n]} &= \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix} \\
 f(Y^{[n]}) &= \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \cdot \\ \cdot \\ f(Y_s^{[n]}) \end{bmatrix} \tag{19}
 \end{aligned}$$

From (19), we get the order of equation (12) and main

equation (13) as $y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix}$ and error constants as

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \cdot \\ \cdot \\ y_r^{[n]} \end{bmatrix}$$

3.2. Zero Stability of the Main Method(s)

3.2.1. Zero Stability of Main Method for SETR

Consider the characteristic equation associated with the main discrete scheme of (11) given by

$$Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]} Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) y^{[n-1]} \tag{20}$$

where $y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]}$ is the eigenvalue(s) of the Jacobian of (1). \oplus and $Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]}$ are the first and second characteristic

polynomials of the scheme in (11) respectively and which is given by

$$y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]} \tag{21}$$

$$M = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \tag{22}$$

Solving (21), we obtained

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h \left[\sum_{j=k-2}^2 \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q} \right] \tag{23}$$

Zero-Stability property requires that the roots of (23) must satisfy $\alpha_j(x)$ and every root with $\beta_j(x)$ must have multiplicity 1.

3.2.2. Zero Stability of Main Method for EETR

Similarly, the first and second characteristic polynomials of the main scheme in (12) is given by

$$j = 0(1) \frac{3}{2}, 2 \tag{24}$$

$$\beta_q(x) \tag{25}$$

From (24), it can easily be shown that the main scheme of the eighth order ETRs is zero stable.

3.3. Zero Stability of the Block Method(s)

Definition 1: The block method SETR is said to be zero-stable provided the roots $\alpha_q(x)$ of the first characteristics polynomial $q = \frac{a}{b}$ specified by

$$\frac{2r+1}{2}, r = 1, 2 \tag{26}$$

satisfies $\phi_0(x) = 1 - \frac{1225}{303} \frac{\xi}{h} + \frac{8035}{1212} \frac{\xi^2}{h^2} - \frac{3419}{606} \frac{\xi^3}{h^3} + \frac{3187}{1212} \frac{\xi^4}{h^4} - \frac{193}{303} \frac{\xi^5}{h^5} + \frac{19}{303} \frac{\xi^6}{h^6}$ and for those roots with $\phi_1(x) = -\frac{22016}{101} \frac{\xi^3}{h^3} + \frac{17268}{101} \frac{\xi^2}{h^2} - \frac{3876}{101} \frac{\xi^5}{h^5} + \frac{13329}{101} \frac{\xi^4}{h^4} - \frac{5040}{101} \frac{\xi}{h} + \frac{436}{101} \frac{\xi^6}{h^6}$ the multiplicity does not exceed two.

Consistency and Convergence: Since each of the block methods (8), (11), (12) and their individual main equations in (13) has order

$$\phi_2(x) = -\frac{4017}{101} \frac{\xi^5}{h^5} + \frac{60687}{404} \frac{\xi^2}{h^2} - \frac{40631}{202} \frac{\xi^3}{h^3} + \frac{475}{101} \frac{\xi^6}{h^6} - \frac{4365}{101} \frac{\xi}{h} + \frac{52203}{404} \frac{\xi^4}{h^4}$$

they are all consistent and by Henrici (1962). Convergence= Consistency and zero stability.

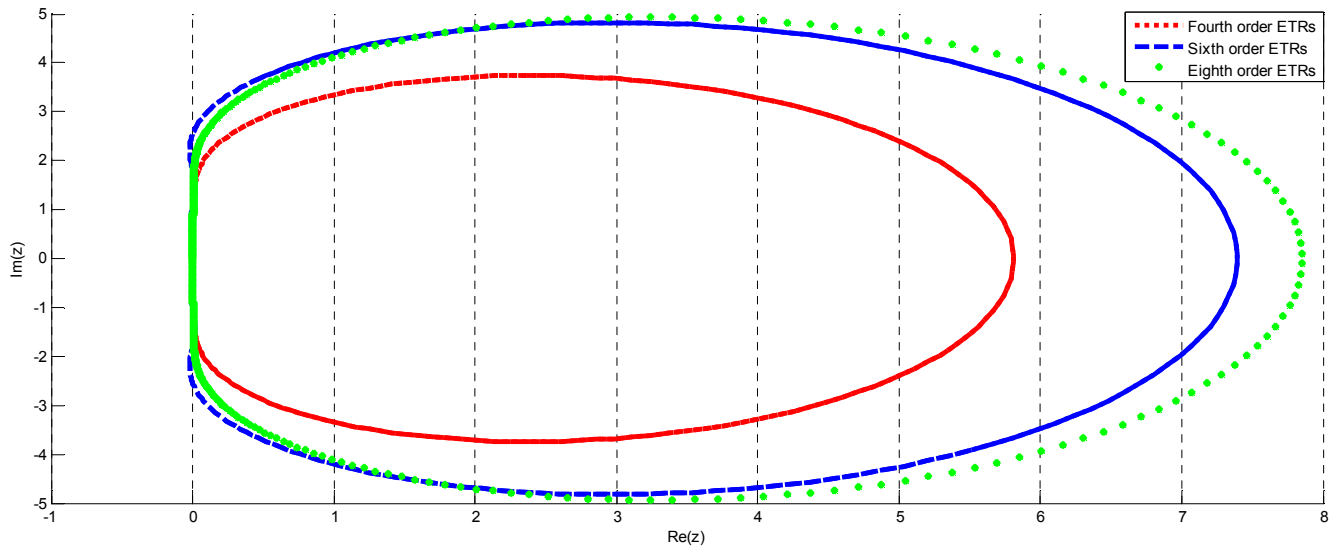


Figure 1. Stability Region for the Fourth, Sixth and Eighth Order Block ETRs Method.

4. Numerical Implementation of FETR, SETR

Problem 1: consider a highly stiff ODE of the form

$$\phi_{\frac{3}{2}}(x) = \frac{29440}{303} \frac{\xi}{h} - \frac{99328}{303} \frac{\xi^2}{h^2} + \frac{23872}{303} \frac{\xi^5}{h^5} - \frac{2752}{303} \frac{\xi^6}{h^6} + \frac{128704}{303} \frac{\xi^3}{h^3} - \frac{79936}{303} \frac{\xi^4}{h^4},$$

$$\psi_1(x) = \frac{11947}{303} \frac{\xi^4}{h^3} - \frac{6953}{101} \frac{\xi^3}{h^2} + \frac{17704}{303} \frac{\xi^2}{h} - \frac{1940}{101} \xi + \frac{364}{303} \frac{\xi^6}{h^5} - \frac{1112}{101} \frac{\xi^5}{h^4}$$

$$\psi_2(x) = \frac{7359}{202} \frac{\xi^4}{h^3} - \frac{8167}{202} \frac{\xi^2}{h} + \frac{5584}{101} \frac{\xi^3}{h^2} + \frac{1155}{101} \xi - \frac{142}{101} \frac{\xi^6}{h^5} + \frac{1166}{101} \frac{\xi^5}{h^4},$$

$$\psi_{\frac{3}{2}}(x) = \frac{704}{303} \frac{\xi^2}{h} - \frac{64}{101} \xi + \frac{704}{303} \frac{\xi^4}{h^3} + \frac{32}{303} \frac{\xi^6}{h^5} - \frac{336}{101} \frac{\xi^3}{h^2} - \frac{80}{101} \frac{\xi^5}{h^4},$$

0	0	0	0	0	0	0	0	0	1
$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$
0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$
0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$-\frac{152}{211}$	$-\frac{257}{39879}$
0	$-\frac{75}{404}$	0	$\frac{675}{808}$	$\frac{15}{101}$	0	$-\frac{675}{1616}$	$\frac{225}{101}$	$-\frac{325}{404}$	$-\frac{9}{1616}$
0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$
0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$
0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$-\frac{152}{211}$	$-\frac{257}{39879}$
0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$
$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$

$$\text{Exact Solution } I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Table 1. Maximum Errors for Example 1.

X	Skwame et al., 2012 th Order Method
0.1	$3.12 A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
0.2	2.49 $I_j = a + jh, j = 0, \dots, N - 1,$
0.3	7.81 $f(x_{n+j}, y(x_{n+j})) e^{-\lambda t}$
0.4	6.24 $y^{[n-1]} y^{[n]}$
0.5	1.95 $y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]} \oplus$
0.6	1.56 $j = 0(1) \frac{3}{2}, 2 \beta_q(x)$
0.7	$4.87 \phi_{\frac{3}{2}}(x) = \frac{29440}{303} \frac{\xi}{h} - \frac{99328}{303} \frac{\xi^2}{h^2} + \frac{23872}{303} \frac{\xi^5}{h^5} - \frac{2752}{303} \frac{\xi^6}{h^6} + \frac{128704}{303} \frac{\xi^3}{h^3} - \frac{79936}{303} \frac{\xi^4}{h^4}$ $\psi_1(x) = \frac{11947}{303} \frac{\xi^4}{h^3} - \frac{6953}{101} \frac{\xi^3}{h^2} + \frac{17704}{303} \frac{\xi^2}{h} - \frac{1940}{101} \xi + \frac{364}{303} \frac{\xi^6}{h^5} - \frac{1112}{101} \frac{\xi^5}{h^4}$
0.8	$3.90 B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix} Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]}$
0.9	1.22 $f : [x_0, x_N] \times R^m \rightarrow R^m I_j = a + jh,$
1.0	9.76 $y(x_{n+j}) f(x_{n+j}, y(x_{n+j}))$
1.1	2.05 $Y^{[n]}, f(Y^{[n]}), y^{[n-1]}$
1.2	2.44 $Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) y^{[n-1]} y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]}$
1.3	7.62 $\alpha_j(x) \beta_j(x)$
1.4	$6.10 \phi_1(x) = -\frac{22016}{101} \frac{\xi^3}{h^3} + \frac{17268}{101} \frac{\xi^2}{h^2} - \frac{3876}{101} \frac{\xi^5}{h^5} + \frac{13329}{101} \frac{\xi^4}{h^4} - \frac{5040}{101} \frac{\xi}{h} + \frac{436}{101} \frac{\xi^6}{h^6}$

X		Skwame et al., 2012 ^{4th} Order Method				
		$\phi_2(x) = -\frac{4017}{101} \frac{\xi^5}{h^5} + \frac{60687}{404} \frac{\xi^2}{h^2} - \frac{40631}{202} \frac{\xi^3}{h^3} + \frac{475}{101} \frac{\xi^6}{h^6} - \frac{4365}{101} \frac{\xi}{h} + \frac{52203}{404} \frac{\xi^4}{h^4}$				
1.5	1.90	$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$			
1.6	1.52	$f : [x_0, x_N] \times R^m \rightarrow R^m \quad I_j = a + jh,$				
1.7	4.76	$y(x_{n+j}) = f(x_{n+j}, y(x_{n+j}))$				
1.8	3.81	$Y^{[n]}, f(Y^{[n]}), y^{[n-1]}$				
1.9	1.19	$Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) y^{[n-1]} \quad y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]}$				
2.0	9.52	$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h \left[\sum_{j=k-2}^2 \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q} \right] \alpha_j(x)$				

X		Skwame et al., 2012 6th Order Method				
0.1	5.00	$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$	$y'(x) = f(x, y),$			
0.2	1.67	$h = \frac{b-a}{N-1} \quad k\text{-step}$				
0.3	5.00	$\lambda \quad Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$				
0.4	3.00	$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix}$	$f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \cdot \\ \cdot \\ f(Y_s^{[n]}) \end{bmatrix}$			
0.5	1.50	$Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]} \quad y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]}$				
0.6	5.00	$\alpha_q(x) \quad q = \frac{a}{b}$				

X	Skwame et al., 2012 6th Order Method	
0.7	$1.50 \psi_2(x) = \frac{7359 \xi^4}{202 h^3} - \frac{8167 \xi^2}{202 h} + \frac{5584 \xi^3}{101 h^2} + \frac{1155 \xi}{101} - \frac{142 \xi^6}{101 h^5} + \frac{1166 \xi^5}{101 h^4}$	$\psi_{\frac{3}{2}}(x) = \frac{704 \xi^2}{303 h} - \frac{64}{101} \xi + \frac{704 \xi^4}{303 h^3} + \frac{32 \xi^6}{303 h^5} - \frac{336 \xi^3}{101 h^2} - \frac{80 \xi^5}{101 h^4}$
0.8	$9.00 Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]} \quad M(z) = V + zB(I - zA)^{-1}U$	
0.9	$4.50 j = 0, \dots, N-1, \quad h = \frac{b-a}{N-1}$	
1.0	$1.50 e^{-\lambda t} \quad \lambda$	
1.1	$4.50 y^{[n]} \quad Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix}$	
1.2	$2.70 \oplus Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]}$	
1.3	$1.35 j = 0(1)\frac{3}{2}, 2 \quad \beta_q(x)$	
1.4	$4.50 \phi_{\frac{3}{2}}(x) = \frac{29440 \xi}{303 h} - \frac{99328 \xi^2}{303 h^2} + \frac{23872 \xi^5}{303 h^5} - \frac{2752 \xi^6}{303 h^6} + \frac{128704 \xi^3}{303 h^3} - \frac{79936 \xi^4}{303 h^4}$	
1.5	$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$	$\Phi(w, z) = \det(wI - M(z))$
1.6	$8.10 j = 0, \dots, N-1, \quad h = \frac{b-a}{N-1}$	
1.7	$4.05 e^{-\lambda t} \quad \lambda$	
1.8	$1.35 y^{[n]} \quad Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix}$	
1.9	$4.05 \oplus Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]}$	
2.0	$2.43 \beta_j(x) \quad j = 0(1)\frac{3}{2}, 2$	

X	FETR	
0.1	$1.5576 y(a) = y_0 \quad I = [x_0, x_N]$	
0.2	$1.7321 \sum_{j=0}^k \alpha_j y_{n+j} = h \left[\sum_{j=0}^k \beta_j(x) f_{n+j} \right] y_{n+j}$	

X	FETR																																																																																																				
0.3	$1.1797 f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]}) Y_1^{[n-1]}, Y_2^{[n-1]}, \dots, Y_r^{[n-1]}$																																																																																																				
0.4	$3.8528 y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix} \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \cdot \\ \cdot \\ y_r^{[n]} \end{bmatrix}$																																																																																																				
0.5	$4.1553 M = \begin{bmatrix} A & U \\ B & V \end{bmatrix} y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h \left[\sum_{j=k-2}^2 \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q} \right]$																																																																																																				
0.6	$2.1700 \frac{2r+1}{2}, r=1,2 \phi_0(x) = 1 - \frac{1225 \xi}{303 h} + \frac{8035 \xi^2}{1212 h^2} - \frac{3419 \xi^3}{606 h^3} + \frac{3187 \xi^4}{1212 h^4} - \frac{193 \xi^5}{303 h^5} + \frac{19 \xi^6}{303 h^6}$																																																																																																				
0.7	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td> </tr> <tr> <td>$-\frac{101}{5040}$</td><td>$-\frac{388}{1008}$</td><td>0</td><td>$\frac{231}{1008}$</td><td>$-\frac{4}{315}$</td><td>0</td><td>$-\frac{97}{112}$</td><td>$\frac{5888}{3024}$</td><td>0</td><td>$-\frac{35}{432}$</td> </tr> <tr> <td>0</td><td>$-\frac{159}{112}$</td><td>$-\frac{303}{56}$</td><td>$-\frac{387}{244}$</td><td>$\frac{3}{56}$</td><td>0</td><td>$\frac{4023}{448}$</td><td>0</td><td>$-\frac{891}{112}$</td><td>$-\frac{11}{448}$</td> </tr> <tr> <td>0</td><td>$-\frac{113}{633}$</td><td>0</td><td>$\frac{530}{1477}$</td><td>$-\frac{256}{4431}$</td><td>$\frac{101}{13293}$</td><td>0</td><td>$\frac{68864}{39879}$</td><td>$\frac{152}{211}$</td><td>$-\frac{257}{39879}$</td> </tr> <tr> <td>0</td><td>$-\frac{75}{404}$</td><td>0</td><td>$\frac{675}{808}$</td><td>$\frac{15}{101}$</td><td>0</td><td>$-\frac{675}{1616}$</td><td>$\frac{225}{101}$</td><td>$-\frac{325}{404}$</td><td>$-\frac{9}{1616}$</td> </tr> <tr> <td>0</td><td>$\frac{366}{101}$</td><td>0</td><td>$-\frac{738}{101}$</td><td>$\frac{192}{101}$</td><td>0</td><td>$\frac{2187}{101}$</td><td>$-\frac{3584}{101}$</td><td>$\frac{1485}{101}$</td><td>$\frac{13}{101}$</td> </tr> </table> <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>0</td><td>$\frac{366}{101}$</td><td>0</td><td>$-\frac{738}{101}$</td><td>$\frac{192}{101}$</td><td>0</td><td>$\frac{2187}{101}$</td><td>$-\frac{3584}{101}$</td><td>$\frac{1485}{101}$</td><td>$\frac{13}{101}$</td> </tr> <tr> <td>0</td><td>$-\frac{113}{633}$</td><td>0</td><td>$\frac{530}{1477}$</td><td>$-\frac{256}{4431}$</td><td>$\frac{101}{13293}$</td><td>0</td><td>$\frac{68864}{39879}$</td><td>$\frac{152}{211}$</td><td>$-\frac{257}{39879}$</td> </tr> <tr> <td>0</td><td>$-\frac{159}{112}$</td><td>$-\frac{303}{56}$</td><td>$-\frac{387}{244}$</td><td>$\frac{3}{56}$</td><td>0</td><td>$\frac{4023}{448}$</td><td>0</td><td>$-\frac{891}{112}$</td><td>$-\frac{11}{448}$</td> </tr> <tr> <td>$-\frac{101}{5040}$</td><td>$-\frac{388}{1008}$</td><td>0</td><td>$\frac{231}{1008}$</td><td>$-\frac{4}{315}$</td><td>0</td><td>$-\frac{97}{112}$</td><td>$\frac{5888}{3024}$</td><td>0</td><td>$-\frac{35}{432}$</td> </tr> </table> $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	0	0	0	0	0	0	0	0	0	1	$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$	0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$	0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$\frac{152}{211}$	$-\frac{257}{39879}$	0	$-\frac{75}{404}$	0	$\frac{675}{808}$	$\frac{15}{101}$	0	$-\frac{675}{1616}$	$\frac{225}{101}$	$-\frac{325}{404}$	$-\frac{9}{1616}$	0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$	0	$\frac{366}{101}$	0	$-\frac{738}{101}$	$\frac{192}{101}$	0	$\frac{2187}{101}$	$-\frac{3584}{101}$	$\frac{1485}{101}$	$\frac{13}{101}$	0	$-\frac{113}{633}$	0	$\frac{530}{1477}$	$-\frac{256}{4431}$	$\frac{101}{13293}$	0	$\frac{68864}{39879}$	$\frac{152}{211}$	$-\frac{257}{39879}$	0	$-\frac{159}{112}$	$-\frac{303}{56}$	$-\frac{387}{244}$	$\frac{3}{56}$	0	$\frac{4023}{448}$	0	$-\frac{891}{112}$	$-\frac{11}{448}$	$-\frac{101}{5040}$	$-\frac{388}{1008}$	0	$\frac{231}{1008}$	$-\frac{4}{315}$	0	$-\frac{97}{112}$	$\frac{5888}{3024}$	0	$-\frac{35}{432}$
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0.8	$5.1155 y'(x) = f(x, y), \quad y(a) = y_0$																																																																																																				
0.9	$4.4037 k - step \sum_{j=0}^k \alpha_j y_{n+j} = h \left[\sum_{j=0}^k \beta_j(x) f_{n+j} \right]$																																																																																																				
1.0	$2.1834 Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]} f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]})$																																																																																																				
1.1	$4.6956 f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \cdot \\ \cdot \\ f(Y_s^{[n]}) \end{bmatrix} \quad y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix}$																																																																																																				

X	FETR
1.2	$1.5467 y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]} \quad M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}$
1.3	$6.8444 \alpha_q(x) \quad q = \frac{a}{b}$
1.4	$1.6501 \psi_2(x) = \frac{7359 \xi^4}{202 h^3} - \frac{8167 \xi^2}{202 h} + \frac{5584 \xi^3}{101 h^2} + \frac{1155 \xi}{101} - \frac{142 \xi^6}{101 h^5} + \frac{1166 \xi^5}{101 h^4} \quad \psi_{\frac{3}{2}}(x) = \frac{704 \xi^2}{303 h} - \frac{64 \xi}{101} + \frac{704 \xi^4}{303 h^3} + \frac{32 \xi^6}{303 h^5} - \frac{336 \xi^3}{101 h^2} - \frac{80 \xi^5}{101 h^4}$
1.5	$1.7794 y'(x) = f(x, y), \quad y(a) = y_0$
1.6	$2.0002 k\text{-step} \sum_{j=0}^k \alpha_j y_{n+j} = h \left[\sum_{j=0}^k \beta_j(x) f_{n+j} \right]$
1.7	$1.4414 Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]} \quad f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]})$
1.8	$8.5234 f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \cdot \\ \cdot \\ f(Y_s^{[n]}) \end{bmatrix} \quad y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix}$
1.9	$2.6913 y^{[n]} = hBf(Y^{[n]}) + Vy^{[n-1]} \quad M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}$
2.0	$5.8666 \beta_q(x) \quad \alpha_q(x)$

X	SETR
0.1	$8.9860 y: [x_0, x_N] \rightarrow R^m \quad f: [x_0, x_N] \times R^m \rightarrow R^m$
0.2	$3.5800 f_{n+j} \quad y(x_{n+j})$
0.3	$2.2793 Y_1^{[n]}, Y_2^{[n]}, \dots, Y_r^{[n]} \quad Y^{[n]}, f(Y^{[n]}),$
0.4	$2.3103 Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]} \quad Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) y^{[n-1]}$
0.5	$9.6111 \alpha_j(x) \quad \beta_j(x)$
0.6	$3.5877 \phi(x) = -\frac{22016 \xi^3}{101 h^3} + \frac{17268 \xi^2}{101 h^2} - \frac{3876 \xi^5}{101 h^5} + \frac{13329 \xi^4}{101 h^4} - \frac{5040 \xi}{101 h} + \frac{436 \xi^6}{101 h^6} \quad \phi_2(x) = -\frac{4017 \xi^5}{101 h^5} + \frac{60687 \xi^2}{404 h^2} - \frac{40631 \xi^3}{202 h^3} + \frac{475 \xi^6}{101 h^6} - \frac{4365 \xi}{101 h} + \frac{52203 \xi^4}{404 h^4}$
0.7	$1.3030 A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B_i = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
0.8	$4.9168 I = [x_0, x_N] \quad y: [x_0, x_N] \rightarrow R^m$
0.9	$1.6269 y_{n+j} \quad f_{n+j}$
1.0	$1.2028 Y_1^{[n-1]}, Y_2^{[n-1]}, \dots, Y_r^{[n-1]} \quad Y_1^{[n]}, Y_2^{[n]}, \dots, Y_r^{[n]}$

X SETR

$$1.1 \quad 4.4549 \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} \quad Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$$

$$1.2 \quad 1.6290 \quad M(z) = V + zB(I - zA)^{-1}U \quad y(x) = \sum_{j=0}^{k-1} \alpha_j(x)y_{n+j} + \alpha_q(x)y_{n+q} + h \left[\sum_{j=k-2}^2 \beta_j(x)f_{n+j} + \beta_q(x)f_{n+q} \right]$$

$$1.3 \quad 6.0645 \quad \frac{2r+1}{2}, r=1,2 \quad \phi_0(x) = 1 - \frac{1225}{303} \frac{\xi}{h} + \frac{8035}{1212} \frac{\xi^2}{h^2} - \frac{3419}{606} \frac{\xi^3}{h^3} + \frac{3187}{1212} \frac{\xi^4}{h^4} - \frac{193}{303} \frac{\xi^5}{h^5} + \frac{19}{303} \frac{\xi^6}{h^6}$$

$$1.4 \quad 2.1259 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & \frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\ 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$1.5 \quad 1.1312 \quad I = [x_0, x_N] \quad y: [x_0, x_N] \rightarrow R^m$$

$$1.6 \quad 4.1787 \quad y_{n+j} \quad f_{n+j}$$

$$1.7 \quad 1.5316 \quad Y_1^{[n-1]}, Y_2^{[n-1]}, \dots, Y_r^{[n-1]} \quad Y_1^{[n]}, Y_2^{[n]}, \dots, Y_r^{[n]}$$

$$1.8 \quad 5.6758 \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix} \quad Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$$

$$1.9 \quad 2.0273 \quad M(z) = V + zB(I - zA)^{-1}U \quad \Phi(w, z) = \det(wI - M(z)).$$

$$2.0 \quad 9.4748 \quad q = \frac{a}{b} \frac{2r+1}{2}, r=1,2$$

Problem 2

$$\begin{aligned} \phi_0(x) &= 1 - \frac{1225 \xi}{303 h} + \frac{8035 \xi^2}{1212 h^2} - \frac{3419 \xi^3}{606 h^3} + \frac{3187 \xi^4}{1212 h^4} - \frac{193 \xi^5}{303 h^5} + \frac{19 \xi^6}{303 h^6}, \\ \phi_1(x) &= -\frac{22016 \xi^3}{101 h^3} + \frac{17268 \xi^2}{101 h^2} - \frac{3876 \xi^5}{101 h^5} + \frac{13329 \xi^4}{101 h^4} - \frac{5040 \xi}{101 h} + \frac{436 \xi^6}{101 h^6}, \\ \phi_2(x) &= -\frac{4017 \xi^5}{101 h^5} + \frac{60687 \xi^2}{404 h^2} - \frac{40631 \xi^3}{202 h^3} + \frac{475 \xi^6}{101 h^6} - \frac{4365 \xi}{101 h} + \frac{52203 \xi^4}{404 h^4}, \\ \phi_{\frac{3}{2}}(x) &= \frac{29440 \xi}{303 h} - \frac{99328 \xi^2}{303 h^2} + \frac{23872 \xi^5}{303 h^5} - \frac{2752 \xi^6}{303 h^6} + \frac{128704 \xi^3}{303 h^3} - \frac{79936 \xi^4}{303 h^4} \end{aligned}$$

Exact Solution $\psi_1(x) = \frac{11947 \xi^4}{303 h^3} - \frac{6953 \xi^3}{101 h^2} + \frac{17704 \xi^2}{303 h} - \frac{1940 \xi}{101} + \frac{364 \xi^6}{303 h^5} - \frac{1112 \xi^5}{101 h^4}$

Table 2. Maximum Errors for Example 1.

X	Exact Solution	Khadijah 2015 4 th Order Method	FETR	Absolute Error
				1.5657
0.2	0.019801326693245	0.818738	0.0198014832636218	$\psi_2(x) = \frac{7359 \xi^4}{202 h^3} - \frac{8167 \xi^2}{202 h} + \frac{5584 \xi^3}{101 h^2} + \frac{1155 \xi}{101} - \frac{142 \xi^6}{101 h^5} + \frac{1166 \xi^5}{101 h^4}$ $\psi_{\frac{3}{2}}(x) = \frac{704 \xi^2}{303 h} - \frac{64 \xi}{101} + \frac{704 \xi^4}{303 h^3} + \frac{32 \xi^6}{303 h^5} - \frac{336 \xi^3}{101 h^2} - \frac{80 \xi^5}{101 h^4}$
				2.0807
				$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{101}{5040} & \frac{388}{1008} & 0 & \frac{231}{1008} & \frac{4}{315} & 0 & \frac{97}{112} & \frac{5888}{3024} & 0 & \frac{35}{432} \\ 0 & \frac{159}{112} & \frac{303}{56} & \frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & \frac{891}{112} & \frac{11}{448} \\ 0 & \frac{113}{633} & 0 & \frac{530}{1477} & \frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & \frac{152}{211} & \frac{257}{39879} \\ 0 & \frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & \frac{675}{1616} & \frac{225}{101} & \frac{325}{404} & \frac{9}{1616} \\ 0 & \frac{366}{101} & 0 & \frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & \frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \end{bmatrix}$
0.4	0.076883653613364	0.670321	0.0768857343339293	$\begin{bmatrix} 0 & \frac{366}{101} & 0 & \frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & \frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ 0 & \frac{113}{633} & 0 & \frac{530}{1477} & \frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & \frac{152}{211} & \frac{257}{39879} \\ 0 & \frac{159}{112} & \frac{303}{56} & \frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & \frac{891}{112} & \frac{11}{448} \\ \frac{101}{5040} & \frac{388}{1008} & 0 & \frac{231}{1008} & \frac{4}{315} & 0 & \frac{97}{112} & \frac{5888}{3024} & 0 & \frac{35}{432} \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

X	Exact Solution	Khadijah 2015 4 th Order Method	FETR	Absolute Error
0.6	0.164729788588728	0.545813	0.164732324486997	$1.2061 A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
0.8	0.273850962926309	0.449325	0.273853652953028	$2.6900 B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$ $\Phi(w, z) = \det(wI - M(z)).$
1.0	0.393469340287367	0.367873	0.393473662912352	$4.3226 \Phi(w, z) = \det(wI - M(z)) \quad \Phi(w, z) = \det(wI - M(z))$

5. Conclusion

We derived block Extended Trapezoidal Rule of First (ETRs) that are A-stable of up to order 8. The methods were shown to compete favorably than other existing methods in terms of accuracy (see tables 1 and 2). Our newly derived methods in block form are shown to have extensive regions of stability and in particular are A-stable up to order 8 and so very suitable for stiff system of ordinary differential equations. The continuous formulation for each step number k is evaluated at the end point of the interval to recover the individual main method for FETR, SETR and EETR in (13) and their counterparts schemes in (8), (11) and (12) respectively, to this end, the idea of additional conditions is discarded. Furthermore, we do not need any pair in our implementation. Our block methods preserve the A-stability property of the trapezoidal rule (refer to figure 1), they are also less expensive in terms of the number of functions evaluation per step.

References

[1] Awari, Y. S. (2017). A-Stable Block ETRs/ETR2s Methods for Stiff ODEs, MAYFEB Journal of Mathematics Vol. 4: Pages 33-52.

[2] Awari, Y. S., Garba, E. J. D., Kumleng, M. G. and Tsaku, N. (2016). Efficient implementation of Some Block Extended Trapezoidal rule of first kind (ETRs) class of method Applied to First Order Initial Value Problems of Differential Equations, Global Journal of Mathematics, Vol. 7 (2).

[3] Biala, T. A., Jator, S. N., Adeniyi, R. B. and Ndikum, P. L. (2015). Block Hybrid Simpson’s Method with Two Off-grid Points for Stiff Systems International Journal of Nonlinear Science Vol. 20 (1), pp. 3-10.

[4] Happy K., Shelly, Arora and Nagaich, R. K. (2013). Solution of Non Linear Singular Perturbation Equation Using Hermite Collocation Method, Applied Mathematical Sciences, Vol. 7, 2013, no. 109, 5397–5408.

[5] Khadijah, M. Abualnaja, (2015). A Block Procedure with Linear Multistep Methods Using Legendre Polynomials for Solving ODEs, Applied Mathematics 6:717-723.

[6] Miletics, E. and Moln’arka, G. (2009). Taylor Series Method with Numerical Derivatives for Numerical Solution of ODE Initial Value Problems, HU ISSN 1418-7108: HEJ Manuscript no.: ANM-030110-B.

[7] Okuonghae, R. I. and Ikhile, M. N. O. (2011). A(α)-Stable Linear Multistep Methods. for Stiff IVPs in ODEs Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 50 (1), 73–90.

[8] Skwame, Y., Sunday, J., Ibijola, E. A. (2012). L-Stable Block Hybrid Simpson’s Methods for Numerical Solution of Initial Value Problems in Stiff Ordinary Differential Equations, International Journal of Pure and Applied Sciences and Technology 11(2):45-54.

[9] Tahmasbi, A. (2008). Numerical Solutions for Stiff Ordinary Differential Equation Systems, International Mathematical Forum, 3, 2008, no. 15, 703 – 711.

[10] Yao, N. M., Akinfenwa, O. A., Jator, S. N. (2011). A linear multistep hybrid Method with continuous coefficients for solving stiff ordinary differential equations, Int. J. Comp. Maths. 5(2): 47-53.