
Applications of statistical physics distributions to several types of income

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Abstract: This paper explores several types of income which have not been explored so far by authors who tackled income and wealth distribution using Statistical Physics. The main types of income we plan to analyze are income before redistribution (or gross income), income of retired people (or pensions), and income of active people (mostly wages). The distributions used to analyze income distributions are Fermi-Dirac distribution and polynomial distribution (as this is present in describing the behavior of dynamic systems in certain aspects). The data we utilize for our analysis are from France and the UK. We find that both distributions are robust in describing these varieties of income. The main finding we consider to be the applicability of these distributions to pensions, which are not regulated entirely by market mechanisms.

Keywords: Fermi-Dirac Distribution, Polynomial Distribution, Gross Income, Pensions, Wages, Cumulative Distribution Function

1. Introduction

Papers that tackled the field of income and wealth distribution used most often so far as a measure for income disposable or net income. This implies that from gross income there are subtracted taxes and there are added transfers from public budget. We plan to see the scope of applicability to several types of income such as gross income (or income before redistribution), income of active people, wages and income of inactive people/pensions. However, these types of income that were measured are not fully equivalent in the cases of wages and income of active people. The most interesting and intriguing part about these data is that some of them depend partially or entirely on state and, therefore, market mechanisms play only a partial role. The only countries which provided this type of data regarding income apart from disposable income (divided in deciles) are France and the UK. We plan to use for the analysis of these types of income Fermi-Dirac and polynomial distributions.

2. Short Literature Review and Theoretical Framework

So far, previous research using these distributions applied

to income and wealth distribution showed clearly that they are robust in their assessment. Fermi-Dirac distribution [1] was applied to income distribution for a group of countries mostly of them developed. The values for coefficient of determination were quite high and the temporal evolution of the coefficients obtained from fitting the data show some similarities with macroeconomic variables chosen to describe such systems. Also, in this paper a first analysis was made on other types of income apart from disposable income. The results showed clearly robustness of this distribution in the analysis of income distribution.

The polynomial distribution was first applied to the same data set as in the previous case and yielded similar results regarding the robustness [2]. The origin of this type of distribution comes from dynamic systems, where polynomials appear to model their behavior in certain aspects.

3. Methodology

Income can be measured using two types of calculation. Mean income is the sum of all individual incomes divided by number of inhabitants in a decile/population [3]. Upper limit on income is the highest individual or household income from a decile of population. Deciles are segments of population divided equally in ten parts, where individuals or

households are ranked increasingly according to their income. Thus, a decile contains 10% of actual population, the first one contain people with lowest income while the tenth decile contains people with highest income. Upper limit on income is a term coined by National Institute of Statistics from Finland [4]. In the case of the UK, the income is expressed in mean income. Instead, the data from France is expressed using upper limit on income, hence the upper income tier contained in the tenth decile is not considered. The data from France is expressed for individuals, while the ones for the UK are expressed for households.

In order to calculate the probability for a certain part of population to have an income, we use the cumulative probability density function. According to this type of probability, we calculate the share of population having an income above a certain threshold. Thus, the probability to have an income higher than zero is 100% (since everyone is assumed to have a certain income). Furthermore, in the case of the first decile the probability that people have a higher income is 90% and so forth. Let us assume x_1, x_2, \dots, x_{10} be such that x_1 is the mean income for the first decile, and so forth up to x_{10} which is the mean income for the tenth decile. The set which contains the plots representing the probability is $S = \{(0, 100\%), (x_1, 90\%), (x_2, 80\%), (x_3, 70\%), (x_4, 60\%), (x_5, 50\%), (x_6, 40\%), (x_7, 30\%), (x_8, 20\%), (x_9, 10\%), (x_{10}, 0\%\}$. In order to fit the data, we used Fermi-Dirac distribution and polynomial distribution.

Fermi-Dirac distribution has its most general form as follows

$$n(\epsilon_i) = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{T}\right) + 1} \quad (1)$$

where n represents the distribution of identical fermions over energy consisting of single-particle states. The parameters to be used for analysis are degeneracy (c), temperature (T), and chemical potential (μ). Therefore, the average number of fermions with a certain energy is calculated by multiplying n with degeneracy g_i [5] such that

$$N(\epsilon_i) = \frac{g_i}{\exp\left(\frac{\epsilon_i - \mu}{T}\right) + 1} \quad (2)$$

We represent graphically the results from Fermi-Dirac distribution using logarithmic values (log-log scale).

Polynomial distribution uses a slightly different approach by utilizing normal values in displaying the results. Unlike Fermi-Dirac distribution, the polynomial distribution can have variable number of coefficients (P_1, P_2 , and/or P_3 as the case may be) according to the choice regarding the degree of the polynomial subject to the goodness of fit to the data. We chose the degree of the polynomials such that the values for the coefficient of determination (resulted from fitting the data) to be above 90%.

The categories of income taken into account are pensions, wages or active people income, and gross income (income before redistribution). In the case of pensions, the data from France are provided without making distinction between

private and public pensions. In case of the UK, the data were provided separately for private and public pensions, and we added them up. It is noteworthy that taken separately they do not fit any distribution. Other types of income considered are income of active people or wages. However, we must show that these two types of income are not exactly the same as income of active people includes other types of income apart from wages such as income from self-employed activities. The third type of income considered is gross income or income before redistribution. This income represents the earnings of population before they pay taxes and receive benefits.

The results are obtained from fitting the data using Matlab toolbox for a confidence interval of 95%. The data we used are from the UK [6] and France [7], since these are the only countries which made these data available to the best of our knowledge.

4. Results

We show the results from fitting the annual data into the appendixes 1-10. In the appendixes 1-4, we exhibit the results from fitting the data using Fermi-Dirac distribution. The results from fitting the data using Fermi-Dirac distribution regarding pensions and income before redistribution for France were shown in [1]. The results from fitting the data using the polynomial distribution are displayed in the appendixes 5-10. In the following, we display graphically few typical examples of annual fitting to the data in the Figures 1-4. It is noteworthy that these examples were chosen such that the values for coefficient of determination for fitting the annual data to be the lowest for the each set of data for which results are displayed in each table.

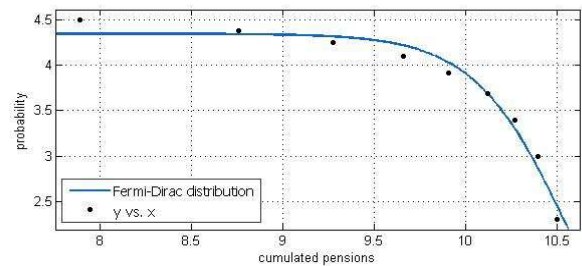


Figure 1. The Fermi-Dirac probability distribution applied to UK pensions in the year 2005.

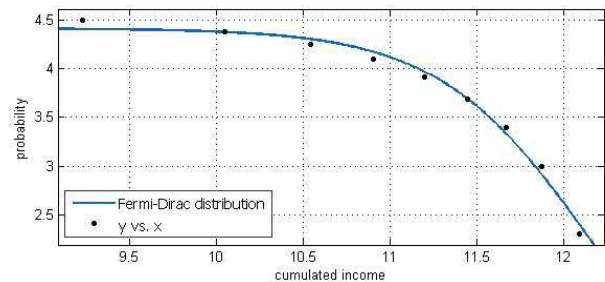


Figure 2. The Fermi-Dirac probability distribution applied to active people's income from France in the year 2004.

Generally, Fermi-Dirac distribution has better goodness of fit compared to polynomial distribution. A possible explanation is that Fermi-Dirac distribution has more coefficients than most of polynomial distributions we used.

Also, the higher the degree for polynomial distribution the higher are the annual values for coefficient of determination. Most of the data sets were described using first degree polynomials. The exception was about the UK wages, for which was possible to fit the data having annual values for coefficient of determination higher than 90% only by using second degree polynomials.

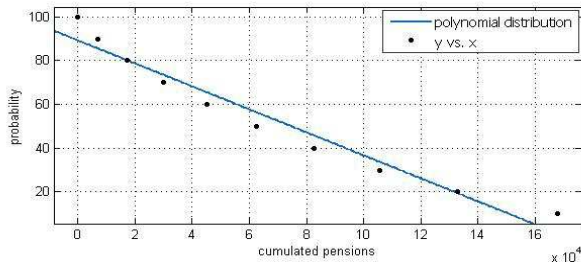


Figure 3. Polynomial probability distribution applied to pensions from France in the year 2003.

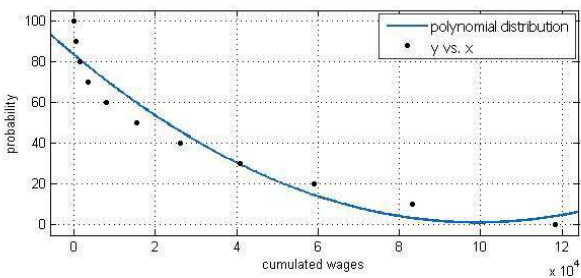


Figure 4. Polynomial distribution applied to UK wages in the year 1993.

These distributions are very robust. Thus, the lowest value for coefficient of determination in the case of Fermi-Dirac distribution is 97.3%. In the case of polynomial distribution, the lowest value regarding coefficient of determination is 89.7%. In the case of UK (which, unlike for France, contains the data regarding upper income segment of population), these distributions show that they are applicable to the entire range of income, namely to upper income segment of population which is thought traditionally to be described only by Pareto distribution.

It is very interesting that income of active people and gross income are described by market mechanisms, while pensions, even though obey the same distributions, they do not observe market mechanisms entirely. Thus, they are managed by public bodies and observe laws that follow some social principles which normally distort the market mechanisms.

5. Conclusions

Both Fermi-Dirac distribution and polynomial distribution describe with high degree of success different types of income. It is our opinion that other types of income

can easily be described by these distributions.

The accuracy of polynomial distribution describing income distribution can be increased by using a higher degree polynomial, which implies an increased number of coefficients.

An important issue arises from the applicability of these distributions to other types of income depending entirely on market mechanisms and to others which, in addition, obey social principles carried out by taxes and benefits system. Further research should explain the share of influence of market mechanisms and social principle on the income distribution in general and on different categories of income in particular.

Appendix

Appendix 1. Coefficients of the Fermi-Dirac distribution fitting income of active people in France.

Year	T	C	μ	R ² (%)
2002	0.4483	4.411	12.18	98.98
2003	0.4441	4.41	12.18	98.96
2004	0.443	4.41	12.17	98.96
2005	0.4425	4.409	12.18	98.96
2006	0.4441	4.41	12.2	98.97
2007	0.4451	4.41	12.22	98.97
2008	0.4409	4.409	12.23	98.97
2009	0.4457	4.41	12.24	98.96

Appendix 2. Coefficients of the Fermi-Dirac distribution fitting gross income in the UK.

Year	T	C	μ	R ² (%)
1977	0.4825	4.386	10.61	98.38
1978	0.4917	4.389	10.75	98.4
1979	0.4999	4.386	10.88	98.3
1980	0.4999	4.384	11.06	98.3
1981	0.5078	4.396	11.18	98.52
1982	0.5086	4.395	11.23	98.47
1983	0.5053	4.401	11.28	98.64
1984	0.5181	4.399	11.34	98.5
1985	0.5242	4.402	11.42	98.55
1986	0.5366	4.403	11.48	98.61
1987	0.5444	4.402	11.58	98.6
1988	0.5513	4.397	11.67	98.52
1989	0.5588	4.398	11.75	98.46
1990	0.5621	4.399	11.85	98.49
1991	0.563	4.4	11.91	98.54
1992	0.5558	4.404	11.94	98.55
1993	0.5601	4.411	11.95	98.73
1995	0.5591	4.41	12	98.63
1996	0.5498	4.41	12.03	98.73
1997	0.565	4.412	12.09	98.74

Year	T	C	μ	R ² (%)
1998	0.5664	4.409	12.14	98.71
1999	0.5589	4.409	12.18	98.76
2000	0.5668	4.407	12.22	98.67
2001	0.5523	4.405	12.28	98.68
2002	0.5654	4.409	12.35	98.74
2003	0.5637	4.415	12.38	98.78
2004	0.5473	4.406	12.4	98.74
2005	0.5451	4.409	12.46	98.78
2006	0.5574	4.413	12.48	98.79
2007	0.5519	4.412	12.53	98.83
2008	0.5488	4.407	12.56	98.72
2009	0.5535	4.411	12.57	98.84
2010	0.5394	4.412	12.58	98.85
2011	0.5436	4.416	12.61	98.86
2012	0.5353	4.416	12.62	98.91

Appendix 3. Coefficients of the Fermi-Dirac distribution fitting pensions in the UK.

Year	T	C	μ	R ² (%)
1977	0.1714	4.363	8.283	97.69
1978	0.1861	4.378	8.448	98.09
1979	0.1553	4.361	8.578	98.15
1980	0.1631	4.372	8.736	98.37
1981	0.1912	4.363	8.94	97.89
1982	0.1886	4.359	8.994	98.07
1983	0.2375	4.365	9.163	97.67
1984	0.2159	4.366	9.206	98.35
1985	0.2398	4.376	9.331	98.56
1986	0.2346	4.368	9.396	98.15
1987	0.221	4.354	9.459	97.71
1988	0.2221	4.355	9.55	97.92
1989	0.2196	4.362	9.61	98.37
1990	0.2451	4.367	9.712	97.87
1991	0.2516	4.363	9.803	98.25
1992	0.2641	4.359	9.864	98.1
1993	0.281	4.356	9.939	97.8
1995	0.2933	4.374	10.02	98.8
1996	0.2862	4.362	10.07	97.77
1997	0.2967	4.36	10.09	97.85
1998	0.2997	4.36	10.17	97.99
1999	0.2755	4.352	10.22	97.85
2000	0.3013	4.362	10.29	97.79
2001	0.2843	4.36	10.33	0.98
2002	0.2604	4.342	10.4	97.12
2003	0.2827	4.35	10.48	97.78
2004	0.2868	4.355	10.55	97.79
2005	0.2606	4.343	10.57	97.3
2006	0.2955	4.363	10.61	98.13
2007	0.2851	4.353	10.65	97.82
2008	0.2847	4.355	10.68	97.99
2009	0.2942	4.349	10.74	97.55
2010	0.3127	4.35	10.83	97.31
2011	0.3288	4.355	10.9	97.5
2012	0.3352	4.353	10.92	97.52

Appendix 4. Coefficients of the Fermi-Dirac distribution fitting wages in the UK.

Year	T	C	μ	R ² (%)
1977	0.5801	4.361	10.34	97.98
1978	0.5974	4.364	10.53	98.01
1979	0.6171	4.355	10.6	97.79
1980	0.612	4.352	10.78	97.72
1981	0.6377	4.366	10.86	98.01
1982	0.6532	4.361	10.9	97.84
1983	0.6584	4.37	10.9	98.18
1984	0.6931	4.36	10.98	97.78
1985	0.6955	4.358	11.05	97.74
1986	0.7185	4.363	11.1	97.92
1987	0.7323	4.356	11.2	97.87
1988	0.7345	4.357	11.31	97.81
1989	0.7321	4.352	11.38	97.59
1990	0.7303	4.356	11.46	97.71
1991	0.7437	4.362	11.5	97.84
1992	0.7506	4.361	11.5	97.79
1993	0.7924	4.374	11.51	98.07
1995	0.7701	4.374	11.57	97.81
1996	0.7593	4.378	11.59	98.08
1997	0.7805	4.38	11.67	98.16
1998	0.7725	4.381	11.74	98.13
1999	0.7607	4.386	11.77	98.27
2000	0.7625	4.381	11.83	98.16
2001	0.7324	4.376	11.9	98.04
2002	0.7689	4.386	11.98	98.23
2003	0.7503	4.391	11.99	98.2
2004	0.7292	4.384	12	98.29
2005	0.7322	4.392	12.07	98.5
2006	0.7392	4.388	12.09	98.2
2007	0.7263	4.393	12.14	98.32
2008	0.7312	4.39	12.17	98.2
2009	0.7341	4.396	12.16	98.48
2010	0.7287	4.402	12.14	98.6
2011	0.7465	4.413	12.17	98.59
2012	0.7441	4.419	12.16	98.79

Appendix 5. Coefficients of the polynomials fitting income of active people in France.

Year	P1	P2	R ² (%)
2002	-0.0004961	91.59	97.18
2003	-0.0004976	91.76	97.3
2004	-0.0005027	91.8	97.33
2005	-0.0004954	91.77	97.33
2006	-0.0004891	91.74	97.29
2007	-0.0004774	91.69	97.26
2008	-0.0004737	91.86	97.38
2009	-0.0004677	91.65	97.25

Appendix 6. Coefficients of the polynomials fitting income before redistribution in France.

Year	P1	P2	R ² (%)
2002	-0.00053	91.97	97.19
2003	-0.0005338	92.15	97.29
2004	-0.0005354	92.32	97.44
2005	-0.0005296	92.19	97.34
2006	-0.0005093	91.88	97.16
2007	-0.0005079	92.01	97.25
2008	-0.0005016	92.05	97.27
2009	-0.0004986	92.03	97.28

Appendix 7. Coefficients of the polynomials fitting income of pensioners in France.

Year	P1	P2	R ² (%)
2003	-0.0005256	89.16	95.78
2004	-0.0005277	89.33	95.92
2005	-0.0005197	89.27	95.87
2006	-0.0005139	89.24	95.83
2007	-0.0005057	89.22	95.84
2008	-0.0004972	89.35	95.93
2009	-0.0004928	89.15	95.82

Appendix 8. Coefficients of the polynomials fitting gross income in the UK.

Year	P1	P2	R ² (%)
1977	-0.0021	85.79	94.45
1978	-0.001833	85.57	94.27
1979	-0.001604	85.07	93.99
1980	-0.001328	84.79	93.71
1981	-0.001175	85.03	93.48
1982	-0.001116	84.98	93.44
1983	-0.001058	85.16	93.23
1984	-0.0009963	84.76	93.06
1985	-0.0009065	84.32	92.43
1986	-0.0008409	83.63	91.75
1987	-0.0007615	83.14	91.34
1988	-0.0006833	82.48	90.79
1989	-0.0006413	82.56	91.11
1990	-0.0005696	82.09	90.34
1991	-0.0005349	82.13	90.43
1992	-0.000527	82.7	90.91
1993	-0.0005163	82.54	90.38
1995	-0.0004942	82.7	90.65
1996	-0.0004816	83.08	90.98
1997	-0.0004505	82.44	90.44
1998	-0.0004247	82.22	90.25
1999	-0.0004047	82.24	90.1
2000	-0.0003848	81.82	89.76
2001	-0.0003668	82.44	90.5
2002	-0.0003394	81.87	89.7
2003	-0.0003348	82.6	90.61
2004	-0.0003252	82.57	90.57
2005	-0.0003088	82.86	90.79
2006	-0.0002999	82.51	90.35
2007	-0.0002842	82.58	90.38
2008	-0.0002791	82.7	90.78
2009	-0.000276	82.64	90.57
2010	-0.0002716	83.01	90.68
2011	-0.0002619	82.89	90.43
2012	-0.0002646	83.51	91.14

Appendix 9. Coefficients of the polynomials fitting pensions in the UK.

Year	P1	P2	R ² (%)
1977	-0.02404	107.2	96.93
1978	-0.02064	107.5	97.34
1979	-0.01792	108.7	95.77
1980	-0.01523	108	95.91
1981	-0.01241	105.1	98.09
1982	-0.01169	104.4	98.03
1983	-0.009715	100.3	99.56
1984	-0.009348	102	99.06
1985	-0.008257	100.7	99.39
1986	-0.007774	101.1	99.49
1987	-0.007186	100.9	99.5
1988	-0.006549	100.8	99.52

Year	P1	P2	R ² (%)
1989	-0.006227	101.5	99.3
1990	-0.005682	100.3	99.68
1991	-0.005139	99.34	99.8
1992	-0.004765	97.67	99.83
1993	-0.004392	96.02	99.73
1995	-0.004116	96.62	99.78
1996	-0.003866	96.02	99.71
1997	-0.003782	95.65	99.69
1998	-0.003487	95.09	99.61
1999	-0.003322	96.16	99.75
2000	-0.003124	95.3	99.67
2001	-0.002992	96.41	99.8
2002	-0.002779	96.63	99.72
2003	-0.002558	95.69	99.72
2004	0.002435	96.15	99.77
2005	-0.002368	97.16	99.79
2006	-0.002251	95.48	99.7
2007	-0.002161	95.62	99.72
2008	-0.002103	95.72	99.7
2009	-0.001981	94.85	99.6
2010	-0.001826	94	99.44
2011	-0.001692	93.2	99.26
2012	-0.001645	92.53	98.98

Appendix 10. Coefficients of the polynomials fitting wages in the UK.

Year	P1	P2	P3	R ² (%)
1977	7.233*10 ⁻⁸	-0.004942	87.94	97
1978	5.013*10 ⁻⁸	-0.004108	87.36	96.84
1979	4.474*10 ⁻⁸	-0.003844	86.12	96.15
1980	3.066*10 ⁻⁸	-0.00318	85.97	96.05
1981	2.773*10 ⁻⁸	-0.003038	86.69	96.23
1982	2.607 *10 ⁻⁸	-0.00293	85.89	95.77
1983	2.628 *10 ⁻⁸	-0.002961	86.41	95.97
1984	2.314*10 ⁻⁸	-0.002749	84.89	94.89
1985	1.994*10 ⁻⁸	-0.002553	84.59	94.7
1986	1.815*10 ⁻⁸	-0.002434	84.21	94.53
1987	1.524*10 ⁻⁸	-0.002218	83.44	94.01
1988	1.229*10 ⁻⁸	-0.001989	83.39	94.04
1989	1.057*10 ⁻⁸	-0.001841	83.14	93.81
1990	8.876*10 ⁻⁹	-0.001696	83.46	93.99
1991	8.219*10 ⁻⁹	-0.001635	83.44	93.99
1992	8.217*10 ⁻⁹	-0.001634	83.24	93.77
1993	8.323*10 ⁻⁹	-0.001651	83.02	93.42
1995	7.375*10 ⁻⁹	-0.001552	83.66	93.88
1996	7.072*10 ⁻⁹	-0.001527	84.2	94.21
1997	6.199*10 ⁻⁹	-0.001426	83.8	93.97
1998	5.224*10 ⁻⁹	-0.001312	83.94	94.12
1999	4.683*10 ⁻⁹	-0.001251	84.4	94.47
2000	4.329*10 ⁻⁹	-0.001195	84.22	94.34
2001	3.703*10 ⁻⁹	-0.001107	84.66	94.78
2002	3.159*10 ⁻⁹	-0.001025	84.27	94.26
2003	3.101*10 ⁻⁹	-0.001018	85.14	94.92
2004	2.921*10 ⁻⁹	-0.0009893	85.09	95.13
2005	2.586*10 ⁻⁹	-0.0009334	85.57	95.38
2006	2.511*10 ⁻⁹	-0.0009175	85.23	94.98
2007	2.216*10 ⁻⁹	-0.0008661	85.8	95.4
2008	2.12*10 ⁻⁹	-0.000844	85.55	95.25
2009	2.174*10 ⁻⁹	-0.0008579	85.81	95.36
2010	2.179*10 ⁻⁹	-0.0008644	86.2	95.58
2011	2.129*10 ⁻⁹	-0.000854	86.42	95.65
2012	2.131*10 ⁻⁹	-0.0008583	86.75	95.76

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