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Ricci flows and topology change in quantum gravity

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Abstract: The topology change in quantum gravity is modeled by a Ricci flow. In this approach we consider the Ricci flow as a statistical system. The metric in the Ricci flow enumerated by a parameter λ is a microscopical statistical state. The probability of every microscopical state is determined by this parameter. The Ricci flow starting from a static wormhole filled with a phantom Sine-Gordon scalar field is investigated numerically.

Keywords: Ricci Flow, Topology Change, Statistical System, Wormhole

1.Introduction

One of the most interesting questions in hypothetical quantum gravity concerns the topology change. The essence of the problem is that on a microscopic level (i.e. at the length scales in the region of the Planck length $\approx 10^{-33}$ cm) a change of a space topology may take place. This process is called as a spacetime foam [1]. Wheeler suggested that on the Planck level, the metric fluctuations are so large that the change of space topology may take place. Such a process occurs in a few steps: (a) two points are identified; (b) one point is deleted; (c) at the boundary of the generated space a sphere is pasted in; (d) the sphere is enlarged (blown up) (see Fig. 1).

The main problem in this model is that, during the topology change, a singularity may be created [2] due fact that during this process some points are deleted and others are inserted. This results in the singularity in the same way as the creation of universe from the Big Bang bring stems from the cosmological singularity. There have been a number of investigations which aim to demonstrate the feasibility of the topology change [3], [4], [5], [6]. In Ref. [7] a numerical investigation of the topology change in an Einstein - matter system was made. Nevertheless, these investigations are still far from being conclusive. We would like to emphasise that some of our findings provide a qualitative (and in the future quantitative) support to speculations concerning the existence of the topology change.

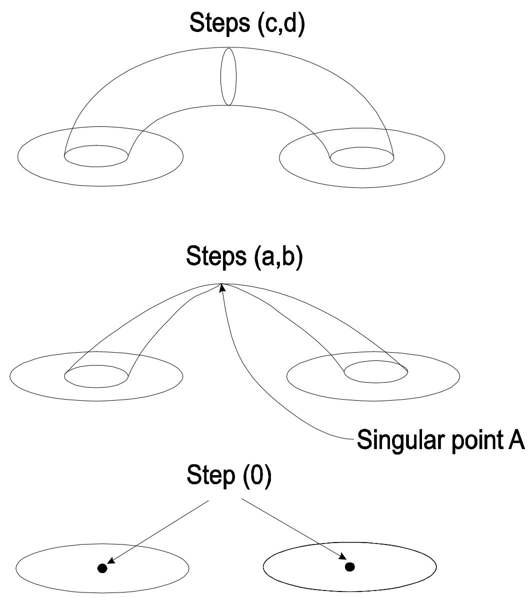

Fig.1*. In step(0) there are two initial points. Steps (a,b) involve the identification of two points with the subsequent deletion of one of them. In steps (c,d) the boundary is pasted in followed by the subsequent blowing up.*

In differential topology, processes known as topological surgery [10] are investigated in order to understand the topological structure of a manifold. In recent years great progress has been made in this direction. The modern mathematical tools for the investigation of topology surgery involves Ricci flows which were introduced by Hamilton [8] over 25 years ago. They play an important role in the proof of the Poincare conjecture [9]. Ricci flows are differential equations for the metric on a manifold M. These equations describe the creation on the manifold of a singularity (or singularities) for a finite value of a parameter λ_0. Fig. 1 (Step (a,b)) gives a schematic picture of the partially singular metric $g(\lambda_0)$ on the manifold M. The Ricci flow evolves from Step (c,d) to Step (a,b) in Fig. 1. The metric $g(\lambda_0)$ is smooth almost everywhere, excluding the point where the two initial points were identified. Strictly speaking, the curvature is locally bounded for almost all λ but blows-up as $\lambda \to \lambda_0$ at one or more points.

Let the metric $g(\lambda)$ be smooth on a domain $\Omega \subset M$. But for $\lambda \to \lambda_0$ the metric becomes singular at one or more points (point A in Fig. 1). If $\Omega \neq \emptyset$, then the main point is that small neighborhoods of the boundary $\partial\Omega$ consist of horns. A horn is a metric on $S^2 \times [0, \delta]$ where the S^2 factor is approximately round of radius $\rho(r)$, with $\rho(r)$ small and $\rho(r)/r \to 0$ as $r \to 0$. Fig. 1 represents a partially singular metric on the smooth manifold $S^2 \times I$ consisting of a pair of horns joined by a degenerate metric: $g(\lambda = \lambda_0)|_{x=A} = \infty$.

The main aim of this work is to model the topology change in a space using Ricci flow mathematics. We assume that for every λ the 3D space-like metric $g(\lambda)$ is realised with probability $\rho(\lambda)$ where the parameter λ describes the evolution of the metric g under the Ricci flow. Usually such probability is calculated from the path integral. The idea presented here is that this probability is determined by the Perelman functional W on a rescaled Ricci flow as $\rho = f\left(\frac{dW}{d\lambda}\right)$ due to the property $\frac{dW}{d\lambda} \geq 0$. Then the Ricci flow is a statistical system where every metric $g(\lambda)$ is a microscopical state.

In this work we continue to investigate the idea that the topology change in quantum gravity is connected with Ricci flows [13]. We consider another type of an initial wormhole from which the Ricci flow is started. The initial wormhole is not asymptotically flat and has an anti de-Sitter asymptotics. The physical idea offered in Ref. [11] is as follows: Let us start with a static wormhole solution in general relativity. Then the Ricci flow from the initial wormhole describes the topology change. The Ricci flow is the sequence of 3D metrics enumerated by the parameter λ. The main idea is that the Ricci flow is a statistical system and every state with the metric $g(\lambda)$ is a microscopical state with the probability determined by the functional (7).

2. Short Introductionto Ricci Flows

Now we would like to present a short introduction to Ricci flows following Ref. [12]. A Ricci flow allows the metric g_{ab} to evolve under

$$\frac{\partial g_{ab}(x^c, \lambda)}{\partial \lambda} = -2R_{ab}(x^c, \lambda), \tag{1}$$

where R_{ab} is the Ricci curvature; λ is a parameter; $a, b, c = 1, 2, , \cdots, n = \dim M$; x^a are the local coordinates on a manifold M. The Ricci flow describes the evolution of the metric g_{ab} as a function of the parameter λ.

Let us introduce the Perelman functional

$$W(g_{ab}, f, \tau) = \int \left[\tau\left(R + |f|^2\right) + f - n\right] u \, dV, \tag{2}$$

where $f : M \to R$ is a smooth function; R is the Ricci scalar; $\tau > 0$ is a scale parameter; $n = \dim M$, and u is defined by

$$u = (4\pi\tau)^{-n/2} e^{-f}. \tag{3}$$

One can show that, if M is closed, and g_{ab}, f and τ satisfy the following equations:

$$\frac{\partial g_{ab}}{\partial \lambda} = -2R_{ab}, \tag{4}$$

$$\frac{d\tau}{d\lambda} = -1, \tag{5}$$

$$\frac{\partial f}{\partial \lambda} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \tag{6}$$

that then the functional W increases according to

$$\frac{d}{d\lambda} W(g_{ab}, f, \tau) = 2\tau \int \left| R_{ab} + \frac{\partial^2 f}{\partial x^a \partial x^b} - \frac{g_{ab}}{2\tau} \right|^2 u dV \geq 0. \tag{7}$$

Under evolution (4)-(6)

$$\frac{d}{d\lambda} \int u dV = 0. \tag{8}$$

This means that

$$\int u dV = const \tag{9}$$

and $u(\lambda)$ represents the probability density of a particle evolving under Brownian motion, backwards in time. This property allows us to define the classical, or ``Boltzman - Shannon" entropy

$$S = -\int_M u \ln u \, dV \tag{10}$$

or a renormalized version of the classical entropy

$$\widetilde{S} = S - \frac{n}{2}\{1 + \ln[4\pi(\lambda_0 - \lambda)]\}. \quad (11)$$

The W – functional applied to this backwards Brownian diffusion on a Ricci flow also arises via the renormalized classical entropy $\widetilde{S}$

$$\mathsf{W}(\lambda) = -\frac{d}{d\lambda}\left(\tau\widetilde{S}\right). \quad (12)$$

3. Topology Change with Ricci Flow

In this section we would like to show that starting from an initial wormhole solution in general relativity one can obtain the sequence of 3D metrics. The sequence converges to the space with a point whose the metric has a singularity in the spirit of Ricci flows. Such a sequence can be elucidated by Steps (c,d) $\longrightarrow$ Steps (a,b) in Fig. 1. In the next subsection we present a wormhole solution filled with a phantom scalar field.

3.1. Wormhole Solution with a Phantom Scalar Field

In this section we follow Ref.[13]. Let us consider a gravitating system with one phantom scalar field φ with the Lagrangian

$$L = -\frac{R}{16\pi G} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi), \quad (13)$$

where R is the scalar curvature, G is the Newton's gravitational constant, and V is the Sine-Gordon potential with the reversed sign

$$V = \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\varphi\right) - 1\right], \quad (14)$$

where m is a mass of the Sine-Gordon scalar field, and λ is a coupling constant. The corresponding energy-momentum tensor is

$$T_i^k = -\partial_i\varphi\partial^k\varphi - \delta_i^k\left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right], \quad (15)$$

and the corresponding field equations are

$$G_i^k = 8\pi G T_i^k,$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}\right) = \frac{\partial V}{\partial\varphi} \quad (16)$$

We seek a wormhole solution of the form

$$\begin{aligned} ds^2 &= e^{2F(r)}dt^2 - \frac{dr^2}{A(r)} \\ &-(r^2 + r_0^2)(d\theta^2 + \sin^2\theta d\varphi^2). \end{aligned} \quad (17)$$

Introducing new dimensionless variables $\phi = (\sqrt{\lambda}/m)\varphi, x = mr$ after some algebraical manipulations one can obtain from (15)–(17) the following equations

$$\begin{aligned} -\frac{A'}{A}x + \frac{1}{A} + \frac{x^2}{x^2 + x_0^2} - 2 &= \\ \frac{x^2 + x_0^2}{A}\beta\left(-\frac{A}{2}\varphi'^2 + \cos\varphi - 1\right) \end{aligned} \quad (18)$$

$$\begin{aligned} &\frac{2x^2}{x^2 + x_0^2} - 2 - \frac{A'}{A}x + 2xF' \\ &= -\beta(x^2 + x_0^2)\varphi'^2, \end{aligned} \quad (19)$$

$$\begin{aligned} &\varphi'' + \left(-\frac{\beta}{2}\frac{x^2 + x_0^2}{x}\varphi'^2 + \frac{A'}{A} + \frac{1}{x} + \frac{x}{x^2 + x_0^2}\right)\varphi' \\ &= \frac{1}{A}\sin\varphi, \end{aligned} \quad (20)$$

where $\beta = m^2/\lambda$. For the wormhole solution we require that

$$A'(0) = F'(0) = \phi(0) = 0. \quad (21)$$

Substituting (22) into (19)–(21) we obtain constraints on the boundary values of the metric and of the scalar field

$$A(0) = 1 + 2\beta x_0^2, \varphi(0) = \pi, \varphi'(0) = \sqrt{\frac{2}{\beta x_0^2}}. \quad (22)$$

Solving numerically equations (19)–(21) with the boundary conditions (23) and with $\beta = 1$, one can obtain the results presented in Fig. s′ 2, 3 and4. One can see that asymptotically (at $x \to \pm\infty$) $\phi \to \pi$ and $e^{2F(x)} \to \frac{2}{3}x^2$. The last asymptotic says us that asymptotically the wormhole is the anti-de Sitter space.

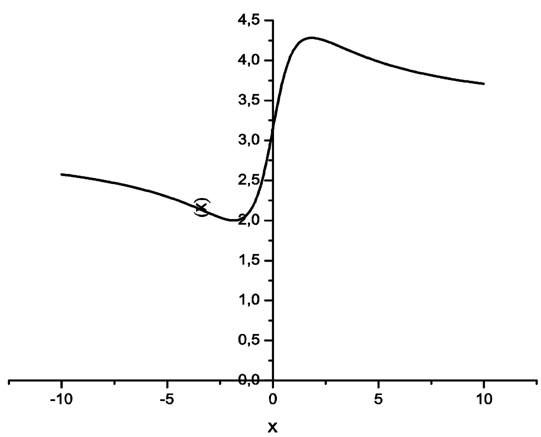

Fig.2. *The profile of the scalar field* $\phi(x)$. $\beta = 1$. *Asymptotically* $\phi \to \pi$.

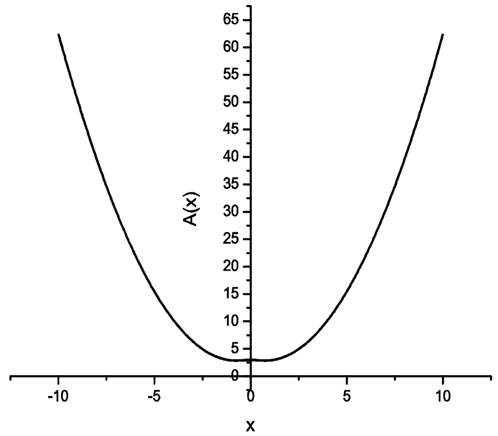

***Fig.3.** The profile of* $A(x)$.

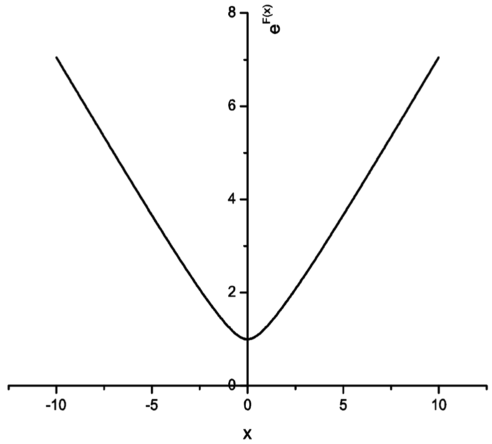

***Fig. 4.** The profile of* $e^{F(x)}$.

3.2. Ricci Flow and Topology Change

In this section we show that the Ricci flow starting from the wormhole obtained in subsection III A leads to a singularity as $\lambda \to \lambda_0$. The standard surgery operation applying at point A (where the singularity arises) leads to the change of topology.

Let us consider the 3D part of the metric (18)

$$\begin{aligned} dl^2 &= e^{2u(r,\lambda)}dr^2 \\ &+e^{2v(r,\lambda)}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \end{aligned} \quad (24)$$

where the initial conditions at $\lambda = 0$, to give us the 3D part of the wormhole metric (18), are

$$u(r,0) = -\frac{1}{2}\ln A(r), \quad (25)$$

$$e^{2v(r,0)} = r^2 + r_0^2. \quad (26)$$

According to section II the Ricci flow is described by

$$\frac{\partial u}{\partial \lambda} = 2e^{-2u}\left(v'' - u'v' + v'^2\right), \quad (27)$$

$$\frac{\partial v}{\partial \lambda} = e^{-2u}\left(v'' - u'v' + 2v'^2\right) - e^{-2v} \quad (28)$$

where $u = u(r,\lambda), v = v(r,\lambda)$. One can see from (27)(28) that

$$\frac{\partial(u - 2v)}{\partial \lambda} = 2e^{-2v}. \quad (29)$$

We will use this expression for the definition ofa boundary condition.

Let us note that there is a soliton solution which is defined as

$$\frac{\partial u}{\partial \lambda} = \frac{\partial v}{\partial \lambda} = 0. \quad (30)$$

In this case the solution of Eqs. (27)(28)satisfies

$$\left(e^v\right) = \pm e^u. \quad (31)$$

One can show that, by introducing a new coordinate x by

$$\pm e^u dr = \left(e^v\right)dr = dx, \quad (32)$$

the metric (24) can be written in the form

$$dl^2 = dx^2 + x^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (33)$$

Immediately, we see that we have obtained a 3D Euclidean space.

4. Numerical Solution

We have not been able to find an analytical solution of Eq's (27) (28). Therefore we have looked for a numerical solution of these equations. The initial and boundary conditions in our investigation have been imposed to be:

$$\begin{aligned} u(r,0) &= -\frac{1}{2}\ln A(r); \\ u(r,\lambda)\big|_{r\to\infty} &= -\frac{1}{2}\ln A(r)\big|_{r\to\infty}; \end{aligned} \quad (34)$$

$$v(r,0) = \frac{1}{2}\ln\left(r^2 + r_0^2\right), \quad \frac{\partial v(0,\lambda)}{\partial r} = 0. \quad (35)$$

For one more boundary condition of Ricci flows (27)(28) we use the expression (29). Since $e^{-2v} \to 0$ we obtain

$$\left.\frac{\partial v(r,\lambda)}{\partial r}\right|_{r\to\infty} = \frac{1}{2}\left.\frac{\partial u(r,\lambda)}{\partial r}\right|_{r\to\infty} \quad (36)$$

Let us note that we are considering a solution with the Z_2 symmetry, i.e. $u(-r,\lambda)=u(r,\lambda)$ and $v(-r,\lambda) = v(r,\lambda)$. For the numerical calculations we can make the following refinements: Equation (21) has the term $1/x$ and, in consequence, we have to start the numerical calculations not from $r = 0$ but from some $r = \delta \neq 0$. Therefore the numerical calculations for the equations set (27)(28)are made for $\delta \leq |r| \leq r_1$. Thus the initial and boundary conditions take the form

$$u(r,0) = -\frac{1}{2}\ln A(r); \quad u(r_1,\lambda) = -\frac{1}{2}\ln A(r_1); \tag{37}$$

$$v(r,0) = \frac{1}{2}\ln\left(r^2 + r_0^2\right); \quad \frac{\partial v(\delta,\lambda)}{\partial r} = 0; \quad \frac{\partial v(r_1,\lambda)}{\partial r} = \frac{1}{2}\frac{\partial u(r_1,\lambda)}{\partial r} \tag{38}$$

The results of the numerical calculations for the Ricci flow are presented in Figs. 5 and 6. From Fig. 6 we see that the radius of the wormhole $e^{v(0,\lambda)} \xrightarrow{\lambda \to \lambda_0} 0$. Simultaneously, at this point, a singularity develops because $e^{u(0,\lambda)} \xrightarrow{\lambda \to \lambda_0} \infty$.

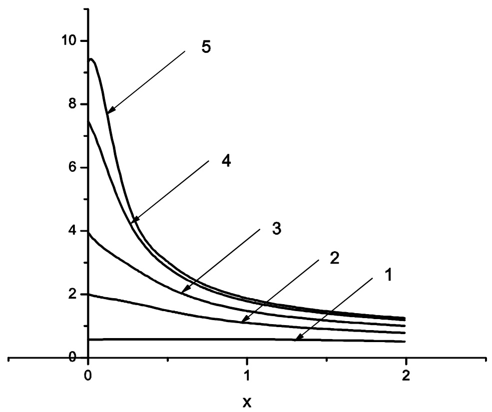

Fig.5. *The profiles of the functions* $e^{u(r,\lambda)}$. *1 -- by* $\lambda = 0$; *2 -- by* $\lambda = 0.3 \cdot \lambda_0$; *3 -- by* $\lambda = 0.6 \cdot \lambda_0$; *4 -- by* $\lambda = 0.9 \cdot \lambda_0$; *5 -- by* $\lambda \approx \lambda_0 = 1.9$.

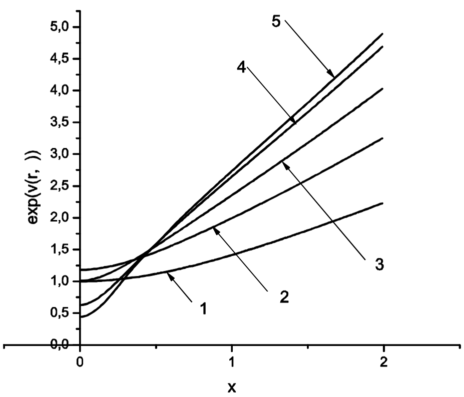

Fig.6. *The profiles of the functions* $e^{v(r,\lambda)}$. *1 -- by* $\lambda = 0$; *2 -- by* $\lambda = 0.3 \cdot \lambda_0$; *3 -- by* $\lambda = 0.6 \cdot \lambda_0$; *4 -- by* $\lambda = 0.9 \cdot \lambda_0$; *5 -- by* $\lambda \approx \lambda_0 = 1.9$.

5. Conclusions

The question of the topology change in gravity is one of the challenges for modern theoretical physics. The basic question is as follows: there exist the topology change in quantum gravity ? If yes then how can one describe this process? Generally accepted point of view is that the process should be described on the path integral language.

In the case of the existence of topological transition the quantum gravity may have an approximate approach to describe such transition. It is possible that this approach is connected with Ricci flows. The reason for such connection is that the Ricci flows describe the topology change by very natural way on mathematical language. We think that in this case (as usually) mathematics has a good imprint on physics. Our point of view is that the path integral in quantum gravity may be approximately described by the Ricci flows. Such approximation can be similar to the saddle point method used by the calculation of the path integral. In this approach the probability of every metric between a static wormhole and the final state with a singularity is defined by a positive functional determined by the Ricci flow.

In this paper we have shown that, starting from a static wormhole created by a phantom Sine-Gordon scalar field, we can obtain a Ricci flow. This Ricci flow describes a statistical system with microscopical states. Every microscopical state is a metric from the Ricci flow enumerated by a the parameter λ. The probability of every microscopical state is controlled by this parameter.

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