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Empirical bayes estimators of parameter and reliability function for compound rayleigh distribution under record data

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Abstract: Based on the record samples, the empirical Bayes estimators of parameter and reliability function for Compound Rayleigh distribution is investigated under the symmetric and asymmetric loss function. In this case the symmetric loss function is squared error and for the asymmetric loss functions, we consider LINEX and general Entropy loss function.
In this paper, we obtain the Bayes estimators of the parameter and reliability function Different from the predecessor, the empirical Bayes estimators of the parameter and reliability function are then derived where hyper-parameter is estimated using maximum likelihood method. In order to investigate the accuracy of the estimation methods, an illustrative example is examined numerically by means of Monte-Carlo simulation.

Keywords: Empirical Bayes Estimate, Compound Rayleigh Distribution, Record Data

1. Introduction

The compound Rayleigh distribution provides a population model which is useful in several areas of statistics, including life testing and reliability. The probability density function and the cumulative distribution function of compound Rayleigh are given, respectively, by

$$f(x) = 2\alpha\beta x(\beta + x^2)^{-(\alpha+1)},\ \ x,\alpha,\beta > 0 \qquad (1)$$

$$F(x) = 1 - \beta^{\alpha}(\beta + x^2)^{-\alpha} \qquad (2)$$

Where β and α are the scale and shape parameters, respectively. The reliability function of the compound Rayleigh model is

$$S(t) = \beta^{\alpha}(\beta + t^2)^{-\alpha},\ t > 0. \qquad (3)$$

Record values and the associated statistics are of interest and important in many real life applications. In industry and reliability studies, many products fail under stress. Chandler [1] introduced the study of record values and documented many of the basic properties of records. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x) For, $n \geq 1$ define

$$U(1) = 1,\quad U(n+1) = \min\left\{j : j > U(n), X_j > X_{U(n)}\right\}$$

The sequence $\{X_{U(n)}\}$ is known as upper record statistics (record times). For more details and applications in the record values, see Ahsanullah(1995).

Let $\mathbf{X} = \left\{X_{U(1)} = X_1, X_{U(2)} = X_2, \ldots, X_{U(n)} = X_n\right\}$ be the first n upper record values arising from a sequences of i.i.d. Compound Rayleigh variables with pdf (1), and distribution function cdf (2). The likelihood function, (see Ahsanullah 1995) is given by

$$L(\mathbf{X} \mid \alpha, \beta) = f\left(X_{U(n)} \mid \alpha, \beta\right) \prod_{i=1}^{n-1} \frac{f\left(X_{U(i)} \mid \alpha, \beta\right)}{1 - F\left(X_{U(i)} \mid \alpha, \beta\right)} \qquad (4)$$

It follows from (1), (2) and (4) that

$$L(\mathbf{X} \mid \alpha, \beta) \propto \alpha^{n} \left(\frac{\beta}{\beta + x_n^2}\right)^{\alpha} \prod_{i=1}^{n} \frac{x_i}{\beta + x_i^2} \qquad (5)$$

2. Bayes Estimation

When parameter β is known, we consider the natural conjugate family of prior densities for parameter α as the following

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha} \quad (6)$$

Combining the likelihood function (5) with the prior pdf (6), we can obtain the posterior density function of α as the following form

$$\pi(\alpha \mid \mathbf{X}) = \frac{(b-T)^{n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha(b-T)} \quad (7)$$

Where $T = \ln\left(\frac{\beta}{\beta + x_n^2}\right)$.

2.1. Bayesian Estimation for α and Reliability Function with LINEX Loss Function

The LINEX loss function for α can be expressed as the following proportional [4]

$$L(\Delta) \propto \exp(k\Delta) - k\Delta - 1;\ k \neq 0$$

Where $\Delta = \hat{\alpha} - \alpha$ and $\hat{\alpha}$ is an estimate of α .The Bayes estimator of α, denoted by $\hat{\alpha}_L$ under the LINEX loss function is given by

$$\hat{\alpha}_L = -\frac{1}{k}\ln E_\alpha\left(\exp(-k\alpha)\right) \quad (8)$$

Under LINEX loss function, we obtain Bayesian estimator of the parameter α by Combining the posterior (7) and Bayes estimator (8) as the following form (Bayesian estimator under LINEX loss function denoted by $\hat{\alpha}_L$)

$$\hat{\alpha}_L = -\frac{1}{k}\ln\left[\int_0^\infty \exp(-k\alpha)\pi(\alpha \mid \mathbf{X})\right] = \frac{n+a}{k}\ln\left(1+\frac{k}{b-T}\right) \quad (9)$$

The Bayes estimate of the reliability, $\hat{S}_L$, based on the LINEX loss function is obtained from (3) and (7) as

$$\begin{aligned}\hat{S}_L &= -\frac{1}{k}\ln\left[E\left(e^{-ke^{\beta^\alpha(\beta+t^2)^{-\alpha}}}\right)\right] \\ &= -\frac{1}{k}\ln\left[\sum_{i=0}^{\infty}\frac{(-k)^i}{\Gamma(i+1)}\left(\frac{b-T}{b-T+iT^*}\right)^{n+a}\right]\end{aligned} \quad (10)$$

Where $T^* = \ln\left(1+\frac{t^2}{\beta}\right)$.

2.2. Bayesian estimation for α and Reliability Function with General Entropy Loss Function

The General Entropy loss function for α can be expressed as the following form [5].

$$L(\delta) = \delta^q - q\ln\delta - 1;\ q \neq 0$$

where $\delta = \frac{\hat{\alpha}}{\alpha}$ and $\hat{\alpha}$ is an estimate of α . The Bayes estimator of α, denoted by $\hat{\alpha}_G$ under the General.

Entropy loss function is given by

$$\hat{\alpha}_G = \left[E\left(\alpha^{-q}\right)\right]^{-\frac{1}{q}} \quad (11)$$

Under General Entropy loss function, we obtain Bayesian estimator of the parameter α by Combining the posterior (7) and Beyas estimator (11) similarly (Bayesian estimator under $\hat{\alpha}_G$)

$$\hat{\alpha}_G = \left[\frac{\Gamma(n+a-q)}{\Gamma(n+a)}\right]^{-\frac{1}{q}}(b-T)^{-1} \quad (12)$$

The Bayes estimate of the reliability, $\hat{S}_G$, based on the General Entropy loss function is obtained from (3) and (7) as

$$\hat{S}_G = \left(\frac{b-T}{b-T-qT^*}\right)^{-\frac{n+a}{q}}. \quad (13)$$

2.3. Bayesian estimation for α and Reliability Function with Squared Error Loss Function

Under the squared error loss function with following form

$$L(\hat{\alpha} - a) = (\hat{\alpha} - \alpha)^2$$

The Bayes estimator of α, denoted by $\hat{\alpha}_S$ is given by

$$\hat{\alpha}_S = E(\alpha \mid \mathbf{X})$$

The Bayes estimate of the parameter and reliability function, based on the Squared error loss function are obtained as

$$\hat{\alpha}_S = \frac{n+a}{b-T} \quad (14)$$

$$\hat{\alpha}_S = \left(\frac{b-T}{b-T+T^*}\right)^{n+a}. \quad (15)$$

3. Empirical Bayes Estimation

Assume that the conjugate family of prior distributions for α is the family of gamma distributions, $\Gamma(a,b)$ with known a and unknown b. The Bayes estimators in (9), (10),(12), (13), (14) and (15) is seen to depend on the parameter b. When the prior parameter b is unknown, we may use the empirical Bayes approach to get its estimate [3]. From (5) and (6), we calculate the marginal pdf of X, with density.

$$f(x\mid b) \propto \int_0^{\infty} \alpha^n \left(\frac{\beta}{\beta + x_n^2}\right)^{\alpha} \prod_{i=1}^{n} \frac{x_i}{\beta + x_i^2} \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\beta} d\alpha \qquad (16)$$

$$\propto \frac{\Gamma(n+a) b^a}{\Gamma(a)(b-T)^{n+a}} \prod_{i=1}^{n} \frac{x_i}{\beta + x_i^2}$$

Assume that a is known, then Based on $f(x \mid b)$, we obtain an estimate, $\hat{b}$, of b. The MLE of b is

$$\hat{b} = -\frac{a}{n} T$$

Now, by substituting $\hat{b}$ for b in the Bayes estimators, we obtain the empirical Bayes estimators as

$$\hat{\alpha}_{EML} = \frac{n+a}{k} \ln\left(1 - \frac{kn}{T(a+n)}\right)$$

$$\hat{\alpha}_{EML} = \left[\frac{\Gamma(n+a-q)}{\Gamma(n+a)}\right]^{-\frac{1}{q}} \left(\frac{-n}{T(a+n)}\right)$$

$$\hat{\alpha}_{EMS} = \frac{-n}{T}$$

$$\hat{S}_{EML} = -\frac{1}{k} \ln\left[\sum_{i=0}^{\infty} \frac{(-k)^i}{\Gamma(i+1)} \left(\frac{T(a+n)}{T(a+n) - niT^*}\right)^{n+a}\right]$$

$$\hat{S}_{EMG} = \left(\frac{T(a+n)}{T(a+n) + nqT^*}\right)^{-\frac{n+a}{q}}$$

$$\hat{S}_{EMG} = \left(\frac{T(a+n)}{T(a+n) - nT^*}\right)^{n+a}.$$

4. Simulation

We used the following steps to generate a Recorded values from the Compound Rayleigh distribution and compute different estimates.

1. For a given values of prior parameters a = 1, b = 0.5 and $\beta = 2$, we generate $\alpha = 1.886493$ from the prior density (6).
2. For given α obtained in step(1), we generate n = (4, 5, 6) upper record values from the Compound Rayleigh distribution with pdf (1) using $x_i = \left(\frac{\beta(1-(1-u_i)^{\frac{1}{\alpha}})}{(1-u_i)^{\frac{1}{\alpha}}}\right)^{\frac{1}{2}}$

Where U_i independent uniform(0,1) random variables.

3. We obtained the estimates N = 2000 times and calculated the MSE given by

$$MSE = \frac{1}{2000} \sum_{i=1}^{2000} (\hat{\varphi}_i - \varphi)^2$$

where $\hat{\varphi}$ is an estimate of φ.

The true values of $S(t)$ in $t = 0.5$ is obtained $S(0.5) = 0.8007577$. The results are summarized in Table 1.

Table 1. *Averaged values of MSEs for estimates of the parameter α and reliability function S(t).*

n	$\hat{\alpha}_S$	$\hat{\alpha}_L(k=0.5)$	$\hat{\alpha}_L(k=-0.5)$	$\hat{\alpha}_G(q=1)$	$\hat{\alpha}_G(q=2)$	$\hat{\alpha}_{EMS}$	$\hat{\alpha}_{EML}(k=0.5)$	$\hat{\alpha}_{EML}(k=-0.5)$	$\hat{\alpha}_{EMG}(q=1)$	$\hat{\alpha}_{EMG}(q=2)$
4	0.2336	0.1916	0.4329	0.3057	0.4664	0.4457	0.3129	1.3736	0.4368	0.5753
5	0.1663	0.1351	0.2898	0.2066	0.3126	0.2712	0.2032	0.5038	0.2782	0.3692
6	0.1240	0.0900	0.2227	0.1238	0.1888	0.1756	0.1262	0.3027	0.1584	0.2158
	$\hat{S}_S$	$\hat{S}_L(k=0.5)$	$\hat{S}_L(k=-0.5)$	$\hat{S}_G(q=1)$	$\hat{S}_G(q=2)$	$\hat{S}_{EMS}$	$\hat{S}_{EML}(k=0.5)$	$\hat{S}_{EML}(k=-0.5)$	$\hat{S}_{EMG}(q=1)$	$\hat{S}_{EMG}(q=2)$
4	0.0018	0.0020	0.0016	0.0021	0.0023	0.0031	0.0030	0.0027	0.0037	0.0041
5	0.0011	0.0015	0.0011	0.0012	0.0013	0.0016	0.0023	0.0015	0.0018	0.0020
6	0.0009	0.0014	0.0010	0.0011	0.0013	0.0013	0.0019	0.0015	0.0015	0.0017

5. Conclusions

Empirical Bayes methods are procedures for statistical inference in which the prior distribution is estimated from the data [6, 7, 8]. This approach stands in contrast to standard Bayesian methods, for which the prior distribution is fixed before any data are observed. In this paper using the Bayes and Empirical Bayes estimation methods, we compute the estimates of the parameter and the reliability function of the Compound Rayleigh distribution when the Record samples are available [5,6].

Our interest here was we anticipated the different estimators to be differently obtained under the symmetric (squared error) and the asymmetric (LINEX and General Entropy) loss functions. We used the MSE for compare the estimators. Based on the results shown in Table 1, one can conclude that the MSE of the Bayesian estimates of α and reliability function are overall the smallest MSE as the as compared with the Empirical Bayes estimates. For parameter α, The MSE of Bayesian estimates and Empirical Bayes estimates squared error loss function have smallest MSE as the as compared with the Bayesian estimates and Empirical Bayes estimates under LINEX and General Entropy loss functions. It is immediate to note that MSE of Bayesian and Empirical Bayes estimators decrease as n increase.

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