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Origin of Heisenberg's Uncertainty Principle

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Abstract: Heisenberg's uncertainty principle states that there is a fundamental limit to the precision with which certain pairs of physical properties of a particle (complementary variables) can be measured simultaneously. Heisenberg's uncertainty principle has indubitable support, but the origin behind this principle is unexplained. If complementary variables of particles are considered as complex numbers—for example, in calculating particle position, a complex vector coordinate space is necessary instead of the Cartesian space—then the origin of lower limit of Heisenberg's uncertainty principle emerges.

Keywords: Heisenberg's Uncertainty Principle, Complex Number, Complex Vector Space, Energy

1. Introduction

If A and B are presumed to be a pair of complementary variables, such as position and momentum, and if ΔA and ΔB are the uncertainties associated with this pair, then Heisenberg's uncertainty principle states [1]

$$\Delta A \Delta B \geq \frac{\hbar}{2}. \quad (1)$$

Although the mathematical origin of Heisenberg's uncertainty principle is known and the principle has been experimentally verified, its physical origin is not known. Questions such as why there is a lower limit of uncertainty for the two complementary variables have not been answered. Certain quantities, such as position, energy and time, are unknown as per Heisenberg's uncertainty principle , except by probabilities, probabilities gives us most probable value for these quantities but physical origin of these probabilities are not known. The aim of this paper is to explain the physical origin of Heisenberg's uncertainty principle and also why there is a lower limit of precision for any complementary pair.

2. Particle in Complex Vector Space

Let us consider a particle of finite extent in a 1-dimensional space as illustrated in Fig. 1; the red bar represents the particle of width δr .

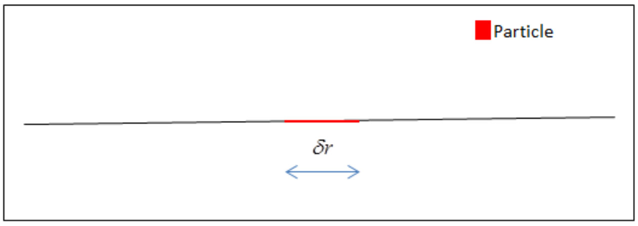

Fig. 1. *Particle in a 1 dimensional space.*

In Fig. 1, we can easily plot coordinates in terms of a 1-dimension line. However, we know that matter/particles are always in state of vibrations [2], hence Fig. 1 is not appropriate because it presupposes a representation in a flat 1-dimensional space. Consider then the same particle in a particular state of vibration as depicted in Fig. 2.

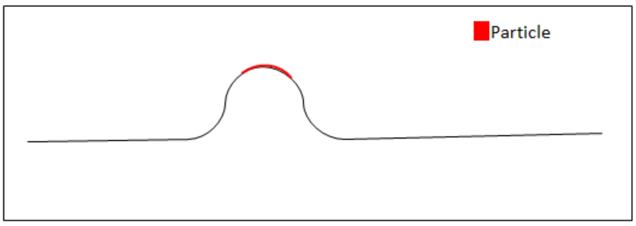

Fig. 2. *Particle in curved space.*

As already mentioned, the particle is in a 1-dimensional space, but plotting coordinates only in terms of one parameter in curved space is not possible. To resolve this problem we can consider the curved path is in an complex plane. This complex plane gives us information of the coordinate in a curved space with the real line giving the coordinate in a normal 1-dimensional space. Hence, the position of a particle in vibration can be expressed in the form of a 1-dimensional; complex vector space ℂ1 = ℂ (this can be viewed as a ℝ2 with the Cartesian unit j generating an imaginary line), which can be called an imaginary line [3].

3. Derivation Elaborating the Origin of Lower Limit in Uncertainty Principle

For this construction, we shall consider the uncertainty principle for the pair of position and momentum observable, If x (position) and p (momentum) are supposed to be a pair of complementary variables and if Δx and Δp are uncertainties associated with this pair, then Heisenberg's uncertainty principle states [4]

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \tag{2}$$

For a particle at rest, a frequency (f) is associated that depends on mass (m):[5]

$$f = \frac{mc^2}{h}. \tag{3}$$

This implies that even at rest the particle is in a state of change. If we apply this change in Fig. 2, one can say that the particle in 1 dimension would act like a vibrating string. Let us denote the length in the real line as δ_r, the length in the imaginary line as δ_i, and the total length as ℓ_x. Total length $|\ell_x|$ will always be constant but δ_r and δ_i will change with time. The total length ℓ_x can be expressed in terms of δ_r and δ_i given in polar coordinates [6]:

$$\ell_x = \delta_r + i\delta_i, \tag{4}$$

with δ_r and δ_i depending on $|\ell_x|$ and an angular frequency as

$$\delta_r = |\ell_x| \sin(\omega t), \tag{5}$$

$$\delta_i = |\ell_x| \cos(\omega t), \tag{6}$$

where

$$\omega = 2\pi f. \tag{7}$$

If we plot δ_r and δ_i in the complex plane, we trace a circle of radius $|\ell_x|$ (Fig 3) [7]. The circumference of the circle is equal to the wavelength of the particle and the wavelength of the particle is given as (if we consider the particle is an electron then this wavelength will be equal to the Compton wavelength) [8]

$$\lambda = \frac{h}{mc}. \tag{8}$$

The radius of the circle $|\ell_x|$ will be $2\pi|l_x| = \lambda$ (because 2π times the radius is equal to the circumference of the circle)

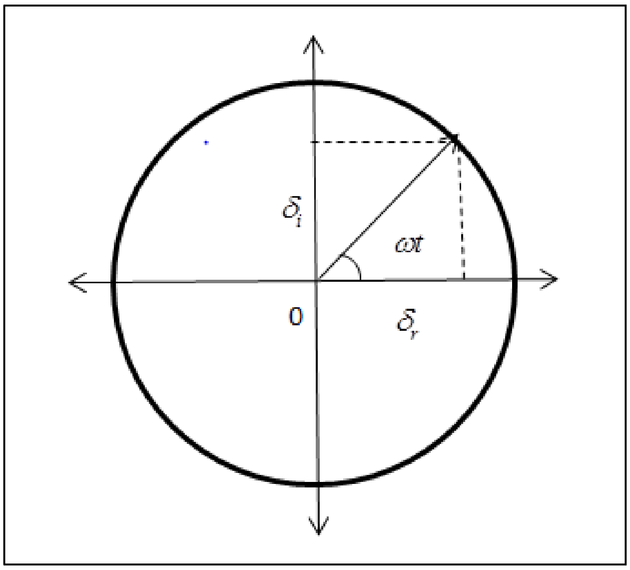

Fig. 3. *Plot of δ_r and δ_i in the complex plane.*

$$|l_x| = \frac{\lambda}{2\pi}. \tag{9}$$

If we try to measure a particle's position which is at rest and located at the origin (see Fig. 4) we will not obtain 0, as $|l_x|$ is non-zero, but rather it will be equal to one-half of $|l_x|$.

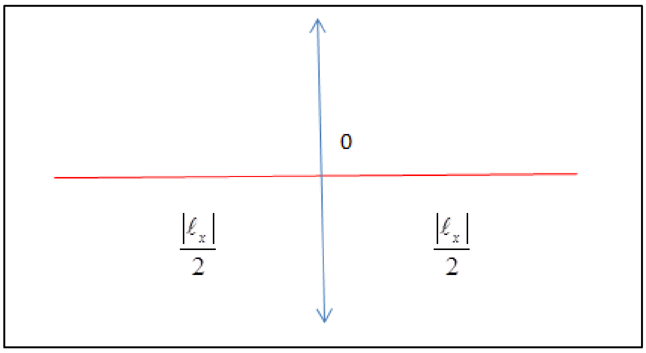

Fig. 4. *Particle at rest situated at the origin.*

Hence we can say that the uncertainty associated with measuring the particle position would be

$$\Delta x = \frac{|\ell_x|}{2} = \frac{\lambda}{4\pi} = \frac{h}{4\pi mc}. \tag{10}$$

The momentum of the particle is given by the de Broglie relation [9]

$$p = \frac{h}{\lambda}. \quad (11)$$

If we compare Eq. (11) with Eq. (8), we find the momentum to be

$$p = mc. \quad (12)$$

When a particle is at rest, the momentum of the particle is zero. However, if we imagine the particle is in a state of vibration then the momentum associated with the particle, even at rest, is non-zero because the particle has some internal velocity associated with vibration.

Hence we can say that the uncertainty in measuring the momentum of the particle is mc

$$\Delta p_x = mc. \quad (13)$$

If we multiply Eqs. (13) and (10), we obtain the following expression

$$\Delta x \Delta p_x = \frac{h}{4\pi mc} mc. \quad (14)$$

$$\Delta x \Delta p_x = \frac{\hbar}{2}. \quad (15)$$

From Eq. (15), we can see that the lower limit of precision for the complementary pair, position and momentum, arises from the particle's vibration in the complex vector space. There is an uncertainty because we measure complementary pairs in a real vector space. If we take the complex vector space into consideration, this so-called uncertainty actually gives us a measurement of the internal measurement of the particle.

4. Proof that Single-Slit Diffraction Is Due to the Complex Vector Space

Diffraction by a single slit is usually explained with the help of Heisenberg's Uncertainty Principle [10]. However, with the introduction of the complex vector space, it is paramount that an account of diffraction be also given. If classical physics is assumed where Heisenberg's Uncertainty Principle does not apply, the output from the slit would be a single peak in intensity, as depicted in Fig. 5. Nevertheless, the real output is an interference pattern illustrated in Fig. 6 [11]

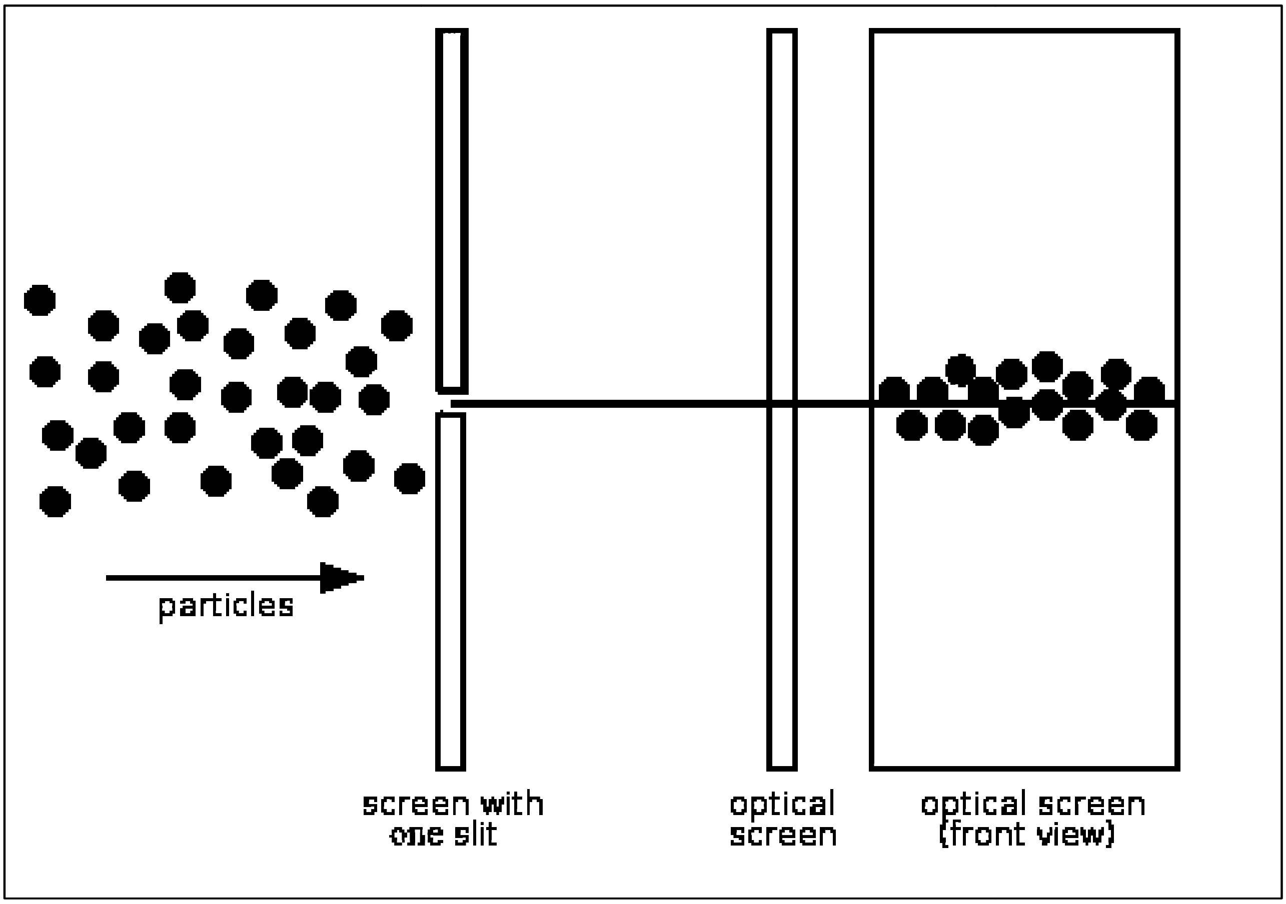

Fig. 5. *Classical physics depiction of the single-slit diffraction output, without Heisenberg's Uncertainty Principle.*

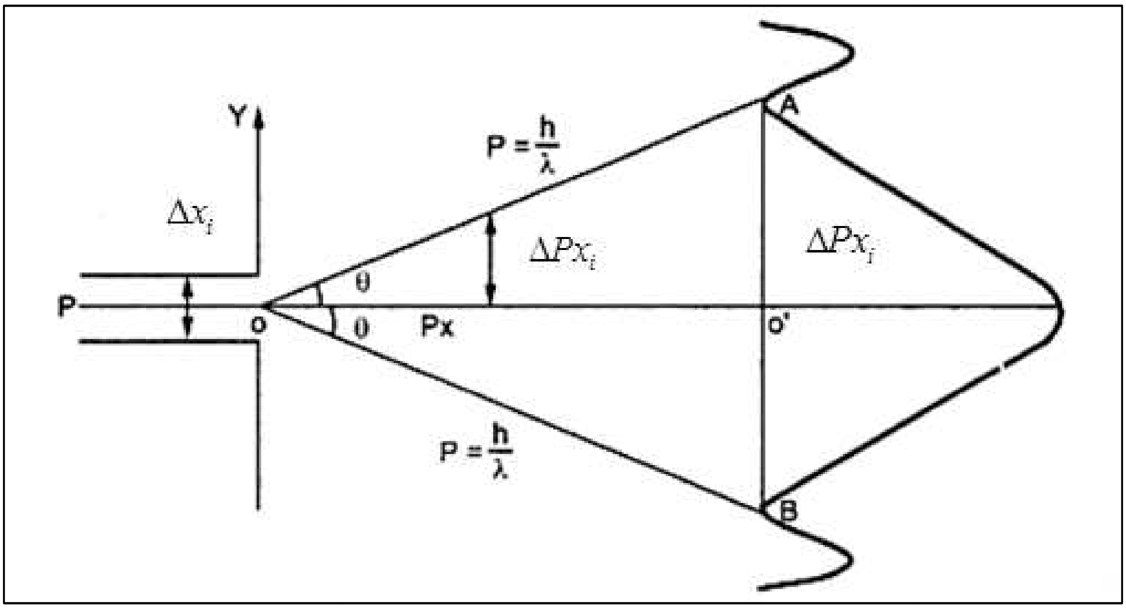

Fig. 6. Actual output of single slit experiment produces interference.

In Fig. 6, we denote the uncertainty in momentum in the y direction as $\Delta P x_i$ and the shift in position as Δx_i, but note that we are considering these variables in a complex vector space. The product of Δx_i and $\Delta P x_i$ for a single slit experiment is given as [12]

$$\Delta x_i \Delta P x_i = h \,. \tag{16}$$

In this experiment only the imaginary component of the complex vector is considered, which is why from Eq. (6) we can say that the uncertainty in position along the imaginary line is given by

$$\Delta x_i = |\ell_x| \cos(\omega t) \,. \tag{17}$$

In Eq. (11) the whole circumference was taken into account; here we need only consider the imaginary line, so $\Delta P x_i$ can be expressed as

$$\Delta P x_i = \frac{h}{|\ell_x| \cos(\omega t)} \,. \tag{18}$$

Multiplying Eqs. (17) and (18), we obtain

$$\Delta x_i \Delta P x_i = h \tag{19}$$

As Eqs (19) and (16) are equal, this opens the possibility that the diffraction produced by a single slit is due to the vibration of the particle in a complex vector space.

5. Derivation of the Intensity of a Single-Slit Diffraction with the Help of a Complex Vector Space

Suppose electrons are fired at a slit of finite width a, as depicted in Fig. 7.

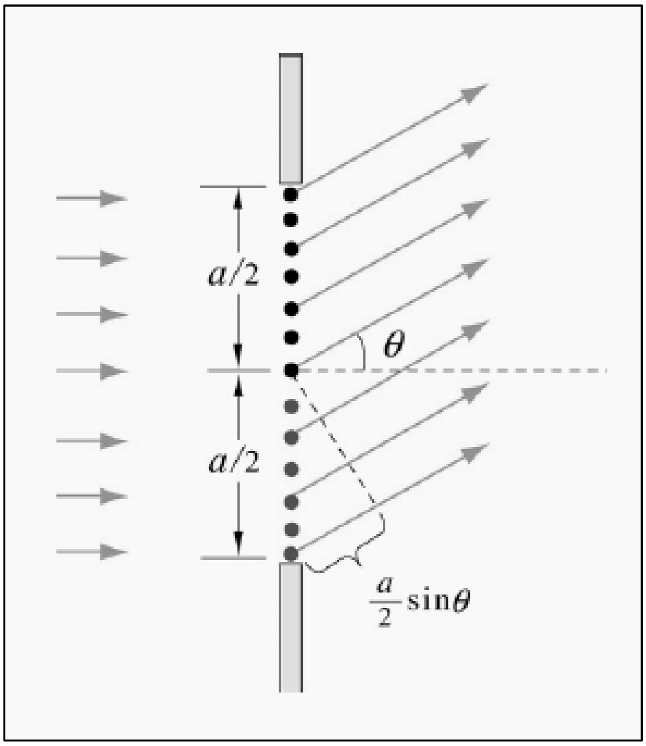

Fig. 7. Diffraction of an electron by slit of width a.

Let us divide the slit into N equal parts of length $\Delta x_i = \frac{a}{N}$. Any two adjacent zones have a relative path length $\delta = \Delta x_i \sin(\theta)$. The relative phase shift $\Delta\beta$ is given by the ratio

$$\frac{\Delta\beta}{2\pi} = \frac{\delta}{\lambda} = \frac{\Delta x_i \sin(\theta)}{\lambda} \,, \tag{20}$$

$$\Delta\beta = \frac{2\pi}{\lambda}\Delta x_i \sin(\theta) \tag{21}$$

Suppose a matter wave originating from point 1 in the slit arrives at a point P on a screen as depicted in Fig 8. The wave can be expressed as

$$(x_r)_1 = |\ell_x|\sin(\omega t) . \tag{22}$$

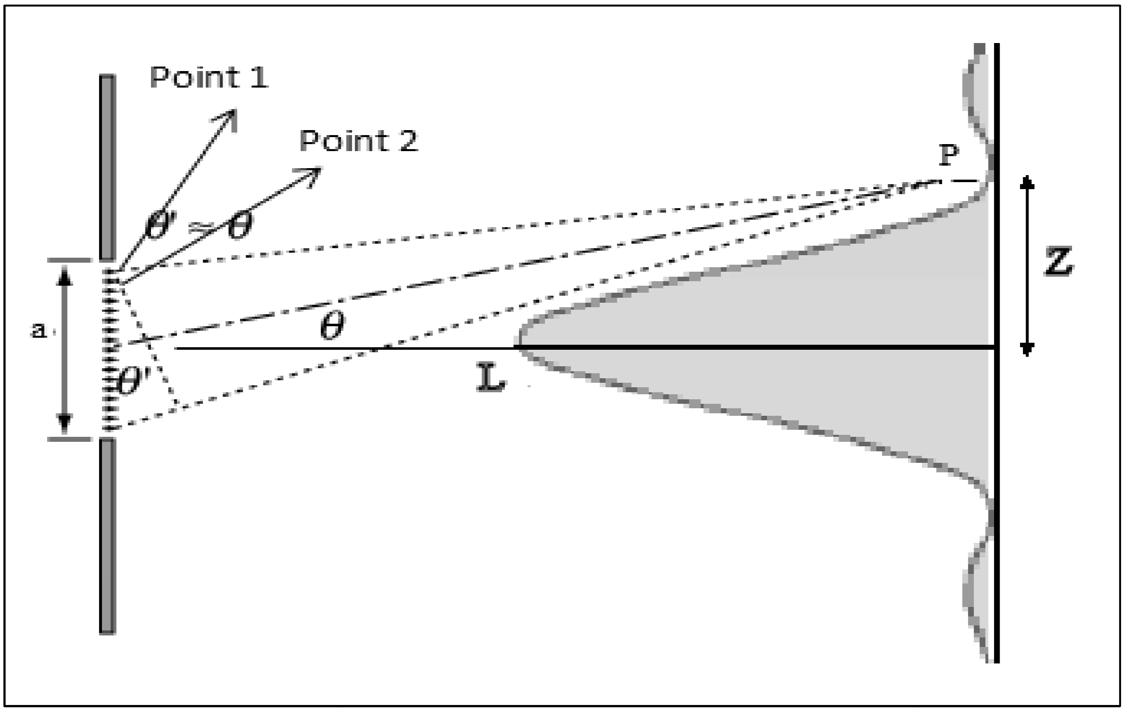

Fig. 8. *Diffraction of an electron by a single slit of width a from two adjacent points.*

A wave from point 2 in the slit (Fig. 8) will have a phase shift of $\Delta\beta$ at point P with respect to point 1, and hence has the wave form

$$(x_r)_2 = |\ell_x|\sin(\omega t + \Delta\beta) . \tag{23}$$

A wave from point N can be expressed as

$$(x_r)_N = |\ell_x|\sin(\omega t + (N-1)\Delta\beta) . \tag{24}$$

At point P on the screen, all these waves are superimposed giving

$$(x_r)_1 + (x_r)_2 + ... + (x_r)_N = |\ell_x|(\sin(\omega t) + \sin(\omega t + \Delta\beta) + ... + \sin(\omega t + (N-1)\Delta\beta)) . \tag{25}$$

The phase shift between the 1st point and the Nth point is

$$\beta = N\Delta\beta = \frac{2\pi}{\lambda}N\Delta x_i \sin(\theta) = \frac{2\pi}{\lambda}a\sin(\theta) . \tag{26}$$

From Appendix A, we know that

$$\left\langle X_r^{\ 2}\right\rangle = \frac{N^2\lambda^2}{8\pi^2}\left[\frac{\sin(\frac{\beta}{2})}{\frac{\beta}{2}}\right]^2 , \tag{27}$$

Substituting the value of β in Eq. (26) into Eq. (27), we get

$$\left\langle X_r^{\ 2}\right\rangle = \frac{N^2\lambda^2}{8\pi^2}\left[\frac{\sin(\frac{\pi}{\lambda}a\sin(\theta))}{\frac{\pi}{\lambda}a\sin(\theta)}\right]^2 . \tag{28}$$

The probability distribution function $\rho(x)$ for particles to strike a point on the screen is given by following expression [14]

$$\rho(x) = \frac{a}{\lambda L}\frac{\sin\left(\frac{\pi a x}{\lambda L}\right)^2}{\left(\frac{\pi a x}{\lambda L}\right)^2} . \tag{29}$$

If we compare Eqs. (28) and (29), we see that $\left\langle X_r^{\ 2}\right\rangle$ is related with $\rho(x)$; this proves that the intensity difference on the screen arising in single-slit diffraction is due to the vibration of the particle in complex vector space.

6. Energy and Time Uncertainty Relationship to Complex Plane

The uncertainty relationship between energy and time is given by Heisenberg's uncertainty principle [15]

$$\Delta E \Delta t = \frac{\hbar}{2}. \quad (30)$$

The energy is correlated with the momentum, and the momentum is expressed in terms of the coordinates of the complex vector space, which is why energy is supposed to be expressed in terms of a complex number (energy is not a vector quantity and being scalar in real plane , energy has to be a complex number when we consider complex vector space). Similarly, time also needs to be expressed in terms of a complex number; the imaginary part of this time is the inverse of the frequency associated with the vibration of the particle or photon whereas the real part of this time is the normal time we measure in terms of the frequency associated with the movement of the particle or photon.

It is important to find the energy relation in this complex space because $E = mc^2$ only accounts for the real component of the energy. We begin by imagining a stationary box in a space; a photon is emitted from one side of the box and absorbed on the other. If the photon's energy is denoted by E and the speed of light by c, then from the Maxwell expression we can write [16]

$$p_{photon} = \frac{E}{c}. \quad (31)$$

Again this term only accounts for the real component of photon's momentum and we know that momentum must be expressed as a complex number. If we perform single-slit diffraction with photons, we obtain a similar pattern as for electrons, or indeed any particle. As photon has no mass and always moves at the speed of light, we can say that both real and imaginary components of photon must be equal. This is why the modified expression of Eq. (46) should be

$$P_{photon} = p_{photon}(1+i) = \frac{E}{c} \quad (32)$$

When the photon initially leaves the box from one side, the box will recoil with speed v and from conversation of momentum, the box should gain the same amount of momentum as the photon. The momentum gained by the box (of mass M) in complex vector space will be

$$p_{box} = Mv + iMv. \quad (33)$$

A photon will take a finite time Δt to reach the other side of the box; by that time the box would have moved a distance Δx, hence the velocity of the box can be written as

$$v = \frac{\Delta x}{\Delta t}. \quad (34)$$

By conservation of momentum, P_{photon} and p_{box} are equal and hence we can say that

$$Mv + iMv = \frac{E}{c}. \quad (35)$$

Substituting the velocity of Eq. (34) into Eq. (35) yields

$$M\frac{\Delta x}{\Delta t}(1+i) = \frac{E}{c}. \quad (36)$$

If the box is of length L, then the time taken by the photon to reach the other side is

$$\Delta t = \frac{L}{c}. \quad (37)$$

Substituting value of Eq. (37) into Eq. (36) gives

$$M\Delta x(1+i) = \frac{EL}{c^2}. \quad (38)$$

Let us assume that photon has mass m and is located at a distance x_2 and the box has position x_1. Then the centre of mass is given by the following expression [17]

$$\bar{x} = \frac{Mx_1 + mx_2}{M+m}. \quad (39)$$

Similarly, the centre of mass when the photon reaches the other side of the box can be expressed as

$$\bar{x}_{right} = \frac{M(x_1 + \Delta x) + mL}{M+m}. \quad (40)$$

We require that the centre of mass of the system does not change; this is why the centre of mass at the start of the experiment ($\bar{x}$) should be equal to the centre of mass at the end of the experiment ($\bar{x}_{right}$).

$$\frac{Mx_1 + mx_2}{M+m} = \frac{M(x_1 + \Delta x) + mL}{M+m}. \quad (41)$$

At the start of the experiment, the photon is at position x_2 =0. Eq. (41) then gives

$$mL = M\Delta x, \quad (42)$$

which substituted into Eq. (38) yields

$$m(1+i) = \frac{E}{c^2}. \quad (43)$$

That is,

$$E = mc^2 + imc^2. \quad (44)$$

The imaginary component in Eq. (44) is due to the vibration motion in the complex plane. This is why we should associate the imaginary component with the wave. From de Broglie's relation, we know that $mc^2 = \hbar\omega$ [18]. Putting this expression in Eq. (44), we get

$$E = mc^2 + i\hbar\omega \tag{45}$$

We can write the expression for the photon's energy by rewriting Eq. (32) as

$$E = pc + ipc\,. \tag{46}$$

Knowing that $pc = \hbar\omega$, we can substitute this expression into the imaginary term of Eq. (46) to obtain

$$E = pc + i\hbar\omega \tag{47}$$

The above Eqs. (45) and (47) infers that matter/photon is both a wave and a particle simultaneously, which was recently proved experimentally [19].

7. Probabilistic Interpretation of Wave Mechanics

Ψ is always connected to factor exp(iα), which disappears when one construct the real probability quantities and consequently is of no importance, which is used to normalize and expressed in terms of probability [20]. Wave functions like electromagnetic wave depends on function of E and H, but Ψ units keeps on changing as per dimensions of space one chose [21]. Ψ is function of one of the complimentary pair being measured, as of now we consider these physical quantities in real terms and hence it leads to uncertainty which in terns relates Ψ with probability. Complementary variable too have complex factors (like momentum, position, energy and time) and hence Ψ too is complex unlike other wave equations like electromagnetic waves and vibrating string. As Ψ depends on physical quantity it units keeps on changing. Probabilistic interpretation of wave function actually gives us information of underlying physical quantity in complex plane.

8. Conclusion

Heisenberg uncertainty principle is boundary condition where complementary variables like (position and time, energy and time) sets limitation on accuracy to find both the variable simultaneously, but as you accurately measure one of the variable you get imaginary plane information of another variable with real plane information, this is not uncertainty but information about another variable in both real plane and complex plane. Thus to define state of any particle or photon position, momentum ,energy , time and other complementary pair should be considered as complex variable rather than real variable to define sate of any particle or photon.

Appendix A: Derivation of the Intensity for Single-Slit Diffraction for Electrons

From basic trigonometry, we know that [13]

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta\,. \tag{48}$$

We can cast Eq. (48) in terms of waves from the successive points on the slit:

$$\cos(\omega t - \frac{\Delta\beta}{2}) - \cos(\omega t + \frac{\Delta\beta}{2}) = 2\sin(\omega t)\sin(\frac{\Delta\beta}{2})\,, \tag{49}$$

$$\cos(\omega t + \frac{\Delta\beta}{2}) - \cos(\omega t + \frac{3\Delta\beta}{2}) = 2\sin(\omega t + \Delta\beta)\sin(\frac{\Delta\beta}{2})\,, \tag{50}$$

For point N, we have

$$\cos(\omega t + (N - \frac{3}{2})\Delta\beta) - \cos(\omega t + (N - \frac{1}{2})\Delta\beta) = 2\sin(\omega t + (N-1)\Delta\beta)\sin(\frac{\Delta\beta}{2})\,. \tag{51}$$

Adding all N terms gives the following result

$$\cos(\omega t + \frac{\Delta\beta}{2}) - \cos(\omega t + (N - \frac{1}{2})\Delta\beta) = 2\sin(\frac{\Delta\beta}{2})(\sin(\omega t) + \sin(\omega t + \Delta\beta) + \ldots.. + \sin(\omega t + (N-1)\Delta\beta)) \tag{52}$$

The terms on the left hand side combine to yield

$$\cos(\omega t + \frac{\Delta\beta}{2}) - \cos(\omega t + (N - \frac{1}{2})\Delta\beta) = 2\sin(\omega t + (N-1)\frac{\Delta\beta}{2})\sin(N\frac{\Delta\beta}{2}) \tag{53}$$

Substituting Eq. (53) into Eq. (52) gives

$$\sin(\omega t) + \sin(\omega t + \Delta\beta) + \ldots.. + \sin(\omega t + (N-1)\Delta\beta) = \frac{\sin(\omega t + (N-1)\frac{\Delta\beta}{2})\sin(N\frac{\Delta\beta}{2})}{\sin(\frac{\Delta\beta}{2})} \tag{54}$$

Substituting Eq. (54) in Eq. (25) yields an expression for the total wave X_i,

$$X_r = (x_r)_1 + (x_r)_2 + ...+ (x_r)_N = \frac{|\ell_x| \sin(\omega t + (N-1)\frac{\Delta\beta}{2}) \sin(N\frac{\Delta\beta}{2})}{\sin(\frac{\Delta\beta}{2})}. \tag{55}$$

Squaring both the sides,

$$X_i^2 = \frac{|\ell_x|^2 \sin(N\frac{\Delta\beta}{2})^2}{\sin(\frac{\Delta\beta}{2})^2} \sin(\omega t + (N-1)\frac{\Delta\beta}{2})^2, \tag{56}$$

and then taking the time average for both sides

$$\left\langle X_r^2 \right\rangle = \frac{|\ell_x|^2 \sin(N\frac{\Delta\beta}{2})^2}{\sin(\frac{\Delta\beta}{2})^2} \left\langle \sin(\omega t + (N-1)\frac{\Delta\beta}{2})^2 \right\rangle \tag{57}$$

gives

$$\left\langle X_r^2 \right\rangle = \frac{1}{2}|\ell_x|^2 \left[\frac{\sin(N\frac{\Delta\beta}{2})}{\sin(\frac{\Delta\beta}{2})} \right]^2, \tag{58}$$

where the time average of $\left\langle \sin(\omega t + (N-1)\frac{\Delta\beta}{2})^2 \right\rangle$ .is $\frac{1}{2}$. Substituting first Eq. (9) in Eq. (58),

$$\left\langle X_r^2 \right\rangle = \frac{\lambda^2}{8\pi^2} \left[\frac{\sin(N\frac{\Delta\beta}{2})}{\sin(\frac{\Delta\beta}{2})} \right]^2, \tag{59}$$

and subsequently Eq. (26) in Eq. (59), we obtain

$$\left\langle X_r^2 \right\rangle = \frac{\lambda^2}{8\pi^2} \left[\frac{\sin(\frac{\beta}{2})}{\sin(\frac{\Delta\beta}{2})} \right]^2. \tag{60}$$

In taking the limit $\Delta\beta \to 0$, we replace $\sin(\frac{\Delta\beta}{2})$ by approximation $\frac{\Delta\beta}{2}$, to obtain

$$X_r = \frac{\lambda^2}{8\pi^2} \left[\frac{\sin(\frac{\beta}{2})}{\frac{\Delta\beta}{2}} \right]^2. \tag{61}$$

Multiplying and dividing the denominator by N^2, we get

$$\left\langle X_r^2 \right\rangle = \frac{N^2\lambda^2}{8\pi^2} \left[\frac{\sin(\frac{\beta}{2})}{\frac{N\Delta\beta}{2}} \right]^2, \tag{62}$$

and then substituting Eq. (26) in Eq. (62) yields

$$\left\langle X_r^2 \right\rangle = \frac{N^2\lambda^2}{8\pi^2} \left[\frac{\sin(\frac{\beta}{2})}{\frac{\beta}{2}} \right]^2, \tag{63}$$

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