Hypermathematics, H_v -Structures, Hypernumbers, Hypermatrices and Lie-Santilli Admissibility

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Abstract: We present the largest class of hyperstructures called H_v -structures. In H_v -groups and H_v -rings, the fundamental relations are defined and they connect the algebraic hyperstructure theory with the classical one. Using the fundamental relations, the H_v -fields are defined and their elements are called hypernumbers or H_v -numbers. H_v -matrices are defined to be matrices with entries from an H_v -field. We present the related theory and results on hypermatrices and on the Lie-Santilli admissibility.

Keywords: Representations, Hope, Hyperstructures, H_v -Structures

1. Introduction to Hypermathematics, the H_v -Structures

Hyperstructure is called an algebraic structure containing at least one hyperoperation. More precisely, a set H equipped with at least one multivalued map $\cdot : H \times H \rightarrow P(H)$, is called hyperstructure and the map hyperoperation, we abbreviate hyperoperation by hope. The first hyperstructure was the hypergroup, introduced by F. Marty in 1934 [25], [26], where the strong generalized axioms of a group were used. We deal with the largest class of hyperstructures called H_v -structures introduced in 1990 [40],[44],[45] which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions:

Definitions 1.1 In a set H with a hope $\cdot : H \times H \rightarrow P(H)$, we abbreviate by WASS the weak associativity: $(xy)z \cap x(yz) \neq \emptyset$, $\forall x, y, z \in H$ and by COW the weak commutativity: $xy \cap yx \neq \emptyset$, $\forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called H_v -semigroup if it is WASS and is called H_v -group if it is reproductive H_v -semigroup:

$$xH = Hx = H, \forall x \in H.$$

The hyperstructure $(R, +, \cdot)$ is called H_v -ring if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, (x+y)z \cap (xz+yz) \neq \emptyset, \forall x, y, z \in R.$$

For definitions, results and applications on H_v -structures, see books [44],[4],[10],[12] and papers [6],[7],[8],[9],[11],[17],[18],[19],[22],[24],[46]. An extreme class is defined as follows [41],[44]: An H_v -structure is very thin iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, a very thin H_v -structure is an H with a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $\text{card} A > 1$, and all the other products, are singletons.

The main tools to study hyperstructures are the so called, fundamental relations. These are the relations β^* and γ^* which are defined, in H_v -groups and H_v -rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [38],[40],[44],[48],[49]. The way to find the fundamental classes is given as follows [44]:

Theorem 1.2 Let $(H, \cdot)$ be an H_v -group and let us denote by U the set of all finite products of elements of H . We define the relation β in H as follows: $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then the fundamental relation β^* is the transitive closure of the relation β .

The main point of the proof is that β guaranties that the following is valid: Take elements x, y such that $\{x, y\} \subset u \in U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Thus, if the 'hyperproducts' of the above

β -classes are 'products', then, they are fundamental classes. Analogously for the γ in H_v -rings.

An element is called single if its fundamental class is a singleton.

Motivation for H_v -structures:

1. The quotient of a group with respect to an invariant subgroup is a group.

2. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.

3. The quotient of a group with respect to any partition is an H_v -group.

In H_v -structures a partial order can be defined [44].

Definition 1.3 Let $(H, \cdot)$, $(H, \otimes)$ be H_v -semigroups defined on the same H . $(\cdot)$ is smaller than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, iff there exists automorphism $f \in \text{Aut}(H, \otimes)$ such that $xy \subset f(x \otimes y)$, $\forall x \in H$.

Then $(H, \otimes)$ contains $(H, \cdot)$ and write $\cdot \leq \otimes$. If $(H, \cdot)$ is structure, then it is called basic and $(H, \otimes)$ is an H_b -structure.

The Little Theorem [26]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v -field one uses the fundamental relation γ^* :

Definition 1.4 [40],[43],[44]. The H_v -ring $(R, +, \cdot)$ is an H_v -field if the quotient R/γ^* is a field.

The elements of an H_v -field are called hypernumbers. Let ω^* be the kernel of the canonical map and from H_v -ring R to R/γ^* ; then we call it reproductive H_v -field if:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*, \forall x \in R-\omega^*.$$

From this definition a new class is defined [51],[56]:

Definition 1.5 The H_v -semigroup $(H, \cdot)$ is called h/v -group if the H/β^* is a group.

An H_v -group is called cyclic [33],[44], if there is an element, called generator, which the powers have union the underline set, the minimal power with this property is the period of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the H_v -group is called single-power cyclic.

To compare classes we can see the small sets. To enumerate and classify H_v -structures, is complicate because we have great numbers. The partial order [44],[47], restrict the problem in finding the minimal, up to isomorphisms, H_v -structures. We have results by Bayon & Lygeros as the following [2],[3]: In sets with three elements: Up to isomorphism, there are 6.494 minimal H_v -groups. The 137 are abelian; 6.152 are cyclic. The number of H_v -groups with three elements is 1.026.462. 7.926 are abelian; 1.013.598 are cyclic, 16 are very thin. Abelian H_v -groups with 4 elements are, 8.028.299.905 from which the 7.995.884.377 are cyclic.

Some more complicated hyperstructures can be defined, as well. In this paper we focus on H_v -vector spaces and there exist an analogous theory on H_v -modules.

Definition 1.6 [44],[50]. Let $(F, +, \cdot)$ be an H_v -field, $(M, +)$ be COW H_v -group and there exists an external hope

$$F \times M \rightarrow P(M): (a, x) \rightarrow ax,$$

such that, $\forall a, b \in F$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then M is called an H_v -vector space over F .

The fundamental relation ε^* is defined to be the smallest equivalence such that the quotient M/ε^* is a vector space over the fundamental field F/γ^* . For this fundamental relation there is an analogous to the Theorem 1.2.

Definitions 1.7 [51],[53],[55]. Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. We say that $h \in H$ absorbs $h \in H$ if we replace h by h and h does not appear in the structure. We say that $h \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by h , the union of the results of x with both h , h , and consider h and h as one class, with representative h , therefore the element h does not appeared in the hyperstructure.

Let $(H, \cdot)$ be an H_v -group, then, if an element h absorbs all elements of its own fundamental class then this element becomes a single in the new H_v -group.

Theorem 1.8 In an H_v -group $(H, \cdot)$, if an element h absorbs all elements of its fundamental class then this element becomes a single in the new H_v -group.

Proof. Let $h \in \beta^*(h)$, then, by the definition of the 'absorb', h is replaced by h that means that $\beta^*(h) = \{h\}$. Moreover, for all $x \in H$, the fundamental property of the product of classes

$$\beta^*(x) \cdot \beta^*(h) = \beta^*(xh) \text{ becomes } \beta^*(x) \cdot h = \beta^*(xh),$$

and from the reproductivity ([44] p.19) we obtain $x \cdot h = \beta^*(xh)$, $\forall x \in \beta^*(x)$. This is the basic property that enjoys any single element [44].

Remark that in case we have a single element then we can compute all fundamental classes.

A well known and large class of hopes is given as follows [33],[37],[39],[44],[20]:

Definitions 1.9 Let $(G, \cdot)$ be a groupoid, then for every subset $P \subset G$, $P \neq \emptyset$, we define the following hopes, called P -hopes: $\forall x, y \in G$

$$P: xPy = (xP)y \cup x(Py),$$

$$P_r: xP_r y = (xy)P \cup x(yP), P_l: xP_l y = (Px)y \cup P(xy).$$

The (G, P) , (G, P_r) and (G, P_l) are called P -hyperstructures. In the case of semigroup $(G, \cdot)$: $xPy = (xP)y \cup x(Py) = xPy$ and (G, P) is a semihypergroup but we do not know about (G, P_r) and (G, P_l) . In some cases, depending on the choice of P , the (G, P_r) and (G, P_l) can be associative or WASS.

A generalization of P -hopes is the following [13],[14]: Let $(G, \cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} \text{ if } x \neq e \text{ and } y \neq e$$

$$x \cdot y \text{ if } x = e \text{ or } y = e$$

we call this hope, P_e -hope. The hyperstructure $(G, \times_P)$ is an abelian H_v -group.

A general definition of hopes, is the following [57],[58]:

Definitions 1.10 Let H be a set with n operations (or hopes) $\otimes_1, \otimes_2, \dots, \otimes_n$ and one map (or multivalued map) $f: H \rightarrow H$, then n hopes $\partial_1, \partial_2, \dots, \partial_n$ on H are defined, called ∂ -hopes by putting

$$x\partial_i y = \{f(x)\otimes_i y, x\otimes_i f(y)\}, \forall x, y \in H, i \in \{1, 2, \dots, n\}$$

or in case where $\otimes_i$ is hope or f is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y) \cup (x\otimes_i f(y)), \forall x, y \in H, i \in \{1, 2, \dots, n\}$$

Let $(G, \cdot)$ groupoid and $f_i: G \rightarrow G, i \in I$, set of maps on G . Take the map $f_\cup: G \rightarrow P(G)$ such that $f_\cup(x) = \{f_i(x) \mid i \in I\}$, call it the union of the $f_i(x)$. We call the union ∂ -hope (∂), on G if we consider the map $f_\cup(x)$. An important case for a map f , is to take the union of this with the identity id . Thus, we consider the map $f \equiv f \cup (\text{id})$, so $f(x) = \{x, f(x)\}, \forall x \in G$, which is called b - ∂ -hope, we denote it by (∂) , so we have

$$x\partial y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G.$$

Remark If $\otimes_i$ is associative then ∂_i is WASS. If ∂ contains the operation $(\cdot)$, then it is b -operation. Moreover, if $f: G \rightarrow P(G)$ is multivalued then the b - ∂ -hopes is defined by using the $f(x) = \{x\} \cup f(x), \forall x \in G$.

Motivation for the definition of ∂ -hope is the derivative where only multiplication of functions is used. Therefore, for functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$, $(\cdot)$ is the derivative.

Example. For all first degree polynomials $g_i(x) = a_i x + b_i$, we have

$$g_1 \partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so it is a hope in the set of first degree polynomials. Moreover all polynomials $x+c$, where c be a constant, are units.

There exists the uniting elements method introduced by Corsini-Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

Definition 1.11 The uniting elements method is the following: Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d . The quotient by this partition G/d is an H_v -structure. Then, quotient out the H_v -structure G/d by the fundamental relation β^* , a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

An interesting application of the uniting elements is when more than one property is desired, because some of the properties lead straight to the classes. The commutativity and the reproductivity property are easily applicable. The following is valid:

Theorem 1.12 [44] Let $(G, \cdot)$ be a groupoid, and

$$F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$$

be a system of equations on G consisting of two subsystems

$$F_m = \{f_1, \dots, f_m\} \text{ and } F_n = \{f_{m+1}, \dots, f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

i.e. the following diagram is commutative

$$\begin{array}{ccccc} G & \xrightarrow{\rho_m} & G/\sigma_m & \xrightarrow{\varphi_m} & G_m \\ \rho \downarrow & & & & \downarrow \rho_n \\ G/\sigma & & & & G_m/\sigma_n \\ \varphi \downarrow & & \cong & & \downarrow \varphi_n \\ (G/\sigma)/\beta^* & \xrightarrow{\quad} & & & (G_m/\sigma_n)/\beta^* \end{array}$$

From the above it is clear that the fundamental structure is very important, and even more so if this is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures [44].

Theorem 1.13 Let $(S, \cdot)$ be a commutative semigroup with one element $w \in S$ such that the set wS is finite. Consider the transitive closure L^* of the relation L defined as follows: xLy iff there exists $z \in S$ such that $zx = zy$.

Then $\langle S/L^*, \circ \rangle / \beta^*$ is finite commutative group, where $(\circ)$ is the induced operation on classes of S/L^* .

For the proof see [5],[44].

An application combining hyperstructures and fuzzy theory, is to replace the 'scale' of Likert in questionnaires by the bar of Vougiouklis & Vougiouklis [69],[70],[21],[27]:

Definition 1.14 In every question substitute the Likert scale with the 'bar' whose poles are defined with '0' on the left end, and '1' on the right end:

$$0 \text{ ————— } 1$$

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm. The hyperstructure theory, offer innovating new suggestions to connect finite groups of objects. These suggestions are obtained from properties and special elements inside the hyperstructure.

2. Hyper-Representations

Representations (abbreviate by rep) of H_v -groups can be faced either by generalized permutations or by H_v -matrices [34],[36],[39],[43],[44],[52],[54],[66]. Reps by generalized permutations can be achieved by using translations [42]. We

present an outline of the hypermatrix rep in H_v -structures and there exist the analogous theory for the h/v -structures.

Definitions 2.1 [44],[66] H_v -matrix is a matrix with entries elements of an H_v -field. The hyperproduct of two H_v -matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined, in the usual manner,

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{ C = (c_{ij}) \mid c_{ij} \in \bigoplus_k a_{ik} \cdot b_{kj} \},$$

and it is a set of $m \times r$ H_v -matrices. The sum of products of elements of the H_v -field is the union of the sets obtained with all possible parentheses put on them, called n -ary circle hope on the hyperaddition.

The hyperproduct of H_v -matrices does not satisfy WASS.

The problem of the H_v -matrix reps is the following:

Definitions 2.2 For a given H_v -group $(H, \cdot)$, find an H_v -field $(F, +, \cdot)$, a set $M_R = \{(a_{ij}) \mid a_{ij} \in F\}$ and a map $T: H \rightarrow M_R: h \rightarrow T(h)$ such that

$$T(h_1 h_2) \cap T(h_1) T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

The map T is called H_v -matrix rep. If $T(h_1 h_2) \subset T(h_1) T(h_2)$, $\forall h_1, h_2 \in H$, then T is called inclusion rep. T is a good rep if $T(h_1 h_2) = T(h_1) T(h_2) = \{T(h) \mid h \in h_1 h_2\}$, $\forall h_1, h_2 \in H$. If T is one to one and good then it is a faithful rep.

The problem of reps is complicated since the hyperproduct is big. It can be simplified in cases such as: The H_v -matrices are over H_v -fields with scalars 0 and 1. The H_v -matrices are over very thin H_v -fields. On 2×2 H_v -matrices, since the circle hope coincides with the hyperaddition. On H_v -fields which contain singles, which act as absorbings.

The main theorem of reps is the following [44],[52]:

Theorem 2.3 A necessary condition in order to have an inclusion rep T of an H_v -group $(H, \cdot)$ by $n \times n$ H_v -matrices over the H_v -field $(F, +, \cdot)$ is the following:

For all classes $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}$$

Thus, every inclusion rep $T: H \rightarrow M_R: a \rightarrow T(a) = (a_{ij})$ induces a homomorphic rep T^* of the group H/β^* over the field F/γ^* by setting

$$T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall \beta^*(a) \in H/\beta^*,$$

where $\gamma^*(a_{ij}) \in F/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T^* is called fundamental induced rep of T .

Denote $\text{tr}_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T: H \rightarrow F/\gamma^*: x \rightarrow X_T(x) = \text{tr}_\varphi(T(x)) = \text{tr} T^*(x)$$

is called fundamental character.

Using special classes of H_v -structures one can have several reps [52],[66]:

Definition 2.4 Let $M = M_{m \times n}$ be vector space of $m \times n$ matrices over a field F and take sets

$$S = \{s_k: k \in K\} \subseteq F, Q = \{Q_j: j \in J\} \subseteq M, P = \{P_i: i \in I\} \subseteq M.$$

Define three hopes as follows

$$S: F \times M \rightarrow P(M): (r, A) \rightarrow rSA = \{(rs_k)A: k \in K\} \subseteq M$$

$$Q_+: M \times M \rightarrow P(M): (A, B) \rightarrow AQ_+B = \{A + Q_j + B: j \in J\} \subseteq M$$

$$P: M \times M \rightarrow P(M): (A, B) \rightarrow APB = \{AP_i B: i \in I\} \subseteq M$$

Then (M, S, Q_+, P) is a hyperalgebra over F called general matrix P -hyperalgebra.

The bilinear hope P , is strong associative and the inclusion distributivity with respect to addition of matrices

$$AP(B+C) \subseteq APB + APC, \forall A, B, C \in M$$

is valid. So $(M, +, P)$ defines a multiplicative hyperring on non-square matrices.

In a similar way a generalization of this hyperalgebra can be defined considering an H_v -field instead of a field and using H_v -matrices instead of matrices.

In the representation theory several constructions are used, one can find some of them as follows [43],[44],[52],[54]:

Construction 2.5 Let $(H, \cdot)$ be H_v -group, then for all $(\oplus)$ such that $x \oplus y \supset \{x, y\}$, $\forall x, y \in H$, the $(H, \oplus, \cdot)$ is an H_v -ring. These H_v -rings are called associated to $(H, \cdot)$ H_v -rings.

In rep theory of hypergroups, in sense of Marty where the equality is valid, there are three associated hyperrings $(H, \oplus, \cdot)$ to $(H, \cdot)$. The $(\oplus)$ is defined respectively, $\forall x, y \in H$, by:

type a: $x \oplus y = \{x, y\}$, type b: $x \oplus y = \beta^*(x) \cup \beta^*(y)$, type c: $x \oplus y = H$

In the above types the strong associativity and strong or inclusion distributivity, is valid.

Construction 2.6 Let $(H, \cdot)$ be an H_v -semigroup and $\{v_1, \dots, v_n\} \cap H = \emptyset$, an ordered set, where if $v_i < v_j$, when $i < j$. Extend $(\cdot)$ in $H_n = H \cup \{v_1, v_2, \dots, v_n\}$ as follows:

$$x \cdot v_i = v_i \cdot x = v_i, v_i \cdot v_j = v_j \cdot v_i = v_j, \forall i < j \text{ and}$$

$$v_i \cdot v_i = H \cup \{v_1, \dots, v_{i-1}\}, \forall x \in H, i \in \{1, 2, \dots, n\}.$$

Then $(H_n, \cdot)$ is an H_v -group, called Attach Elements Construction, and $(H_n, \cdot)/\beta^* \cong Z_2$, where v_n is single [51],[55].

Some problems arising on the topic, are:

Open Problems.

a. Find standard H_v -fields to represent all H_v -groups.

b. Find reps by H_v -matrices over standard finite H_v -fields analogous to Z_n .

c. Using matrices find a generalization of the ordinary multiplication of matrices which it could be used in H_v -rep theory (see the helix-hope [68]).

d. Find the 'minimal' hypermatrices corresponding to the minimal hopes.

e. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

Recall some definitions from [68],[16],[32]:

Definitions 2.7 Let $A = (a_{ij}) \in M_{m \times n}$ be $m \times n$ matrix and $s, t \in N$ be natural numbers such that $1 \leq s \leq m$, $1 \leq t \leq n$. Then we define a characteristic-like map $\text{cst}: M_{m \times n} \rightarrow M_{s \times t}$ by corresponding to the matrix A , the matrix $A_{\text{cst}} = (a_{ij})$ where $1 \leq i \leq s$, $1 \leq j \leq t$. We call

it cut-projection of type st. We define the mod-like map $st: M_{m \times n} \rightarrow M_{s \times t}$ by corresponding to A the matrix $Ast=(a_{ij})$ which has as entries the sets

$$a_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in \mathbb{N}, i+\kappa s \leq m, j+\lambda t \leq n\}.$$

Thus we have the map

$$st: M_{m \times n} \rightarrow M_{s \times t}: A \rightarrow Ast=(a_{ij}).$$

We call this multivalued map helix-projection of type st. So Ast is a set of $s \times t$ -matrices $X=(x_{ij})$ such that $x_{ij} \in a_{ij}, \forall i, j$.

Let $A=(a_{ij}) \in M_{m \times n}$, $B=(b_{ij}) \in M_{u \times v}$ matrices and $s=\min(m, u)$, $t=\min(n, v)$. We define a hope, called helix-addition or helix-sum, as follows:

$$\oplus: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{s \times t}):$$

$$(A, B) \rightarrow A \oplus B = Ast + Bst = (a_{ij}) + (b_{ij}) \subset M_{s \times t},$$

where

$$(a_{ij}) + (b_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in a_{ij} \text{ and } b_{ij} \in b_{ij}\}.$$

And define a hope, called helix-multiplication or helix-product, as follows:

$$\otimes: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{m \times v}): (A, B) \rightarrow A \otimes B = Ams \cdot Bsv = (a_{ij}) \cdot (b_{ij}) \subset M_{m \times v},$$

where

$$(a_{ij}) \cdot (b_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in a_{ij} \text{ and } b_{ij} \in b_{ij}\}.$$

Remark. In $M_{m \times n}$ the addition of matrices is an ordinary operation, therefore we are interested only in the ‘product’. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined [62], as well. This algebra is an H_v -Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

For more results on the topic see [16],[32],[61],[62].

In the following we denote E_{ij} any type of matrices which have the ij -entry 1 and in all the other entries we have 0.

Example 2.8 Consider the 2×3 matrices of the following form,

$$A_\kappa = E_{11} + \kappa E_{21} + E_{22} + E_{23}, B_\kappa = \kappa E_{21} + E_{22} + E_{23}, \forall \kappa \in \mathbb{N}.$$

$$\text{Then we obtain } A_\kappa \otimes A_\lambda = \{A_{\kappa+\lambda}, A_{\kappa+1}, B_{\kappa+\lambda}, B_{\kappa+1}\}$$

$$\text{Similarly, } B_\kappa \otimes A_\lambda = \{B_{\kappa+\lambda}, B_{\kappa+1}\}, A_\kappa \otimes B_\lambda = B_\lambda = B_\kappa \otimes B_\lambda.$$

Thus the set $\{A_\kappa, B_\lambda \mid \kappa, \lambda \in \mathbb{N}\}$ becomes an H_v -semigroup which is not COW because for $\kappa \neq \lambda$ we have

$$B_\kappa \otimes B_\lambda = B_\lambda \neq B_\kappa = B_\lambda \otimes B_\kappa,$$

however

$$(A_\kappa \otimes A_\lambda) \cap (A_\lambda \otimes A_\kappa) = \{A_{\kappa+\lambda}, B_{\kappa+\lambda}\} \neq \emptyset, \forall \kappa, \lambda \in \mathbb{N}.$$

All elements B_λ are right absorbing and B_1 is a left scalar,

because $B_1 \otimes A_\lambda = B_{\lambda+1}$ and $B_1 \otimes B_\lambda = B_\lambda$, A_0 is a unit.

3. Hyper-Lie-Algebras

Lie-Santilli admisibility

The general definition of an H_v -Lie algebra over an H_v -field is given as follows [61],[62]:

Definition 3.1 $(L, +)$ be H_v -vector space on H_v -field $(F, +, \cdot)$, $\phi: F \rightarrow F/\gamma^*$ the canonical map and $\omega_F = \{x \in F: \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Moreover, let ω_L be the core of the canonical map $\phi': L \rightarrow L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^* . Consider the bracket (commutator) hope:

$$[\cdot, \cdot]: L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is called an H_v -Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset,$$

$$\forall x, x_1, x_2, y, y_1, y_2 \in L \text{ and } \lambda_1, \lambda_2 \in F$$

$$(L2) [x, x] \cap \omega_L \neq \emptyset, \forall x \in L$$

$$(L3) ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L$$

Example 3.2 Consider all traceless matrices $A=(a_{ij}) \in M_{2 \times 3}$, in the sense that $a_{11} + a_{22} = 0$. In this case, the cardinality of the helix-product of any two matrices is 1, or 2^3 , or 2^6 . These correspond to the cases: $a_{11} = a_{13}$ and $a_{21} = a_{23}$, or only $a_{11} = a_{13}$ either only $a_{21} = a_{23}$, or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or 2^6 , or 2^{12} , resp. We remark that, from the definition of the helix-projection, the initial 2×2 , block guaranties that in the result there exists at least one traceless matrix.

From this example it is obvious the following:

Theorem 3.3 Using the helix-product the Lie-bracket of any two traceless matrices $A=(a_{ij})$, $B=(b_{ij}) \in M_{m \times n}$, $m < n$, contain at least one traceless matrix.

Last years, hyperstructures have a variety of applications in mathematics and other sciences. The hyperstructures theory can now be widely applicable in industry and production, too. In several books [4],[10],[12] and papers [1],[11],[17],[23],[31],[35],[50],[67],[70] one can find numerous applications.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed [28] a ‘lifting’ of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called e-hyperfields [29],[30],[59],[60],[64],[13],[14],[15] which are used in physics or biology. The H_v -fields can give

e-hyperfields which can be used in the isotopy theory for applications.

The IsoMathematics Theory is very important subject in applied mathematics. It is a generalization by using a kind of the Rees analogous product on matrix semigroup with a sandwich matrix, like the P-hopes. It contains the classical theory but also can find easy solutions in different branches of mathematics. To compare this novelty we give two analogous examples: (1) The unsolved, from ancient times, problems in Geometry was solved in a different branch of mathematics, the Algebra with the genius Galois Theory. (2) With the Representation Theory one can solve problems in Lie Algebras and to transfer these in Lie Groups using the exponential map, and the opposite. One very important thing of the IsoMathematics Theory is that admits generalizations, as well. Two very important of them are the following: First, is the so called Admissible Lie-Santilli Algebras [28],[30],[62],[65] by using again a kind of Rees sandwich product. Second, is that one can extend this theory into the multivalued case, i.e. into H_v -structures.

Definitions 3.4 A hyperstructure $(H, \cdot)$ containing a unique scalar unit e , is called e-hyperstructure. We assume that $\forall x$, there is an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following are valid:

$(F, +)$ is abelian group with the additive unit 0, $(\cdot)$ is WASS,

$(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$, there exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a strong e-hyperfield.

A general construction based on the partial ordering of the H_v -structures:

Construction 3.5 [13],[14],[15],[30] Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G , a large number of hopes $(\otimes)$ by extended $(\cdot)$, as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

Then $(G, \otimes)$ becomes an H_v -group, in fact is H_b -group which contains the $(G, \cdot)$. The H_v -group $(G, \otimes)$ is an e-hypergroup. Moreover, if $\forall x, y$ such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup.

Definition 3.6 Let $(H_0, +, \cdot)$ be the attached, by one element, H_v -field of the H_v -semigroup $(H, \cdot)$. Thus, for $(H, \cdot)$, take an element v outside of H , and extend $(\cdot)$ in $H_n = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, v \cdot v = H, \forall x \in H.$$

$(H_n, \cdot)$ is an H_v -group, called Attach Elements Construction, and $(H_n, \cdot) / \beta^* \cong Z_2$, where v , is single. If $(H, \cdot)$ has a left and right scalar unit e then $(H_0, +, \cdot)$ is an e-hyperfield, the attached H_v -field of $(H, \cdot)$.

Remark. The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate

e-hyperstructures. However, the most useful are the ones where only few products are enlarged.

Example 3.7 Take the finite-non-commutative quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$. Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting $i = \{i, -i\}$, $j = \{j, -j\}$, $k = \{k, -k\}$ we may define the $(*)$ hope by the Cayley table:

*	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

The hyperstructure $(Q, *)$ is strong e-hypergroup because 1 is scalar unit and the elements $-1, i, -i, j, -j, k$ and $-k$ have unique inverses the elements $-1, -i, i, -j, j, -k$ and k , resp., which are the inverses in the basic group. Thus, from this example one can have more strict hopes.

In [30],[62],[65] a kind of P-hopes was introduced which is appropriate to extent the Lie-Santilli admissible algebras in hyperstructures:

The general definition is the following:

Construction 3.8 Let $(L = M_{m \times n}, +)$ be an H_v -vector space of $m \times n$ hyper-matrices over the H_v -field $(F, +, \cdot)$, $\phi: F \rightarrow F/\gamma^*$, the canonical map and $\omega_F = \{x \in F: \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* , ω_L be the core of the canonical map $\phi': L \rightarrow L/\epsilon^*$ and denote again by 0 the zero of L/ϵ^* . Take any two subsets $R, S \subseteq L$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:

$$[,]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = xR^t y - yS^t x.$$

Notice that $[x, y]_{RS} = xR^t y - yS^t x = \{r^t y - y s^t x \mid r \in R \text{ and } s \in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict':

$$(a) R = \{e\} \text{ then } [x, y]_{RS} = xy - yS^t x = \{xy - y s^t x \mid s \in S\}$$

$$(b) S = \{e\} \text{ then } [x, y]_{RS} = xR^t y - yx = \{x r^t y - yx \mid r \in R\}$$

$$(c) R = \{r_1, r_2\} \text{ and } S = \{s_1, s_2\} \text{ then}$$

$$[x, y]_{RS} = xR^t y - yS^t x =$$

$$\{x r_1^t y - y s_1^t x, x r_1^t y - y s_2^t x, x r_2^t y - y s_1^t x, x r_2^t y - y s_2^t x\}$$

4. Galois H_v -Fields and Low Dimensional H_v -Matrices

Recall some results from [63], which are referred to finite H_v -fields which we will call, according to the classical theory, Galois H_v -fields. Combining the uniting elements procedure

with the enlarging theory we can obtain stricter structures or hyperstructures. So enlarging operations or hopes we can obtain more complicated structures.

Theorem 4.1 In the ring $(Z_n, +, \cdot)$, with $n=ms$ we enlarge the multiplication only in the product of elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then

$$(Z_n, +, \otimes) / \gamma^* \cong (Z_m, +, \cdot).$$

Proof. First we remark that the only expressions of sums and products which contain more, than one, elements are the expressions which have at least one time the hyperproduct $0 \otimes m$. Adding to this special hyperproduct the element 1, several times we have the equivalence classes mod m . On the other side, since m is a zero divisor, adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, γ^* -classes form a ring isomorphic to $(Z_m, +, \cdot)$.

Remark. In the above theorem we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case the elements 0 and 1 remain scalars, so they are referred in e -hyperstructures.

From the above theorem it is immediate the following:

Corollary 4.2 In the ring $(Z_n, +, \cdot)$, with $n=ps$ where p is a prime number, we enlarge the multiplication only in the product of the elements $0 \cdot p$ by setting $0 \otimes p = \{0, p\}$ and the rest results remain the same. Then the hyperstructure $(Z_n, +, \otimes)$ is a very thin H_v -field.

The above theorem provides the researchers with H_v -fields appropriate to the rep theory since they may be smaller or minimal hyperstructures.

Remarks 4.3 The above theorem in connection with Uniting Elements method leads to the fact that in H_v -structure theory it is able to equip algebraic structures or hyperstructures with properties as associativity, commutativity, reproductivity. This equipment can be applied independently of the order of the desired properties. This is crucial point since some properties are easy to be applied, so we can apply them first, and then the difficult ones. For example from an H_v -ring we first go to an H_v -integral domain, by uniting the zero divisors, and then to the H_v -field by reaching the reproductivity.

Construction 4.5 (Galois H_v -fields) In the ring $(Z_n, +, \cdot)$, with $n=ps$ where p is prime, enlarge only the product of the elements 2 by $p+2$, i.e. $2 \cdot (p+)$, by setting $2 \otimes (p+2) = \{2, p+2\}$ and the rest remain the same. Then $(Z_n, +, \otimes)$ is a COW very thin H_v -field where 0 and 1 are scalars and we have:

$$(Z_n, +, \otimes) / \gamma^* \cong (Z_p, +, \cdot).$$

Proof. Straightforward.

Remark 4.6 Galois H_v -fields of the above type are the most appropriate in the representation theory since the cardinality of the products is low. Moreover, one can use more enlargements using elements of the same fundamental class, therefore, one can have several cardinalities. The low dimensional reps can be based on the above Galois H_v -fields, since they use infinite H_v -fields although the fundamental fields are finite.

References

- [1] R. Anderson, A.A. Bhalekar, B. Davvaz, P.S. Muktibodh, V.M. Tangde, A.S. Muktibodh, T. Vougiouklis, An introduction to Santilli's isodual theory of antimatter and the open problem of detecting antimatter asteroids, NUMTA Bulletin, 6, 2012-13, 1-33.
- [2] R. Bayon, N. Lygeros, Les hypergroupes abéliens d'ordre 4, Proceedings: Structure Elements of Hyper-structures, Spanidis Press, 2005, 35-39.
- [3] R. Bayon, N. Lygeros, Advanced results in enumeration of hyperstructures, J. Algebra, 320, 2008, 821-835.
- [4] P. Corsini, V. Leoreanu, Applications of Hypergroup Theory, Kluwer Academic Publ., 2003.
- [5] P. Corsini, T. Vougiouklis, From groupoids to groups through hypergroups, Rendiconti Mat. VII, 9, 1989, 173-181.
- [6] B. Davvaz, On H_v -subgroups and anti fuzzy H_v -subgroups, Korean J. Comp. Appl. Math. V.5, N.1, 1998, 181-190.
- [7] B. Davvaz, On H_v -rings and Fuzzy H_v -ideals, J. Fuzzy Math. V.6, N.1, 1998, 33-42.
- [8] B. Davvaz, Fuzzy H_v -submodules, Fuzzy sets and Systems, 117, 2001, 477-484.
- [9] B. Davvaz, A brief survey of the theory of H_v -structures, 8th AHA, Greece, Spanidis, 2003, 39-70.
- [10] B. Davvaz, Polygroup Theory and Related Systems, World Scientific, 2013.
- [11] B. Davvaz, W. Dudek, T. Vougiouklis, A generalization of n -ary algebraic systems, Communications in Algebra, 37, 2009, 1248-1263.
- [12] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, Int. Academic Press, USA, 2007.
- [13] B. Davvaz, R.M. Santilli, T. Vougiouklis, Studies of multi-valued hyper-structures for the characterization of matter-antimatter systems and their extension, Algebras, Groups and Geometries 28(1), 2011, 105-116.
- [14] B. Davvaz, R.M. Santilli, T. Vougiouklis, Multi-valued Hypermathematics for characterization of matter and antimatter systems, J. Comp. Methods in Sciences and Engineering 13, 2013, 37-50.
- [15] B. Davvaz, R.M. Santilli, T. Vougiouklis, Mathematical prediction of Ying's twin universes, American J. of Modern Physics, 4(3), 2015, 5-9.
- [16] B. Davvaz, S. Vougioukli, T. Vougiouklis, On the multiplicative H_v -rings derived from helix hyperoperations, Util. Math., 84, 2011, 53-63.
- [17] B. Davvaz, T. Vougiouklis, N -ary hypergroups, Iranian J. of Science & Technology, Transaction A, V.30, N.A2, 2006, 165-174.
- [18] A. Dramalidis, Geometrical H_v -structures, Proceedings: Structure Elements of Hyper-structures, Spanidis Press, 2005, 41-51.

- [19] S. Hoskova, J. Chvalina, Abelizations of proximal H_v -rings using graphs of good homomorphisms and diagonals of direct squares of hyperstructures, 8th AHA Congress, Spanidis Press, 2003, 147-158.
- [20] A. Iranmanesh, M.N. Iradmusa, H_v -structures associated with generalized P-hyperoperations., *Bul. Iranian Math. Soc.* V.24, N1, 1998, 33-45.
- [21] P. Kambaki-Vougiouklis, A. Karakos, N. Lygeros, T. Vougiouklis, Fuzzy instead of discrete, *Annals of Fuzzy Math. Inf. (AFMI)*, V.2, N.1, 2011, 81-89.
- [22] V. Leoreanu-Fotea, Ivo Rosenberg, B. Davvaz, T. Vougiouklis, A new class of n-ary hyperoperations, *European J. Combinatorics*, V.44, 2015, 265-273.
- [23] N. Lygeros, T. Vougiouklis, The LV-hyperstructures, *Ratio Math.*, 25, 2013, 59-66.
- [24] A. Madanshekar, H_v -structures associated with PQ-hyperoperations., *J. of Disc. Math. Sc. & Crypt.* V.6, N 2-3, 2003, 199-205.
- [25] F. Marty, Sur un généralisation de la notion de groupe, 8^{ème} Congrès Math. Scandinaves, Stockholm, (1934), 45-49.
- [26] F. Marty, Sur les groupes et hypergroupes attachés à une fraction rationnelle, *Annales Ec. Norm. Sup.*, V 53, (1936), 83-123.
- [27] P. Nikolaidou, T. Vougiouklis, H_v -structures and the Bar in questionnaires, *Italian J. Pure and Appl. Math.* N.29, 2012, 341-350.
- [28] R.M. Santilli, *Hadronic Maths, Mechanics and Chemistry*, Vol. I, II, III, IV and V, Int. Academic Press, USA, 2008.
- [29] R.M. Santilli, T. Vougiouklis, Isotopies, Genotopies, Hyperstructures and their Appl., *Proc. New Frontiers Hyperstructures and Related Algebras*, Hadronic, 1996, 1-48.
- [30] R.M. Santilli, T. Vougiouklis, Lie-admissible hyperalgebras, *Italian J. Pure Appl. Math.*, N.31, 2013, 239-254.
- [31] S. Spatalis, A. Dramalides, T. Vougiouklis, On H_v -group Rings, *Algebras, Groups and Geometries*, 15, 1998, 47-54.
- [32] S. Vougiouklis, H_v -vector spaces from helix hyperoperations, *Int. J. Math. Anal. (New Series)*, 1(2), 2009, 109-120.
- [33] T. Vougiouklis, Cyclicity in a special class of hypergroups, *Acta Un. Car. – Math. Et Ph.*, V.22, N1, 1981, 3-6.
- [34] T. Vougiouklis, Representations of hypergroups, Hypergroup algebra, *Proc. Convegno: ipergroupi, altre strutture multivoche appl.* Udine, 1985, 59-73.
- [35] T. Vougiouklis, On affine Kac-Moody Lie algebras, *Commentationes Math. Un. Car.*, V.26, 2, 1985, 387-395.
- [36] T. Vougiouklis, Representations of hypergroups by hypermatrices, *Rivista Mat. Pura Appl.*, N 2, 1987, 7-19.
- [37] T. Vougiouklis, Generalization of P-hypergroups, *Rend. Circ. Mat. Palermo*, S.II, 36, 1987, 114-121.
- [38] T. Vougiouklis, Groups in hypergroups, *Annals of Discrete Math.* 37, 1988, 459-468.
- [39] T. Vougiouklis, On representations of algebraic multivalued structures, *Rivista Mat. Pura Appl.*, N.7, 1990, 87-92.
- [40] T. Vougiouklis, The fundamental relation in hyperrings. The general hyperfield, *Proc. 4th AHA*, World Scientific, 1991, 203-211.
- [41] T. Vougiouklis, The very thin hypergroups and the S-construction, *Combinatorics '88, Incidence Geometries Combinatorial Str.*, 2, 1991, 471-477.
- [42] T. Vougiouklis, Representations of hypergroups by generalized permutations, *Algebra Universalis*, 29, 1992, 172-183.
- [43] T. Vougiouklis, Representations of H_v -structures, *Proc. Int. Conf. Group Theory 1992*, Timisoara, 1993, 159-184.
- [44] T. Vougiouklis, *Hyperstructures and their Representations*, Monographs in Math., Hadronic Press, 1994.
- [45] T. Vougiouklis, Some remarks on hyperstructures, *Contemporary Math.*, Amer. Math. Society, 184, 1995, 427-431.
- [46] T. Vougiouklis, A new class of hyperstructures, *J.C.I.&S.S.*, V.20, N.1-4, 1995, 229-239.
- [47] T. Vougiouklis, H_v -groups defined on the same set, *Discrete Math.* 155, 1996, 259-265.
- [48] T. Vougiouklis, Constructions of H_v -structures with desired fundamental structures, *New frontiers in Hyperstructures*, Hadronic Press, 1996, 177-188.
- [49] T. Vougiouklis, On H_v -fields, 6th AHA, Prague 1996, Democritus Univ. Press, 1997, 151-159.
- [50] T. Vougiouklis, Convolutions on WASS hyperstructures, *Discrete Math.* 174, 1997, 347-355.
- [51] T. Vougiouklis, Enlarging H_v -structures, *Algebras and Combinatorics*, ICAC'97, Hong Kong, Springer-Verlag, 1999, 455-463.
- [52] T. Vougiouklis, On H_v -rings and H_v -representations, *Discrete Math.*, Elsevier, 208/209, 1999, 615-620.
- [53] T. Vougiouklis, On hyperstructures obtained by attaching elements, *Proc. C. Caratheodory in his...origins*, Hadronic Press, 2001, 197-206.
- [54] T. Vougiouklis, Finite H_v -structures and their representations, *Rend. Seminario Mat. Messina S.II*, V.9, 2003, 245-265.
- [55] T. Vougiouklis, Attach, remove, absorb and merge elements, *Proc. 8th AHA Congress*, Spanidis Press, 2003, 251-260.
- [56] T. Vougiouklis, The h/v-structures, *J. Discrete Math. Sciences and Cryptography*, V.6, 2003, N.2-3, 235-243.
- [57] T. Vougiouklis, A hyperoperation defined on a groupoid equipped with a map, *Ratio Math. on line*, N.1, 2005, 25-36.
- [58] T. Vougiouklis, ∂ -operations and H_v -fields, *Acta Math. Sinica*, (Engl. Ser.), V.24, N.7, 2008, 1067-1078.
- [59] T. Vougiouklis, The Santilli's theory 'invasion' in hyperstructures, *Algebras, Groups and Geometries* 28(1), 2011, 83-103.
- [60] T. Vougiouklis, The e-hyperstructures, *J. Mahani Math. Research Center*, V.1, N.1, 2012, 13-28.
- [61] T. Vougiouklis, The Lie-hyperalgebras and their fundamental relations, *Southeast Asian Bull. Math.*, V.37(4), 2013, 601-614.

- [62] T. Vougiouklis, Lie-admissible hyperalgebras, Italian J. Pure Appl. Math., N.31, 2013,
- [63] T. Vougiouklis, From H_v -rings to H_v -fields, Int. J. Algebraic Hyperstructures Appl. Vol.1, No.1, 2014, 1-13.
- [64] T. Vougiouklis, On the iso H_v -numbers, Hadronic J., Dec.5, 2014, 1-18.
- [65] T. Vougiouklis, Lie-Santilli Admissibility using P-hyper-operations on matrices, Hadronic J. Dec.7, 2014, 1-14.
- [66] T. Vougiouklis, Hypermatrix representations of finite H_v -groups, European J. Combinatorics, V.44 B, 2015, 307-315.
- [67] T. Vougiouklis, Quiver of hyperstructures for Ying's twin universes, American J. Modern Physics, 4(1-1), 2015, 30-33.
- [68] T. Vougiouklis, S. Vougiouklis, The helix hyperoperations, Italian J. Pure Appl. Math., 18, 2005, 197-206.
- [69] T. Vougiouklis, P. Kambaki-Vougioukli, On the use of the bar, China-USA Business Review, V.10, N.6, 2011, 484-489.
- [70] T. Vougiouklis, P. Kambakis-Vougiouklis, Bar in Questionnaires, Chinese Business Review, V.12, N.10, 2013, 691-697.