On Generalized Interval Valued Fuzzy Soft Matrices

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Abstract: Interval valued fuzzy soft set was a combination of the interval valued fuzzy set and soft set, while in generalized interval valued fuzzy soft set a degree was attached with the parameterization of fuzzy sets in defining an interval valued fuzzy soft set. In this paper we introduced the concept of generalized interval valued fuzzy soft matrices. We discussed some of its types and some operations. We also discussed about the similarity of two generalized interval valued fuzzy soft matrices.

Keywords: Interval Valued Fuzzy Soft Set (IVFSS), Generalized Fuzzy Soft Set, Generalized IVFSS, Similarity

1. Introduction

A lot of problems in our real life in economics, social sciences, medical sciences, environmental sciences and engineering etc. involve various uncertainties. Many theories have been developed to deal with these uncertainties. Some of these theories are probability theory [1], fuzzy set theory (FST) [2], rough set theory (RST) [3], interval mathematics [4] and intuitionistic fuzzy set theory (IFST) [5] etc. Molodtsov [6] pointed out that all these theories have some inherent difficulties. He proposed soft set theory (SST) to overcome these difficulties. It was a generic mathematical tool for dealing problems having uncertainty. Later Maji and Biswas [7] defined soft subset and soft super set. They also defined absolute soft set and null soft set. They introduced some operations on soft set and De Morgan’s laws are also verified by them. Ali et al [8] pointed out some errors of the previous work and introduced some new operations. They further studied more and discussed some algebraic structures of soft sets. Maji et al. [9] proposed fuzzy soft set (FSS), an improvement of the SST by combining (FST) and (SST). Roy and Maji [10] gave an application of fuzzy soft set in decision making. Yang et al. [11] introduced the interval-valued fuzzy soft set (IVFSS) which was a combination of the IVFS and SST.

Majumdar and Samanta [12] introduced the concept of generalized fuzzy soft sets (GFSS). B. K. Saikia et al. [13] defined generalized fuzzy soft matrix (GFSM) and applied it to a decision making (DM) problem. Shawkat Alkhazaleh and Abdul Razak Salleh [14] introduced generalized interval valued fuzzy soft set (GIVFSS). In their generalization of FSS, they attached a degree with the parameterization of fuzzy sets in defining an IVFSS. They discussed various operations and properties of GIVFSS. Some of these are GIVFS subset, GIVFS equal set, generalized null interval valued fuzzy soft set (GNIVFS), generalized absolute interval valued fuzzy soft set (GAIVFS), compliment of GIVFSS, union of GIVFSS’s and intersection of GIVFSS’s. They defined AND and OR operations on GIVFSS and similarity measure of two GIVFSS’s. They also give some applications of GIVFSS in DM problem and medical diagnosis.


In this paper we extended the concept of GIVFSS and introduced generalized interval valued fuzzy soft matrix (GIVFSM). We defined different types of GIVFSM’s and...
studied some properties. We also discussed some operators on the basis of weights and some of their properties.

2. Some Basic Definitions

Definition 2.1. [11] Let X be a universal set, P be set of parameters and I^P be the set of all fuzzy subsets of X. Let A ⊆ P. A pair (F, A) is a fuzzy soft set over X, where

F: A → I^P.

Definition 2.2. [11] An IVFSS X on a universe U is a mapping such that

X: U → Int([0, 1]),

where Int([0, 1]) represents the set of all closed subintervals of [0, 1], the set of all interval valued fuzzy sets on U is denoted by P(U). Suppose that X ∈ P(U), ∀x ∈ U

µ_x(x) = [µ^-x(x), µ^+x(x)]

is the degree of membership x to X, where µ^-x(x) and µ^+x(x) are the lower and upper degrees of membership of x to X respectively, such that

0 ≤ µ^-x(x) ≤ µ^+x(x) ≤ 1

Definition 2.3. [26] Let X be the universal set and P be the set of parameters. Suppose that A ⊆ P and (F, A) is a fuzzy soft set. Then the matrix form of the fuzzy soft set (F, A) is given as

A = [a_{ij}]_{m×n}, i = 1, 2, 3, ..., m and j = 1, 2, 3, ..., n

where

a_{ij} = \{(u_{ij}(p_{i}), p_{j} ∈ A

\text{if } a_{ij} ∈ A

\text{if } a_{ij} ∉ A

\}

\forall i, j.

Here u_{ij} (p_{i}) denotes the membership of p_{i} in the fuzzy soft set F(p_{i}).

Definition 2.4. [13] Let X be the universal set, P be the set of parameters and A ⊆ P. Let (F_{A}, A) be a GFSS over (X, P). A subset of X × P, R_A = \{(x, p), p ∈ P, x ∈ F_A (p)\} is a relation form of (F_{A}, P), where

µ_{R_A}: X × P → [0, 1] and λ_{R_A}: X × P → [0, 1],

such that

R_A: (x, p) ∈ [0, 1], ∀ x ∈ X, p ∈ P

and

λ_{R_A}: (x, p) ∈ [0, 1], ∀ x ∈ X, p ∈ P

If [µ_{ij}, λ_{ij}]_{m×n} = µ_{R_A} \times x \times p \in P, then we can define an m × n generalized fuzzy soft matrix (GFSM) of GFSS over (X, P) as

[µ_{ij}, λ_{ij}]_{m×n} = \begin{bmatrix}
(µ_{11}, λ_{11}) & (µ_{12}, λ_{12}) & \cdots & (µ_{1n}, λ_{1n}) \\
(µ_{21}, λ_{21}) & (µ_{22}, λ_{22}) & \cdots & (µ_{2n}, λ_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
(µ_{m1}, λ_{m1}) & (µ_{m2}, λ_{m2}) & \cdots & (µ_{mn}, λ_{mn})
\end{bmatrix}

Definition 2.5. [16] Let U = \{c_1, c_2, c_3, ..., c_m\} be the Universe set and E be the set of parameters given by E = \{e_1, e_2, e_3, ..., e_n\}. Let A be a subset of E and (F, A) be an interval valued fuzzy soft set over U and F is a mapping given by F: A→I^U, where I^U denotes the collection of all Interval valued fuzzy subsets of U. Then the Interval valued fuzzy soft set can expressed in matrix form as

A = [a_{ij}]_{m×n} or \tilde{A} = [a_{ij}] i = 1, 2, ..., m, j = 1, 2, ..., n.

Where

a_{ij} = \begin{cases}
(µ_{ij}(e_i), µ(e_i)) & \text{if } e_i ∈ A \\
(0,0) & \text{if } e_i ∉ A
\end{cases}

The interval [µ_{ij}(c_i), µ_{ij}(e_i)] represents the membership of c_i in the Interval valued fuzzy set F(e_i).

If µ_{ij}(c_i) = µ_{ij}(e_i) then the Interval- valued fuzzy soft matrix (IVFSM) reduces to a FSM.

Definition 2.6. [11] Let U be an initial Universe set and E be the set of parameters, let

A ⊆ E. A pair (F, A) is called Interval valued fuzzy soft set over U where F is a mapping given by F: A→I^U, where I^U denotes the collection of all Interval valued fuzzy subsets of U.

Definition 2.7. [14] Let U be the Universal set and E be the set of parameters. Let A ⊆ E and µ be a fuzzy subset of A. Let F: A→P(U) and µ: A→I = [0, 1] where P(U) is the collection of all Interval valued fuzzy subsets of U.

Define a function F_µ: A→P(U)×I, such that

F_µ(e) = (F(e), µ(e))

F_µ(e_i) = (F(e_i)(x), µ(e_i)), ∀ (e_i).

where F(e_i)(x) is an interval value and is called degree of membership of an element x to F(e) and µ(e) is called the degree of possibility of this belongingness. Then F_µ is a GIVFSS.

3. Generalized Interval Valued Fuzzy Soft Matrices

Definition 3.1 Let U be the Universal set and E be the set of parameters. Let A ⊆ E and µ be a fuzzy subset of A. Let F: A→P(U) and µ: A→I = [0, 1] where P(U) is the collection of all Interval valued fuzzy subsets of U. A function F_µ: A→P(U)×I defined as

F_µ(e) = (F(e), µ(e))

Then the generalized interval valued fuzzy soft set F_µ can expressed in matrix form as

(F_µ(e_i), µ) = [a_{ij}]_{m×n}

where

a_{ij} = \begin{cases}
(F(e_i)(x_i), µ(e_i)) & \text{if } e_i ∈ A \\
(0,0) & \text{if } e_i ∉ A
\end{cases}

\forall i = 1, 2, ..., m, j = 1, 2, ..., n.

i.e. [a_{ij}]_{m×n}
\[\begin{bmatrix}
(F(e_1)(x_1), \mu(e_1)) & (F(e_1)(x_2), \mu(e_1)) & \cdots & (F(e_1)(x_m), \mu(e_1)) \\
(F(e_2)(x_1), \mu(e_2)) & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots \\
(F(e_n)(x_1), \mu(e_n)) & \cdots & \cdots & (F(e_n)(x_m), \mu(e_n))
\end{bmatrix}\]

where \(F(e_i)(x_1) = [F(e_{i1})(x_1), F(e_{i2})(x_1)]\) represents the membership of \(e_i\) in the GIVFSS \(F_{\mu}(e_i)\), such that

\[0 \leq F(e_i)(x_1) \leq F(e_i)(x_1) \leq 1.\]

If \(F(e_i)(x_1) = \mu(e_i)\) then the GIVFSS reduces to a GFSM.

Definition 3.2. Let \(F_{\mu}\) and \(G_{\lambda}\) be two GIVFSM’s. Then \(F_{\mu}\) is called GIVFSS sub matrix of \(G_{\lambda}\) if

\[F(e_i)(x_1) \leq G(e_{k})(x_1) \text{ and } \mu(e_i) \leq \lambda(e_k), \forall i = 1, 2, \ldots, m \text{ and } j, k = 1, 2, \ldots, n\]

Example 3.3. Consider a set of three motorcycles \(U = \{b_1, b_2, b_3\}\) and a set of parameters,

\(E = \{e_1, e_2, e_3\}\), where \(e_1, e_2\) and \(e_3\) stands for cheap, expansive and comfortable respectively. The GFSM’s \(F_{\mu}\) and \(G_{\lambda}\) are defined as

\[F_{\mu}(e_1) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

\[F_{\mu}(e_2) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

\[F_{\mu}(e_3) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

and

\[G_{\lambda}(e_1) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

\[G_{\lambda}(e_2) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

\[G_{\lambda}(e_3) = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

and

\[F_{\mu} = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

\[G_{\lambda} = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}\]

It is clear that \(F_{\mu}\) is GIVFS sub matrix of \(G_{\lambda}\).

Definition 3.4. Let \(F_{\mu}\) and \(G_{\lambda}\) be two GIVFSS’s. Then \(F_{\mu}\) is called GIVFSS equal matrix of \(G_{\lambda}\) if

\[F(e_i)(x_1) = G(e_{k})(x_1) \text{ and } \mu(e_i) = \lambda(e_k), \forall i = 1, 2, \ldots, m \text{ and } j, k = 1, 2, \ldots, n\]

Definition 3.5. Let \(F_{\mu}\) and \(G_{\lambda}\) be two GIVFSS’s. Then \(F_{\mu}\) is called GIVFSS sub matrix of \(G_{\lambda}\) if

\[F(e_i)(x_1) \leq G(e_{k})(x_1) \text{ and } \mu(e_i) \leq \lambda(e_k), \forall e, F(e_i)(x_1) < G(e_{k})(x_1) \text{ and } \mu(e_i) < \lambda(e_k), \text{ for at least one } e\]

Definition 3.6. Let \(F_{\mu}\) and \(G_{\lambda}\) be two GIVFSS’s. Then \(F_{\mu}\) is called GIVFSS sub matrix of \(G_{\lambda}\) if

\[F(e_i)(x_1) \leq G(e_{k})(x_1) \text{ and } \mu(e_i) < \lambda(e_k), \forall i = 1, 2, \ldots, m \text{ and } j, k = 1, 2, \ldots, n\]

Definition 3.7. A GIVFSS \(F_{\mu}\) is called GIVFSS rectangular matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m \neq n, \forall i, j\]

Definition 3.8. A GIVFSS \(F_{\mu}\) is called GIVFSS square matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m = n, \forall i, j\]

Definition 3.9. A GIVFSS \(F_{\mu}\) is called GIVFSS diagonal matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m \neq n, \forall i, j\]

Definition 3.10. A GIVFSS \(F_{\mu}\) is called GIVFSS scalar matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m = n, \forall i, j\]

Definition 3.11. A GIVFSS \(F_{\mu}\) is called GIVFSS row matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m = n, \forall i, j\]

Definition 3.12. A GIVFSS \(F_{\mu}\) is called GIVFSS column matrix if

\[F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ and } m = n, \forall i, j\]

Definition 3.13. Let \(F_{\mu}\) be a GIVFSS, then scalar multiple of \(F_{\mu}\) by a scalar \(k\) is defined as

\[kF_{\mu} = \begin{bmatrix} kF(e_1)(x_1), k\mu(e_1) \end{bmatrix} \text{ and } 0 \leq k \leq 1\]

Definition 3.14. A GIVFSS \(A_{\mu}\) is called generalized absolute IVFSM, if

\[A_{\mu} = \begin{bmatrix} \mu(e_1)(x_1) \end{bmatrix} \text{ and } A(e_1)(x_1) = 1 \text{ and } \mu(e_1) = 1, \forall i, j\]

Definition 3.15. A GIVFSS \(\phi_{\mu}\) is called generalized null IVFSM, if

\[\phi_{\mu} = \begin{bmatrix} \phi(e_1)(x_1) \end{bmatrix} \text{ and } \phi(e_1)(x_1) = 0 \text{ and } \mu(e_1) = 0, \forall i, j\]

4. Some Operations on GIVFSS’s

Definition 3.1. Let \(F_{\mu} = \begin{bmatrix} F(e_1)(x_1), \mu(e_1) \end{bmatrix} \text{ be a GIVFSS, where }\]

\[F(e_1)(x_1) = \begin{bmatrix} F(e_{i1})(x_1), F(e_{i2})(x_1) \end{bmatrix}, \text{ then compliment of } \]

\[F_{\mu} = G_{\lambda} \text{ is denoted by } F_{\mu}^{c} = G_{\lambda} \text{ and is given by}\]

\[F_{\mu}^{c} = G_{\lambda} = \begin{bmatrix} 1 - F(e_{i1})(x_1), 1 - F(e_{i2})(x_1) \end{bmatrix}, \forall i, j.\]
Example 4.2. Consider a GIVFSM $F_\mu$ as in example 3.3,

\[
F_\mu = \left[ \begin{array}{ccc}
(0.1,0.4,0.3) & (0.4,0.7,0.3) & (0.2,0.4,0.3) \\
(0,2,0.3,0.4) & (0,3,0,4) & (0,2,0,5,0,4) \\
(0,3,0,5,0,1) & (0,1,0,2,0,1) & (0,0,2,0,1) 
\end{array} \right]
\]

Then compliment of $F_\mu$ is given by

\[
F_\mu^c = \left[ \begin{array}{ccc}
([0.6,0.9],[0.7]) & ([0.3,0.6,0.7]) & ([0.6,0.8,0.7]) \\
([0.7,0.8],[0.6]) & ([0.7,1,0.6]) & ([0.5,0.8,0.6]) \\
([0.5,0.7],[0.9]) & ([0.6,0.9],[0.9]) & ([0.8,1,0.1]) 
\end{array} \right]
\]

Example 3.3,

\[
F_\mu = \left[ \begin{array}{ccc}
0.1,0.4,0.3 & 0.4,0.7,0.3 & 0.2,0.4,0.3 \\
0.2,0.3,0.4 & 0.3,0.4,0.4 & 0.2,0.5,0.4 \\
0.3,0.5,0.1 & 0.1,0.2,0.1 & 0.0,2,0,1 
\end{array} \right]
\]

Proposition 4.3. Let $F_\mu$ be a GIVFSM, then $(F_\mu^c)^c = F_\mu$

Proof:

Since

\[
F_\mu^c = G_\lambda = \left( [F(e_{ju}(x_i), F(e_{jx}(x_i), 1 - \mu(e_j)) \right)
\]

then $(F_\mu^c)^c = G_\lambda^c$

but from definition, $G_\lambda = \left( [F(e_{ju}(x_i), F(e_{jx}(x_i), 1 - \mu(e_j)) \right)$

Definition 4.4. The union of two GIVFSM’s $F_\mu = [F(e_j(x_i), \mu(e_j))_{m\times n}$ and $G_\lambda = [G(e_k(x_i), \lambda(e_k))_{m\times n}$, denoted by $F_\mu \cup G_\lambda$ is a GIVFSM $[H(e_j(x_i), \gamma(e_j))_{m\times n}$, such that

\[
\gamma(e_j) = s(\mu(e_j), \lambda(e_k)), \text{ where } s \text{ is an } s-norm.
\]

Example 4.5. Consider two GIVFSM’s $F_\mu$ and $G_\lambda$ as in example 3.3,

\[
F_\mu = \left[ \begin{array}{ccc}
(0.1,0.4,0.3) & (0.4,0.7,0.3) & (0.2,0.4,0.3) \\
(0.2,0.3,0.4) & (0.3,0.4,0.4) & (0.2,0.5,0.4) \\
(0.3,0.5,0.1) & (0.1,0.2,0.1) & (0.0,2,0,1) 
\end{array} \right]
\]

and

\[
G_\lambda = \left[ \begin{array}{ccc}
(0.3,0.5,0.4) & (0.5,0.8,0.4) & (0.3,0.6,0.4) \\
(0.4,0.5,0.5) & (0.2,0.4,0.5) & (0.4,0.6,0.5) \\
(0.5,0.7,0.3) & (0.3,0.5,0.3) & (0.1,0.3,0.3) 
\end{array} \right]
\]

Then the union of $F_\mu$ and $G_\lambda$ is given by

\[
F_\mu \cup G_\lambda = \left[ \begin{array}{ccc}
(0.3,0.5,0.4) & (0.5,0.8,0.4) & (0.3,0.6,0.4) \\
(0.4,0.5,0.5) & (0.2,0.4,0.5) & (0.4,0.6,0.5) \\
(0.5,0.7,0.3) & (0.3,0.5,0.3) & (0.1,0.3,0.3) 
\end{array} \right]
\]

Proposition 4.6. Let $F_\mu$ be a GIVFSM, then

\[
F_\mu \cup F_\mu = F_\mu
\]

Proof: From Definition, we have

\[
[ F(e_j)(x_i), \mu(e_j)]_{m\times n} \cup [ F(e_j)(x_i), \mu(e_j)]_{m\times n}
= [ H(e_j)(x_i), \gamma(e_j)]_{m\times n}
\]

Such that

\[
= \sup(F(e_{jx})(x_i), \sup(F(e_{ju})(x_i))) = [ F(e_j)(x_i), \mu(e_j)]_{m\times n}
= F_\mu
\]

Proposition 4.7. Let $F_\mu = [ F(e_j)(x_i), \mu(e_j)]_{m\times n}$ and $G_\lambda = [ G(e_k)(x_i), \lambda(e_k)]_{m\times n}$ be two GIVFSM’s, then

a) $F_\mu \cup G_\lambda = G_\lambda \cup F_\mu$

b) $F_\mu \cap A_\mu = A_\mu$

c) $F_\mu \cup \Phi_\mu = F_\mu$

Proof:

(a) From Definition, we have

\[
[ F(e_j)(x_i), \mu(e_j)]_{m\times n} \cup [ G(e_k)(x_i), \lambda(e_k)]_{m\times n}
= [ H(e_j)(x_i), \gamma(e_j)]_{m\times n}
\]

Such that

\[
= \sup(F(e_{jx})(x_i), \sup(F(e_{ju})(x_i))) = [ F(e_j)(x_i), \mu(e_j)]_{m\times n}
= F_\mu
\]

But $H_\gamma = F_\mu \cup G_\lambda = G_\lambda \cup F_\mu$ (since union of GIVFSM’s is commutative)

and $\gamma(e_j) = s(\mu(e_j), \lambda(e_k)) = s(\lambda(e_k), \mu(e_j))$, (since $s-norm$ is commutative)

Then,

\[
F_\mu \cup G_\lambda = G_\lambda \cup F_\mu
\]

The proof of (b) and (c) are straight forward from definition.

Definition 4.8. The intersection of two GIVFSM’s $F_\mu = [ F(e_j)(x_i), \mu(e_j)]_{m\times n}$ and $G_\lambda = [ G(e_k)(x_i), \lambda(e_k)]_{m\times n}$, denoted by $F_\mu \cap G_\lambda$ is a GIVFSM $[ H(e_j)(x_i), \gamma(e_j)]_{m\times n}$ such that

\[
[ H(e_j)(x_i), \gamma(e_j)]_{m\times n}
= \inf(F(e_{jx})(x_i), \inf(F(e_{ju})(x_i))) = [ F(e_j)(x_i), \mu(e_j)]_{m\times n}
\]

\[
= [ F(e_j)(x_i), \mu(e_j)]_{m\times n}
\]

\[
= F_\mu
\]

Example 4.9. Consider two GIVFSM’s $F_\mu$ and $G_\lambda$ as in example 3.3,

\[
F_\mu = \left[ \begin{array}{ccc}
(0.1,0.4,0.3) & (0.4,0.7,0.3) & (0.2,0.4,0.3) \\
(0.2,0.3,0.4) & (0.3,0.4,0.4) & (0.2,0.5,0.4) \\
(0.3,0.5,0.1) & (0.1,0.2,0.1) & (0.0,2,0,1) 
\end{array} \right]
\]

and

\[
G_\lambda = \left[ \begin{array}{ccc}
(0.3,0.5,0.4) & (0.5,0.8,0.4) & (0.3,0.6,0.4) \\
(0.4,0.5,0.5) & (0.2,0.4,0.5) & (0.4,0.6,0.5) \\
(0.5,0.7,0.3) & (0.3,0.5,0.3) & (0.1,0.3,0.3) 
\end{array} \right]
\]

Then the union of $F_\mu$ and $G_\lambda$ is given by

\[
F_\mu \cup G_\lambda = \left[ \begin{array}{ccc}
(0.3,0.5,0.4) & (0.5,0.8,0.4) & (0.3,0.6,0.4) \\
(0.4,0.5,0.5) & (0.2,0.4,0.5) & (0.4,0.6,0.5) \\
(0.5,0.7,0.3) & (0.3,0.5,0.3) & (0.1,0.3,0.3) 
\end{array} \right]
\]


Proposition 4.10. Let $F_{\mu}$ be a GIVFSM, then
\[ F_{\mu} \cap F_{\mu} = F_{\mu} \]
Proof: From Definition, we have
\[ F(e_j(x_i), \mu(e_j))_{m \times n} \cap [F(e_j(x_i), \mu(e_j))_{m \times n} = [H(e_j(x_i), \gamma(e_j))_{m \times n} \]
such that
\[ H(e_j(x_i), \gamma(e_j))_{m \times n} = \inf(F(e_{jL}), F(e_{jU})), \inf(F(e_{jL}), F(e_{jU}))_{m \times n} \]
\[ = [F(e_j(x_i), \mu(e_j))_{m \times n} = F_{\mu} \]
Proposition 4.11. Let $G_{\lambda}$ be two GIVFSM's, then
\[ G_{\lambda} = \left\{ \begin{array}{l}
(\{0.3,0.5\}, 0.4) \\
(\{0.5,0.8\}, 0.4) \\
(\{0.3,0.6\}, 0.4) \\
(\{0.4,0.5\}, 0.5) \\
(\{0.2,0.4\}, 0.5) \\
(\{0.4,0.6\}, 0.5) \\
(\{0.5,0.7\}, 0.3) \\
(\{0.3,0.5\}, 0.3) \\
(\{0.1,0.2\}, 0.1) \\
(\{0.0,0.2\}, 0.1) \\
\end{array} \right. \]
Then the intersection of $F_{\mu}$ and $G_{\lambda}$ is given by
\[ F_{\mu} \cap G_{\lambda} = \left\{ \begin{array}{l}
(\{0.1,0.4\}, 0.3) \\
(\{0.4,0.7\}, 0.3) \\
(\{0.2,0.4\}, 0.3) \\
(\{0.2,0.3\}, 0.4) \\
(\{0.3,0.3\}, 0.4) \\
(\{0.2,0.5\}, 0.4) \\
(\{0.3,0.5\}, 0.1) \\
(\{0.1,0.2\}, 0.1) \\
(\{0.0,0.2\}, 0.1) \\
\end{array} \right. \]
Proposition 4.12. Let $F_{\mu}$ be a GIVFSM, then
\[ F_{\mu} \cap F_{\mu} = F_{\mu} \]
Proof: From Definition, we have
\[ F(e_j(x_i), \mu(e_j))_{m \times n} \cap [F(e_j(x_i), \mu(e_j))_{m \times n} = [H(e_j(x_i), \gamma(e_j))_{m \times n} \]
\[ = [\inf(F(e_{jL}), F(e_{jU})), \inf(F(e_{jL}), F(e_{jU}))_{m \times n} \]
\[ = [F(e_j(x_i), \mu(e_j))_{m \times n} = F_{\mu} \]
Proposition 4.13. Let $G_{\lambda}$ be two GIVFSM's, then
\[ G_{\lambda} = \left\{ \begin{array}{l}
(\{0.3,0.5\}, 0.4) \\
(\{0.5,0.8\}, 0.4) \\
(\{0.3,0.6\}, 0.4) \\
(\{0.4,0.5\}, 0.5) \\
(\{0.2,0.4\}, 0.5) \\
(\{0.4,0.6\}, 0.5) \\
(\{0.5,0.7\}, 0.3) \\
(\{0.3,0.5\}, 0.3) \\
(\{0.1,0.2\}, 0.1) \\
(\{0.0,0.2\}, 0.1) \\
\end{array} \right. \]
Then the intersection of $F_{\mu}$ and $G_{\lambda}$ is given by
\[ F_{\mu} \cap G_{\lambda} = \left\{ \begin{array}{l}
(\{0.1,0.4\}, 0.3) \\
(\{0.4,0.7\}, 0.3) \\
(\{0.2,0.4\}, 0.3) \\
(\{0.2,0.3\}, 0.4) \\
(\{0.3,0.3\}, 0.4) \\
(\{0.2,0.5\}, 0.4) \\
(\{0.3,0.5\}, 0.1) \\
(\{0.1,0.2\}, 0.1) \\
(\{0.0,0.2\}, 0.1) \\
\end{array} \right. \]
\[
(\mu(e_j) \cup (\lambda(e_k) \cap \gamma(e_j))) = \max\{\mu(e_j), (\lambda(e_k) \cap \gamma(e_j))\}
\]
\[
= \max\{\mu(e_j), \min(\lambda(e_k) \cap \gamma(e_j))\}
\]
\[
= \min\{\max(\mu(e_j), (\lambda(e_k)) \cap \gamma(e_j)), \max(\mu(e_j), (\lambda(e_j)) \cap (\gamma(e_j)))\}
\]
\[
= \mu(e_j) \cup (\lambda(e_k)) \cap (\gamma(e_j))
\]
\[
= (\mu(e_j) \cup (\lambda(e_k))) \cap ((\mu(e_j) \cup (\gamma(e_j)))
\]
(b) The proof is similar to proof of (a).

**Definition 4.15.** Let \(F_\mu = [F(e_j)(x_i), \mu(e_j)]\) and \(G_\lambda = [G(e_k)(x_i), \lambda(e_k)]\) be two GIVFSM's, then AND product of \(F_\mu\) and \(G_\lambda\) is denoted by \(F_\mu \land G_\lambda\) and is defined as

\[
F_\mu \land G_\lambda = H_f
\]

where \(H_f = [H(e_j)(x_i), \gamma(e_j)]\)

\[
F_\mu = \left\{ \begin{array}{ccc}
(0.1, 0.4, 0.3) & (0.4, 0.7, 0.3) & (0.2, 0.4, 0.3) \\
(0.2, 0.3, 0.4) & (0.3, 0.4, 0.4) & (0.2, 0.5, 0.4) \\
(0.3, 0.5, 0.1) & (0.1, 0.2, 0.1) & (0.1, 0.2, 0.1)
\end{array} \right. 
\]

and

\[
G_\lambda = \left\{ \begin{array}{ccc}
(0.3, 0.5, 0.4) & (0.5, 0.8, 0.4) & (0.3, 0.6, 0.4) \\
(0.4, 0.5, 0.5) & (0.2, 0.4, 0.5) & (0.4, 0.6, 0.5) \\
(0.5, 0.7, 0.3) & (0.3, 0.5, 0.3) & (0.1, 0.3, 0.3)
\end{array} \right. 
\]

Then \(F_\mu \land G_\lambda\) is given by

\[
F_\mu \land G_\lambda = \left\{ \begin{array}{ccc}
(0.1, 0.4, 0.3) & (0.4, 0.7, 0.3) & (0.2, 0.4, 0.3) \\
(0.1, 0.4, 0.3) & (0.2, 0.4, 0.3) & (0.2, 0.4, 0.3) \\
(0.1, 0.4, 0.3) & (0.3, 0.5, 0.3) & (0.1, 0.3, 0.3) \\
(0.2, 0.3, 0.4) & (0.3, 0.4, 0.4) & (0.2, 0.5, 0.4) \\
(0.2, 0.3, 0.4) & (0.3, 0.4, 0.4) & (0.2, 0.5, 0.4) \\
(0.2, 0.3, 0.3) & (0.3, 0.3, 0.3) & (0.1, 0.3, 0.3) \\
(0.3, 0.5, 0.1) & (0.1, 0.2, 0.1) & (0.1, 0.2, 0.1) \\
(0.3, 0.5, 0.1) & (0.1, 0.2, 0.1) & (0.1, 0.2, 0.1) \\
(0.3, 0.5, 0.1) & (0.1, 0.2, 0.1) & (0.1, 0.2, 0.1)
\end{array} \right. 
\]

Definition 4.19. Let \(F_\mu = [F(e_j)(x_i), \mu(e_j)]\) and \(G_\lambda = [G(e_k)(x_i), \lambda(e_k)]\) be two GIVFSM’s, then OR product of \(F_\mu\) and \(G_\lambda\), denoted by \(F_\mu \lor G_\lambda\) is defined by

\[
F_\mu \lor G_\lambda = H_y
\]

where

\[
H_y = [H(e_j)(x_i), \gamma(e_j)]
\]

such that

\[
H(e_j)(x_i) = \frac{F(e_j)(x_i) + G(e_k)(x_i)}{2}
\]

\[
\gamma(e_j) = \frac{\mu(e_j) + \lambda(e_k)}{2}
\]
Definition 4.20. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then weighted arithmetic mean of $F_{\mu}$ and $G_{\lambda}$, denoted by $F_{\mu} \oplus ^w G_{\lambda}$ is defined by $$H^w(e_j)(x_i) = \left[ H^w(e_{jL})(x_i), H^w(e_{jU})(x_i) \right]$$

$$H^w(e_{jL})(x_i) = \frac{w_1 F(e_{jL})(x_i) + w_2 G(e_{jL})(x_i)}{w_1 + w_2} \quad H^w(e_{jU})(x_i) = \frac{w_1 F(e_{jU})(x_i) + w_2 G(e_{jU})(x_i)}{w_1 + w_2}$$

Where

$$H^w_{\gamma} = \left[ H^w(e_j)(x_i), \gamma \right]$$

and

$$\gamma^w(e_j) = \frac{w_1 \mu(e_j) + w_2 \lambda(e_k)}{w_1 + w_2}$$

such that

Definition 4.21. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then geometric mean of $F_{\mu}$ and $G_{\lambda}$, denoted by $F_{\mu} \oplus ^G G_{\lambda}$ is defined by

$$F_{\mu} \oplus ^G G_{\lambda} = \left( \left[ F(e_{jL})(x_i) \cdot G(e_{kL})(x_i) \right]^{\frac{1}{w_1 + w_2}}, \left[ F(e_{jU})(x_i) \cdot G(e_{kU})(x_i) \right]^{\frac{1}{w_1 + w_2}} \right)$$

Definition 4.22. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then weighted geometric mean of $F_{\mu}$ and $G_{\lambda}$, denoted by $F_{\mu} \oplus ^{wG} G_{\lambda}$ is defined by

$$F_{\mu} \oplus ^{wG} G_{\lambda} = \left\{ \left[ F(e_{jL})(x_i) \cdot G(e_{kL})(x_i) \right]^{\frac{1}{w_1 + w_2}}, \left[ F(e_{jU})(x_i) \cdot G(e_{kU})(x_i) \right]^{\frac{1}{w_1 + w_2}} \right\}^{\frac{1}{\mu(e_j) \cdot \lambda(e_k)}}$$

Definition 4.23. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then harmonic mean of $F_{\mu}$ and $G_{\lambda}$, denoted by $F_{\mu} \oplus ^H G_{\lambda}$ is defined by

$$F_{\mu} \oplus ^H G_{\lambda} = \left( \left[ F(e_{jL})(x_i) \cdot G(e_{kL})(x_i) \right]^{\frac{w_1}{w_1 + w_2}}, \left[ F(e_{jU})(x_i) \cdot G(e_{kU})(x_i) \right]^{\frac{w_2}{w_1 + w_2}} \right)$$

Definition 4.24. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then weighted harmonic mean of $F_{\mu}$ and $G_{\lambda}$, denoted by $F_{\mu} \oplus ^{wH} G_{\lambda}$ is defined by

$$F_{\mu} \oplus ^{wH} G_{\lambda} = \left( \left[ F(e_{jL})(x_i) \cdot G(e_{kL})(x_i) \right]^{\frac{w_1}{w_1 + w_2}}, \left[ F(e_{jU})(x_i) \cdot G(e_{kU})(x_i) \right]^{\frac{w_2}{w_1 + w_2}} \right)$$

5. Similarity Between Two GIVESM's

Definition 5.1. Let $F_{\mu} = [F(e_j)(x_i), \mu(e_j)]_{m \times n}$ and $G_{\lambda} = [G(e_k)(x_i), \lambda(e_k)]_{m \times n}$ be two GIVFSM's, then similarity between $F_{\mu}$ and $G_{\lambda}$, denoted by $S(F_{\mu}, G_{\lambda})$, is defined by

$$S(F_{\mu}, G_{\lambda}) = H_{\gamma},$$

where $H_{\gamma} = [H(e_{jL})(x_i), \gamma, H(e_{jU})(x_i), \gamma]$.

$$H(e_{jL})(x_i) = \min \left( \phi \left( F(e_{jL}), G(e_{kL}) \right), \phi \left( F(e_{jL}), G(e_{kU}) \right) \right)$$

$$H(e_{jU})(x_i) = \max \left( \phi \left( F(e_{jL}), G(e_{kL}) \right), \phi \left( F(e_{jU}), G(e_{kU}) \right) \right)$$

Such that

$$\phi \left( F(e_{jL}), G(e_{kL}) \right) = 0, \text{if} \ F(e_{jL}) = 0, \forall e_j$$

$$\phi \left( F(e_{jL}), G(e_{kL}) \right) = \sum_i \max \left\{ \min \left( F(e_{jL}), G(e_{kL}) \right) \right\}, \text{otherwise}$$

and

$$\phi \left( F(e_{jU}), G(e_{kU}) \right) = \sum_i \max \left\{ \min \left( F(e_{jU}), G(e_{kU}) \right) \right\}$$
be used to solve decision making problems in situations where uncertainty involved.

References


6. Conclusion

We have introduced the concept of GIVFSM’s in this paper. Some of its types are defined. Some basic operations like union, intersection, compliment, AND operation and OR operation have been defined and exemplified. Arithmetic mean, geometric mean, harmonic mean and their weighted means are also defined and some properties of these operators are discussed. Furthermore, similarity between two GIVFSM’s is discussed. To future concern, GIVFSM’s can


