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Generalized estimation of missing observations in nonlinear time series model using state space representation

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Abstract: The aim of the study was to formulate a Time Series Model to be used in obtaining optimal estimates of missing observations. State space models and Kalman filter were used to handle irregularly spaced data. A non-Bayesian approach where the missing values were treated as fixed parameters**.** Simulated AR (1) data and corresponding estimated missing values were generated using a computer programme. Values were withheld and then estimated as though they were missing. The results revealed that simple exposition of state space representation for commonly used Time Series Models can be formulated.

Keywords: Model, Linear, Non-Linear, Simulated, Non-Bayesian

1. Introduction

One of the unfortunate facts facing data analysts is missing data. Data that are known to have been observed erroneously can be categorized as missing. Erroneous data can also wreak havoc with the estimation and forecasting of time series models.

In the past, estimation of missing observations has been considered among others by [1, 2, 3, 4, 5, 6]. In particular, [7] and [8] developed state space method for dealing with missing observations in the long- memory context. More recently, [9] have given statistical analysis of incomplete long-range dependent data and the application of these procedures to the analysis of the annual minimum water levels of Nile River.

This paper addresses both theoretical and methodological issues related to the estimation of missing observations. Estimates are calculated by means of state space models and Kalman filter.

2. State Space Models

The linear state space system is given by, $\theta_{t+1} = \alpha_t\theta_t + \beta_t\mu_{t+1}, y_t = A_{t-1}\theta_t + \beta_{t-1}\upsilon_t$(2.1) where, θ_t and μ_t are $p \times 1$ vectors, y_t and υ_t are $q \times 1$ vectors, $\alpha_t$and β_t are p x p matrices, and A_t and $B_t$are matrices of dimensions $p \times p$ and $q \times q$ respectively $\{y_t\}$ represents the observed time series, whereas $\alpha_t, A_t, \beta_t, B_t$ are known matrices of nonrandom function. The vectors $\{\mu_t\}, \{\upsilon_t\}$ are independent each being a sequence of independent normal random vectors, having components with zero mean and unit variance. In order to handle various deviations which may occur in practice several generalization of (2.1) have been suggested.

In this paper we consider the model in (2.1) with random coefficients. We allow the coefficients in (2.1) to depend on past observations as follows;

$\alpha_t = \alpha(t, F_t^y), \beta_t = \beta(t, F_t^y), A_{t-1} = A(t-1, F_{t-1}^y)$ and $B_{t-1} = B(t-1, F_{t-1}^y)$, where F_t^y denotes the σ-field generated by observationsup to time t.

We refer to (2.1) under this settings as the generalized model. This generalized model encompasses some of the non linear time series models that have been proposed in the literature.

(i) ARCH models: Supposed that $\alpha_t = \alpha, A_{t-1} = A$ and $B_t \cong 0$ so that,

$$\theta_{t+1} = \alpha\theta_t + \beta_t\mu_{t+1}, y_t = A\theta_t$$

This is the ARCH model described in [10]

(ii) Dynamic linear state space models: When$\{\alpha_t\}, \{\beta_t\}$ and $\{B_t\}$ are constant matrices and A_{t-1} is a matrix of knownfunctions at $t-1$ (i.e. A_{t-1} is F_t^y measurable) the generalized model (2.1) becomes,

$$\theta_t = \alpha\theta_t + \mu_t, y_t = A_{t-1}\theta_t + v_t$$

Which is the state space model described in [11]

(iii) Doubly stochastic time series model (cf.Tjøstheim (1986)) [12]: When $\alpha_t = 1, \beta_t = 1\mu_{t+1} = \varepsilon_{t+1} - \varepsilon_{t-1}$ and $B_t = 1$, then (2.1) becomes

$$\theta_{t+1} = \theta_t + \mu_{t+1}, y_t = A_{t-1}\theta_t + v_t.$$

This corresponds to the doubly stochastic time series model

$$\theta_{t+1} = \theta_t + \varepsilon_t + \varepsilon_{t-1}, y_t = \theta_t f\left(t, F_{t-1}^y\right) + e_t,$$

considered in [13].

When$\left(t, F_{t-1}^y\right) = y_{t-1}$, this turns out to be a special case of the random Coefficient Autoregression (RCA) model of [14]. Moreover if we take, $\theta_t = \alpha_{t-1}\theta_{t-1} + \mu_t,\ y_t = \theta_t$ with $\alpha_{t-1} = \phi + \pi \exp\left(-\gamma y_{t-1}^2\right)$ then the generalized model (2.1) describes the exponential autoregressive model of [15].

Theorem 2.1 This theorem gives prediction and fixed point smoothing algorithms for the generalize model (2.1)

Let,$\hat{\theta}_t = E\left(\theta_t \backslash F_{t-1}^y\right)$, $\Sigma_t = \mathrm{E}\left[\left(\theta_t - \hat{\theta}_t\right)\left(\theta_t - \hat{\theta}_t\right)^{\mathrm{T}} \backslash Ft{-}1y\right.$

Then,

$$\hat{\theta}_{t+1} = \alpha_t\hat{\theta}_t + k_t[y_t - \hat{y}_t],$$

$$\Sigma_{t+1} = \beta_t{\beta_t}^T + (\alpha_t - k_tA_{t-1})\Sigma_t(\alpha_t - k_tA_{t-1})^T + k_tB_{t-1}B_{t-1}^Tk_t^T$$

Where,$k_t = \alpha_t\Sigma_tA_{t-1}^T[A_{t-1}\Sigma_tA_{t-1}^T + B_{t-1}B_{t-1}^T]^+$and $\hat{y}_t = E\left(y_t \backslash F_{t-1}^y\right)$; $M^T$and M^+ denote the transpose and the pseudo inverse respectively of a matrix M.

Proof of this theorem is a straight forward extension of results as presented by [16].

Theorem 2.2

For$t > t_j$, let $\tilde{\theta}_{t_j\backslash t} = E\left(\theta_{t_j} \backslash F_t^y\right)$be the estimate of θ_{t_j} base on the observations up to time t, $\tilde{\Sigma}_t$be the covariance matrix, $\tilde{\Sigma}_t = E\left[\left(\theta_{t_j} - \hat{\theta}_t\right)\left(\theta_{t_j} - \hat{\theta}_{t_j}\right)^T \backslash F_{t-1}^y\right]and\Sigma_t^* = \mathrm{E}\left[\left(\theta_{t_j} - \hat{\theta}_{t_j\backslash t}\right)\left(\theta_{t_j} - \hat{\theta}_{t_j\backslash t}\right)^T \backslash F_{t-1}^y\right]$

Then,$\tilde{\theta}_{t_j\backslash t} = \tilde{\theta}_{t_j\backslash t-1} + \tilde{k}_t\left(y_t - A_{t-1}\hat{\theta}_t\right), t > t_j$

Where $\tilde{k}_t = \tilde{\Sigma}_tA_{t-1}^T[A_{t-1}\Sigma_tA_{t-1}^T + B_{t-1}B_{t-1}^T]^+$,

$$\tilde{\Sigma}_{t+1} = \tilde{\Sigma}_t[\alpha_t - k_tA_{t-1}]^T, \Sigma_{t-1}^* = \Sigma_t \text{ for } t < t_j \text{ and} \Sigma_t^* = \Sigma_{t-1}^* - \tilde{\Sigma}_tA_{t-1}^T\tilde{k}_t^T, t \geq t_j.$$

Proof of this theorem is a straight forward generalization of results in [16] or [17].

This research used state space models methodology which can handle irregularly spaced data. It also used the Kalman filtering technique and a non-Bayesian approach where the missing values were treated as fixed parameters.

3. Results and Discussions

Autoregressive conditionally heteroscedastic (ARCH) type models and application to missing data

There are two different approaches of estimating missing values in time series;

1. A Bayesian approach: Which uses Kalman filtering technique.
2. A non-Bayesian approach: Where the missing values are treated as fixed parameters.

This paper uses the Kalman type recursive approach to estimate the missing values by replacing them with normal random variables. This type of approach may be viewed as one which uses a prior for the parameter which replaces the missing value.

3.1. One Missing Observation

Now we indicate an appropriate way to modify a given non linear time series to reflect the fact that the observation at time m is missing.Let $\{X_t\}$be a time series in which X_m is missing and $X_n' = \{X_1, X_2, \ldots, X_{m+1}, \ldots, X_n\}$

If we know the first two conditional moments $E[X_{t+1} \backslash Ftx$ and $varXt{+}1\backslash Ftx$, then $Xt{+}1$ can be written as$X_{t+1} = E[X_{t+1} \backslash F_t^x] + X_{t+1} - E[X_{t+1} \backslash F_t^x]$(1)

Suppose that the time series X_t satisfies$E[X_{t+1} \backslash F_t^x] = \alpha_{t-1}X_t$and $X_{t+1} - E[X_{t+1} \backslash F_t^x] = \beta_{t-1}\mu_{t+1}$ (2)

Where α_{t-1} and β_{t-1} are F_{t-1}^x measurable and $\{\mu_t\}$ is an i.i.d. $\sim N(0,1)$ sequence. Then X_{t+1} has ARCH representation.

$$X_{t+1} = \alpha_{t-1}X_t + \beta_{t-1}\mu_{t+1} \quad (3)$$

The restriction in (2) is introduced to apply the recursive approach. Now we consider the estimation of a missing observation as a parameter estimation problem in a particular formulation of the generalized model (2.1).

$$\theta_{t+1} = \alpha_{t-1}\theta_t + \beta_{t-1}\mu_{t+1}, Var[X_{t+1} \backslash F_t^x](4)$$

$$y_t = A_{t-1}\theta_t + B_{t-1}v_t \text{ with } A_{m-1} = 0, B_{m-1} = 1, A_t = 1, t \neq m-1; B_t = 0, t \neq m-1;$$

Then $Y = \left(X_1, X_2, \ldots, X_{m-1}, v_m, X_{m+1}, X_{m+2}, \ldots, X_n\right)$ is the extended observed series. Here v_m is a random variable replacing the missing observation.

Such a formulation was also considered in Abraham and Thavaneswaran (1991) [1]

Using theorem 2.1 and 2.2 we have equations,

$k_t = \alpha_{t-1}\Sigma_tA_{t-1}(A_{t-1}^2\Sigma_t + B_{t-1}^2)^+$ and $\Sigma_{t+1} = \beta_{t-1}^2 + (\alpha_{t-1} - k_tA_{t-1})^2\Sigma_t + k_t^2B_{t-1}^2$.

Substituting the values of $A_t,\ A_{t-1}, B_t$and B_{t-1} gives

$$k_t = \alpha_{t-1}\Sigma_t[\Sigma_t]^+$$

$$k_t = \alpha_{t-1} \quad (5)$$

For $t \neq m$ and

$$\Sigma_{t+1} = \beta_{t-1}^2 \quad (6)$$

Alsofor $t = m$, $k_m = \alpha_{m-1}\Sigma_m A_{m-1}(A^2{}_{m-1}\Sigma_m +$ $Bm-12+$,

But $A_{m-1} = 0$, hence $k_m = 0$and $\Sigma_{m+1} = \beta_{m-1}^2 +$ $(\alpha_{m-1} - k_m A_{m-1})^2\Sigma_m + k_m^2 B_{m-1}^2$but $k_m = 0$ to give $\Sigma_{m+1} = \beta_{m-1}^2 + \alpha_{m-1}^2\Sigma_m$ (7)

Where from (6) when$t = m$ we have$\Sigma_{m+1} = \beta_{m-1}^2$ so that $\Sigma_m = \beta_{m-2}^2$.

Substituting for $\Sigma_m$in (7) we have $\Sigma_{m+1} = \beta_{m-1}^2 +$ $\alpha_{m-1}^2\beta_{m-2}^2$(8)

Also from theorem 2.2 we have $\tilde{\Sigma}_{t-1} = \tilde{\Sigma}_t(\alpha_{m-1} -$ $kmAm-1$but km=0 so that

$$\tilde{\Sigma}_{t-1} = 0 \quad (9)$$

and $\tilde{\Sigma}_{m+1} = \tilde{\Sigma}_m(\alpha_{m-1} - k_m A_{m-1})$. But $k_m = 0$so that $\tilde{\Sigma}_{m+1} = \tilde{\Sigma}_m\alpha_{m-1}^2$,also $\tilde{\Sigma}_m = \Sigma_m = \beta_{m-2}^2$.

Hence,$\tilde{\Sigma}_{m+1} = \alpha_{m-1}^2\beta_{m-2}^2$ (10)

We also have from theorem (2.2)$\tilde{\theta}_{t_j\backslash t} = \tilde{\theta}_{t_j\backslash t-1} +$ $\tilde{k}_t(y_t - A_{t-1}\hat{\theta}_t)$, but in this case$t = m$ and $A_t = 1$ gives,

$\tilde{\theta}_{m\backslash t} = \tilde{\theta}_{m\backslash t-1} + \tilde{k}_t(y_t - A_{t-1}\hat{\theta}_t)$,and at $t = m + 1$

$$\tilde{\theta}_{m\backslash m+1} = \tilde{\theta}_{m\backslash m} + \tilde{k}_{m+1}(y_{m+1} - \hat{\theta}_{m+1}) \quad (11)$$

Since, $\theta_t = y_t = X_t$.To find$\tilde{\theta}_{m\backslash m}$ we have that $\tilde{\theta}_{m\backslash m} =$ $E[\theta_m \backslash F_m^y] = E[(\alpha_{m-2}\theta_{m-1} + \beta_{m-2}\mu_m)\backslash F_m^y]$

$= \alpha_{m-2}\hat{\theta}_{m-1} = \alpha_{m-2}X_{m-1}$.

Next,$\hat{\theta}_{m+1} = E[\theta_{m+1}\backslash F_{m-1}^y] = E[(\alpha_{m-2}\theta_m +$ $\beta m-1\mu m+1\backslash Fmy$which gives $\theta m-1=\alpha m-1\theta m$ and

$\hat{\theta}_m = \alpha_{m-2}\hat{\theta}_{m-1} = \alpha_{m-2}X_{m-1}$.Combining this gives $\hat{\theta}_{m+1} = \alpha_{m-1}\alpha_{m-2}X_{m-1}$.

Moreover, $\tilde{k}_t = \tilde{\Sigma}_t A_{t-1}^T[A_{t-1}\Sigma_t A_{t-1}^T + B_{t-1}B_{t-1}^T]^+$

$$\tilde{k}_t = \frac{\tilde{\Sigma}_t}{\Sigma_t} \quad (12)$$

Since $A_{t-1} = 1, B_{t-1} = 0$ and at $t = m + 1$(12) leads to

$$\tilde{k}_{m+1} = \frac{\tilde{\Sigma}_{m+1}}{\Sigma_{m+1}} = \frac{\alpha_{m-1}\beta_{m-2}^2}{\beta_{m-1}^2 + \alpha_{m-1}^2\beta_{m-2}^2} \text{from (8) and (10). (13)}$$

Thus the estimate of the mth observation based on X_{m-1} is $\tilde{X}_{m\backslash m+1} = \alpha_{m-2}X_{m-1} + \frac{\alpha_{m-1}\beta_{m-2}^2}{\beta_{m-1}^2 + \alpha_{m-1}^2\beta_{m-2}^2}[X_{m+1} -$ $\alpha m-1\alpha m-2Xm-1$(14)

This simplifies to

$$\tilde{X}_{m\backslash m+1} = \frac{\alpha_{m-2}\beta_{m-1}^2 X_{m-1} + \alpha_{m-1}\beta_{m-2}^2 X_{m+1}}{\beta_{m-1}^2 + \alpha_{m-1}^2\beta_{m-2}^2}$$

Moreover, for a nonlinear model of the form

$X_{t+1} = \phi X_t + \mu_{t+1}$ in which the mth observation X_m is missing, the estimate of X_m based on F_{m+1}^y is given by

$$\tilde{X}_{m\backslash m+1} = \frac{\phi X_{m-1}}{1 + \phi^2 X_{m-1}^2}[X_{m-2} + X_{m+1}]$$

Autoregressive models with deterministic time varying coefficient:

Model of the form

$$X_t - \alpha(t,\phi)X_{t-1} = \mu_t \quad (15)$$

have been found to be quite useful, in particular in signal processing [18] as in 3.15, it can be shown that the estimate $\tilde{X}_{m\backslash m+1}$ of the missing observation based on $F_{m+1}^y$is given by;

$$\tilde{X}_{m\backslash m+1} = \frac{\alpha(m+1,\phi)X_{m+1} + \alpha(m,\phi)X_{m-1}}{1 + \alpha^2(m+1,\phi)}$$

Bilinear models:

Consider the model$X - \phi X = cu_t + \beta X_{t-2}u_t$

The estimation of missing observation, X_m, can be obtained by writing the model as $X_t = \alpha_{t-2}X_{t-1} + \beta_{t-2}u_t$

Where$\alpha_{t-2} = \phi$ and $\beta_{t-2} = c + \beta X_{t-2}$. Hence the estimate of X_m, $\tilde{X}_{m\backslash m+1} = E[X_m\backslash F_{m+1}^y]$ and is given by

$$\tilde{X}_{m\backslash m+1} = \frac{\phi\beta_{m-2}^2 X_{m+1} + \phi\beta_{m-1}^2 X_{m-1}}{\beta_{m-1}^2 + \phi^{2\beta_{m-2}^2}}$$

3.2. Two Consecutive Missing Observations

(a) The estimate of mth observation based onX_{m+2}.

We consider a slightly modified form of the model (3) in which we let $\beta_t^2 = \sigma^2$, $X_{t+1} = \alpha_{t-2}X_{t-1} + \beta_{t-2}u_{t+1}$

Here X_m and X_{m+1} missing andα_t is F_t^xmeasurable. The problem is to estimate X_m based on the available data

$$X'_n = (X_1, X_2, \dots, X_{m-1}, X_{m+2}, \dots, X_n).$$

The corresponding state space model may be written as

$$\left.\begin{aligned} \theta_{t+1} &= \alpha_{t-1}\theta_t + \beta_{t-1}\mu_{t+1} \\ X_t &= A_{t-1}\theta_t \\ y_t &= A_{t-1}\theta_t + B_{t-1}v_t \end{aligned}\right\} (16)$$

Where $A_{m-1} = A_m = 0, B_{m-1} = B_m = 1$, $A_t = 1$, for $t \neq m$; $B_t = 0, t \neq m, m-1$

Then $Y = (X_1 X_2, \dots, X_{m-1}, v_m, X_{m+1}, X_{m+2}, \dots, X_n)$ is the extended observed series.

Here v_m is a random variable replacing the missing observation. By theorem (2.1) and (2.2) it can be shown that for $t \neq m, m+1, k_t = \alpha_{t-2}$ and $k_m = k_{m+1} = k_{m+2} = 0$.

Also from theorem 2.2 we have the following equations

$$\tilde{k}_t = \tilde{\Sigma}_t A_{t-1}[A_{t-1}^2 + \beta_{t-1}^2]^+, \tilde{\Sigma}_{t+1} = \tilde{\Sigma}_t(\alpha_{t-1} - k_t A_{t-1}).$$

Hence we have $\tilde{k}_t = \tilde{\Sigma}_t[\tilde{\Sigma}_t]^+ = \frac{\tilde{\Sigma}_t}{\Sigma_t} = 1$ (17)

which gives$\tilde{\Sigma}_{t+1} = \tilde{\Sigma}_t\alpha_{t-1}$ (18)

and because $A_m = A_{m-1} = 0$; $B_m = B_{m-1} = 1$; $A_t =$ $1, B_t = 0$ at $t \neq m, m+1$ we also get

$$\tilde{k}_m = \tilde{k}_{m+1} = 0 \quad (19)$$

It can be shown that $\tilde{\Sigma}_{m+2} = \alpha_{m-1}\alpha_{m-2}\beta_{m-3}^2$(20)

We have from theorem (2.2) we have the following equation $\tilde{\theta}_{m|t} = \tilde{\theta}_{m|t-1} + \tilde{k}_t(y_t - A_{t-1}\hat{\theta}_t)$,

Which by taking expectation and simplifying gives

$$\hat{\theta}_{m+2} = \alpha_{m-1}\hat{\theta}_{m+1}, \hat{\theta}_{m+1} = \alpha_{m-2}\hat{\theta}_m \text{ and } \hat{\theta}_m = \alpha_{m-3}\hat{\theta}_{m-1} = \alpha_{m-3}X_{m-1}$$

which combines to give

$$\hat{\theta}_{m+2} = \alpha_{m-1}\alpha_{m-2}\alpha_{m-3}X_{m-1}.$$

Thus the estimate of mthobservation based on X_{m+2} is

$$\tilde{X}_{m|m+2} = \alpha_{m-3}X_{m-1} + \tilde{k}_{m+2}(X_{m+2} - \alpha_{m-1}\alpha_{m-2}\alpha_{m-3}X_{m-1})(21)$$

But $\tilde{k}_{m+2} = \frac{\alpha_{m-1}\alpha_{m-2}\beta^2_{m-3}}{\beta^2_{m-1}\alpha^2_{m-1}\beta^2_{m-2}+\alpha^2_{m-1}\alpha^2_{m-2}\beta^2_{m-3}}$ (22)

Hence (21)becomes,$\tilde{X}_{m|m+2} = \alpha_{m-3}X_{m-1} + \frac{\alpha_{m-1}\alpha_{m-2}\beta^2_{m-3}}{\beta^2_{m-1}\alpha^2_{m-1}\beta^2_{m-2}+\alpha^2_{m-1}\alpha^2_{m-2}\beta^2_{m-3}}(X_{m+2} - \alpha m{-}1\alpha m{-}2\alpha m{-}3Xm{-}1,$

Which simplifies to give$\tilde{X}_{m|m+2} = \frac{\alpha_{m-3}(\beta^2_{m-1}\alpha^2_{m-1}\beta^2_{m-2})X_{m-1}\alpha^2_{m-1}\alpha^2_{m-2}\beta^2_{m-3}X_{m+2}}{\beta^2_{m-1}\alpha^2_{m-1}\beta^2_{m-2}+\alpha^2_{m-1}\alpha^2_{m-2}\beta^2_{m-3}}$(23)

Which is the required estimate of mth observation based onX_{m+2}.

(b) The estimate of the $(m+1)th$observation based on X_{m+2}.

We estimateX_{m+1}observation based on available data$X'_n = (X_1, X_2, \dots, X_{m-1}, \hat{X}_m, X_{m+2}, \dots, X_n)$, where $\hat{X}_m$ is the estimate of X_m from the previous steps.

The corresponding state space models becomes,

$$\left.\begin{aligned} \theta_{t+1} &= \alpha_{t-4}\theta_t + \beta_{t-4}u_{t+1} \\ X_t &= A_{t-1}\theta_t \\ y_t &= A_{t-1}\theta_t + B_{t-1}v_t \end{aligned}\right\}(24)$$

In this case we have $A_m = 0, B_m = 1$; and $B_t = 0, A_t = 1$ for $t \neq m+1$. The extended data will be, $Y = (X_1, X_2, \dots, X_{m-1}, \hat{X}_m, v_{m+1}, X_{m+2}, \dots, X_n)$where v_{m+1}a normal random variable is replacing the missing observation. This estimate is now treated as if only one observation is missing in the data.

Using theorems 2.1 we have the modified form as,

$\Sigma_{t+1} = \beta^2_{t-3} + (\alpha_{t-3} - k_tA_{t-1})^2\Sigma_t + k_t^2B^2_{t-1}andk_t = \alpha_{t-3}\Sigma_tA_{t-1}[A^2_{t-1}\Sigma_t - B^2_{t-1}]^+$, then

$k_t = \alpha_{t-3}\Sigma_t[\Sigma_t]^+ = \alpha_{t-3}$ (25)

and $\Sigma_{t+1} = B^2_{t-3} + (\alpha_{t-3} - k_t)^2\Sigma_t$

But from (25) $k_t = \alpha_{t-3}$, hence$\Sigma_{t+1} = B^2_{t-3}$(26)

Also from theorem (2.2) we have

$\tilde{k}_t = \tilde{\Sigma}_tA_{t-1}[A^2_{t-1}\Sigma_t - B^2_{t-1}]^+$and $\tilde{\Sigma}_{t+1} = \tilde{\Sigma}_t(\alpha_{t-3} - ktAt{-}1T$. Since, $At{=}1$, $Bt{=}0$; $t{\neq}m{+}1$ then this becomes$\tilde{k}_t = \Sigma_t[\Sigma_t]^+ = \frac{\tilde{\Sigma}_t}{\Sigma_t}$as in (17) and $\tilde{\Sigma}_{t+1} = 0$(27)

Since $A_{t-1} = 1$ and $k_t = \alpha_{t-3}$.

Setting $t = m+1$ $A_m = 0, B_m = 1$ we obtain $k_{m+1} = 0$ (28)

and $\Sigma_{m+1} = \beta^2_{t-2} + \alpha^2_{m-2}\Sigma_{t+1}$.But from(26) at $t = m$, $\Sigma_{m+2} = \beta^2_{m-3}$

Hence,

$\Sigma_{m+2} = B^2_{m-2} + \alpha^2_{m-2}\beta^2_{m-3}$(29)

and $\tilde{\Sigma}_{m+2} = \tilde{\Sigma}_{m+2}(\alpha_{m-2} - k_{m+1}A_m)$ at $A_m = 0, k_{m+1} = 0$, we have;

$\tilde{\Sigma}_{m+2} = \alpha_{m-2}\tilde{\Sigma}_{m+1}$ and $\tilde{k}_{m+1} = 0$. Also, $\tilde{\Sigma}_{m+2} = \alpha_{m-2}\tilde{\Sigma}_{m+1} = \alpha_{m-2}\Sigma_{m+1} = \alpha_{m-2}\beta^2_{m-3}$

$\tilde{\Sigma}_{m+2} = \alpha_{m-2}\beta^2_{m-3}$ (30)

But at$t \geq t_j, \tilde{\theta}_{tj|t} = \tilde{\theta}_{t_j|t-1} + \tilde{k}_t(y_t - \hat{\theta}_t)$ since $A_t = 1$.

In this case if we let $t_j = m+1$ and $t = m+2$ we have$\tilde{\theta}_{m+1|m+2} = \tilde{\theta}_{m+1|m+1} + \tilde{k}_{m+2}(y_{m+2} - \hat{\theta}_{m+2})$

Where,$\tilde{\theta}_{m+1|m+1} = E[\theta_{m+1}|F^y_{m+1}] = E[(\alpha_{m-3}\theta_m + \beta m{-}3um{+}1|Fm{+}1y$

Which gives$\tilde{\theta}_{m+1|m+1} = \alpha_{m-3}\hat{X}_m$.

Next$\hat{\theta}_{m+2} = E[\theta_{m+2}|F^y_{m+1}] = E[(\alpha_{m-2}\theta_{m+1} + \beta m{-}2um{+}2|Fm{+}1y{=}\alpha m{-}2E\theta m{+}1|Fm{+}2y{=}\alpha m{-}2\theta m{+}1and\hat{\theta}_{m+1} = \alpha_{m-2}\hat{\theta}_m$ which combines to give $\hat{\theta}_{m+2} = \alpha_{m-2}\alpha_{m-3}\hat{\theta}_m = \alpha_{m-2}\alpha_{m-3}\alpha_{m-4}\hat{X}_m$, $\hat{\theta}_m = \hat{X}_m$ since the data has been observed up to time $t = m$.

Hence,

$$\hat{X}_{m+1|m+2} = \alpha_{m-3}\hat{X}_m + \tilde{k}_{m+2}(X_{m+2} - \alpha_{m-2}\alpha_{m-3}\hat{X}_m)(31)$$

But,$\tilde{k}_{m+2} = \frac{\tilde{\Sigma}_{m+2}}{\Sigma_{m+2}} = \frac{\alpha_{m-2}\beta^2_{m-3}}{\beta^2_{m-2}+\alpha^2_{m-2}\beta^2_{m-3}}$ (32)

Replacing for$\tilde{k}_{m+2}$ in (31) we obtain$\tilde{X}_{m+1|m+2} = \alpha_{m-3}\hat{X}_m + \frac{\alpha_{m-2}\beta^2_{m-3}}{\beta^2_{m-2}+\alpha^2_{m-2}\beta^2_{m-3}}(X_{m+2} - \alpha_{m-2}\alpha_{m-3}\hat{X}_m)$.

Simplifying this gives$\tilde{X}_{m+1|m+2} = \frac{\alpha_{m-3}(\beta^2_{m-2})\hat{X}_m+\alpha_{m-2}\beta^2_{m-3}X_{m+2}}{\beta^2_{m-2}+\alpha^2_{m-2}\beta^2_{m-3}}$ (33)

3.3. Three Consecutive Missing Observations

a) The estimate of mth observation based on X_{m+3}.

The modified state space model is given by

$$\left.\begin{aligned} \theta_{t+1} &= \alpha_{t-3}\theta_t + \beta_{t-3}u_{t+1} \\ X_t &= A_{t-1}\theta_t \\ y_t &= A_{t-1}\theta_t + B_{t-1}v_t \end{aligned}\right\}(42)$$

In this case we have three consecutive missing observationsX_m, X_{m+1} and X_{m+2} for which we need to estimate X_m before we estimate the remaining two observations respectively.The initial observation set is $X'_n = (X_1, X_2, \dots, X_{m-1}, X_{m+3}, \dots, X_n)$. The extended observed series is

$Y = (X_1, X_2, \dots, X_{m-1}, v_m, X_{m+3}, \dots, X_n)$ wherev_m is a normal random variable replacing the missing observation.

Usingtheorems 2.1 and2.2we have the modified set of equations

$$k_t = \alpha_{t-3}\Sigma_tA_{t-1}[A^2_{t-1}\Sigma_t\beta^2_{t-1}]^+ \ (34)$$

and $\Sigma_{t+1} = \beta^2_{t-3} + (\alpha_{t-3} - k_tA_{t-1})^2\Sigma_t + k_t^2B^2_{t-1}$(44)

In this case$A_{m+1} = A_m = A_{m+1} = 0, B_{m-1} = B_m = B_{m+1} = 1; A_t = 1, B_t = 0; t \neq m, m+1, m+2$.

This implies that,$k_t = \alpha_{t-3}$and$\Sigma_{t+1} = \beta^2_{t-3}$.

It can be shown that $\Sigma_{m+3} = \beta^2_{m-1}\alpha^2_{m-1}\beta^2_{m-2} + \alpha^2_{m-1}\alpha^2_{m-2}\beta_{m-3} + \alpha^2_{m-1}\alpha^2_{m-2}\alpha^2_{m-3}\beta^2_{m-4}$ (35)

and $\tilde{\Sigma}_{m+3} = \alpha_{m-1}\alpha_{m-2}\alpha_{m-3}\beta^2_{m-4}$

We have$\tilde{\theta}_{m\backslash m+3} = \tilde{\theta}_{m\backslash m+2} + \tilde{k}_{m+3}(y_{m+3} - \hat{\theta}_{m+3})$ (36)

$$\tilde{k}_{m+3} = \frac{\tilde{\Sigma}_{m+3}}{\Sigma_{m+3}} = \frac{\alpha_{m-1}\alpha_{m-2}\alpha_{m-3}\beta^2_{m-4}}{\beta^2_{m-1}+\alpha^2_{m-1}\beta^2_{m-2}+\alpha^2_{m-1}\alpha^2_{m-2}\beta^2_{m-3}+\alpha^2_{m-1}\alpha^2_{m-2}\alpha^2_{m-3}\beta^2_{m-4}} \ (37)$$

Making successive substitution in (36) we get

$\tilde{X}_{m|m+3} =$
$\alpha_{m-4}X_{m-1} +$
$am-1am-2am-3$
$\beta m-42\beta m-12+am-12\beta m-22+am-12am-22\beta m-32+am-12am-22am-32\beta m-42Xm+3-am-1am-2am-3am-4\ Xm-1.$

Simplifying this gives

$\tilde{X}_{m|m+3} =$
$am-4\beta m-12+am-12\beta m-22+am-12am-22\beta m-32Xm-1+am-12am-22am-32\beta m-42Xm+3\beta m-12+am-12\beta m-22+am-12am-22\beta m-32+am-12am-22am-32\beta m-42$(38)

(b) The estimate of $(m+1)th$ observation based on X_{m+3}

This is treated as two missing observations as seen earlier the modified state space representations are

$$\left.\begin{aligned} \theta_{t+1} &= \alpha_{t-4}\theta_t + \beta_{t-4}u_{t+1} \\ X_t &= A_{t-1}\theta_t \\ y_t &= A_{t-1}\theta_t + B_{t-1}v_t \end{aligned}\right\} \quad (39)$$

The initial observation is $X_N' = (X_1, X_2, \ldots, X_{m-1}, \hat{X}_m, X_{m+3}, \ldots, X_n)$, wesee that X_{m+1} and X_{m+2} are missing hence, we need to estimate X_{m+1}. Then the extended observed series is $Y = (X_1, X_2, \ldots, X_{m-1}, \hat{X}_m, v_{m+1}, X_{m+3}, \ldots, X_n)$, here $\hat{X}_m$is estimate of X_m baseonX_{m+3} andv_{m+1} is a normalrandom variable replacing the missing observation.

Using theorem 2.1 and 2.2 as before we have.

$A_m = A_{m+1} = 0, B_m = B_{m+1} = 1; A_t = 1; B_t = 0;$ $t \neq m+1, m+2$, $k_t = \alpha_{t-4}$and

$\Sigma_{t+1} = \beta_{t-4}^2$ (40)

Also $k_{m+1} = k_{m+2} = 0$.

Next we have $\Sigma_{m+2} = \beta_{m-2}^2 + \alpha_{m-3}^2\Sigma_{m+1}and\Sigma_{m+3} = \beta_{m-2}^2 + \alpha_{m-2}^2\Sigma_{m+2}$

Hence $\Sigma_{m+2} = \beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\Sigma_{m+1}$(41)

From (40) at $t = m, \Sigma_{m+1} = \beta_{m-4}^2$, hence(41) becomes

$$\Sigma_{m+3} = \beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2 \quad (42)$$

From theorem 2.2 we have

$\tilde{k}_t = \frac{\tilde{\Sigma}_t}{\Sigma_t}$ and$\tilde{\Sigma}_{t+1} = 0$. Also $\tilde{\Sigma}_{t+2} = \alpha_{m-3}\tilde{\Sigma}_{t+1}and\tilde{\Sigma}_{t+3} = \alpha_{m-2}\tilde{\Sigma}_{t+2}$

So that,$\tilde{\Sigma}_{t+3} = \alpha_{m-2}\alpha_{m-3}\tilde{\Sigma}_{t+1} = \alpha_{m-2}\alpha_{m-3}\Sigma_{m+1}$, where$\Sigma_{m+1} = \beta_{m-4}^2$.

Hence,$\tilde{\Sigma}_{t+3} = \alpha_{m-2}\alpha_{m-3}\ \beta_{m-4}^2$(43)

Also, $\tilde{k}_t = \frac{\tilde{\Sigma}_t}{\Sigma_t}$ so that,$\tilde{k}_{m+3} = \frac{\tilde{\Sigma}_{m+3}}{\Sigma_{m+3}}$

Where from equations (42) and (43) we get$\tilde{k}_{m+3} = \frac{\alpha_{m-2}\alpha_{m-3}\ \beta_{m-4}^2}{\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2}$ (44)

Then$\tilde{\theta}_{t_j|t} = \tilde{\theta}_{t_{j|t-1}} + \tilde{k}_t(y_t - A_{t-1}\hat{\theta}_t), t > t_j$ (45)

But $A_t = 1$ and $t_j = m+1$ and $t = m+3$ so that,

$$\tilde{\theta}_{m+1|m+3} = \tilde{\theta}_{m+1|m+2} + \tilde{k}_{m+3}\ (y_{m+3} - \hat{\theta}_{m+1})(46)$$

We have $\tilde{\theta}_{m+1|m+2} = E[\theta_{m+1}|F_{m+2}^y] = E[(\alpha_{m-4}\theta_m + \beta_{m-4}\mu_{m+1})|F_{m+2}^y] = \alpha_{m-4}E(\theta_m|F_{m+2}^y) = \alpha_{m-4}\hat{\theta}_m,$

But $\hat{\theta}_m = \hat{X}_m$ since we have made an observation up to time $t = m$ hence,$\tilde{\theta}_{m+1|m+2} = \alpha_{m-4}\hat{X}_m$.

Next,$\hat{\theta}_{m+3} = E[\theta_{m+3}|F_{m+2}^y] = E[(\alpha_{m-2}\theta_{m+2} +$ $\beta m-2um+3|Fm+2y=am-2\theta m+2$

$\theta_{m+2} = \alpha_{m-3}\hat{\theta}_{m+1}, \hat{\theta}_{m+1} = \alpha_{m-4}\hat{\theta}_m = \alpha_{m-4}\hat{X}_m$ and these combines to give

$\hat{\theta}_{m+3} = \alpha_{m-2}\alpha_{m-3}\hat{\theta}_{m+1} = \alpha_{m-2}\alpha_{m-3}\alpha_{m-4}\hat{\theta}_m$, which gives

$\hat{\theta}_{m+3} = \alpha_{m-2}\alpha_{m-3}\alpha_{m-4}\hat{X}_m$ (47)

Where$\hat{X}_m$is the estimate of X_m obtained ealier in the same section. Hence (46) becomes,

$$\hat{X}_{m+1|m+3} = \alpha_{m-4}\hat{X}_m + \tilde{k}_{m+3}(X_{m+3} - \alpha_{m-2}\alpha_{m-3}\alpha_{m-4}\ \hat{X}_m)$$

$$= \alpha_{m-4}\ \hat{X}_m + \frac{\alpha_{m-2}\alpha_{m-3}\ \beta_{m-4}^2}{\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2}(X_{m+3} - \alpha_{m-2}\alpha_{m-3}\alpha_{m-4}\ \hat{X}_m)$$

Simplifying this gives

$$\hat{X}_{m+1|m+3} = \frac{\alpha_{m-4}(\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2)\hat{X}_m + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2X_{m+3}}{\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2} \quad (48)$$

c) The estimate of $(m+2)th$ observation based on X_{m+3}.

This is treated as one missing observation since$X_m$andX_{m+1} has been estimated. Then we have the new set of observations as

$X_N' = (X_1, X_2, \ldots, X_{m-1}, \hat{X}_m, \hat{X}_{m+1}, X_{m+3}, \ldots, X_n)$the new state space representations are;

$$\left.\begin{aligned} \theta_{t+1} &= \alpha_{t-5}\theta_t + \beta_{t-5}u_{t+1} \\ X_t &= A_{t-1}\theta_t \\ y_t &= A_{t-1}\theta_t + B_{t-1}v_t \end{aligned}\right\} \quad (67)$$

Then $X_N' = (X_1, X_2, \ldots, X_{m-1}, \hat{X}_m, \hat{X}_{m+1}, v_{m+2}, X_{m+3}, \ldots, X_n)$ is the extended observed series. Here $\hat{X}_m$ and $\hat{X}_{m+1}$ are the estimates of X_m and X_{m+1} from the previous steps. v_{m+2} is a normal random variable replacingthe missing observation. In this case $A_{m+2} = 0, B_{m+2} = 1; A_t = 1, B_t = 0; t \neq m+2$. Again from the theorems 2.1 and 2.2 we have

$$k_t = \alpha_{t-5}, \Sigma_{t+1} = \beta_{t-5}^2(49)$$

$$k_{m+2} = 0, \Sigma_{m+3} = \beta_{m-3}^2 + \alpha_{m-3}^2\Sigma_{m+2} \ (50)$$

Setting$t = m$in (49) we obtain $\Sigma_{t+1} = \beta_{m-5}^2and\Sigma_{t+2} = \beta_{m-4}^2$.

Hence, (50) now becomes $\Sigma_{m+3} = \beta_{m-3}^2 +$

$\alpha_{m-3}^2\beta_{m-4}^2$(51)

Also $\tilde{k_t} = \frac{\tilde{\Sigma}_t}{\Sigma_t}$so that, $\tilde{k}_{m+3} = \frac{\tilde{\Sigma}_{m+3}}{\Sigma_{m+3}}$ (52)

$$\tilde{k}_{m+2} = 0, \tilde{\Sigma}_{m+3} = \alpha_{m-3}\tilde{\Sigma}_{m+2} = \alpha_{m-3}\Sigma_{m+2}.$$

Thus, $\tilde{\Sigma}_{m+3} = \alpha_{m-3}\beta_{m-4}$(53)

So that, $\tilde{k}_{m+3} = \frac{\alpha_{m-3}\,\beta_{m-4}^2}{\beta_{m-3}^2+\alpha_{m-3}^2\beta_{m-4}^2}$(54)

Then, $\tilde{\theta}_{t_j|t} = \tilde{\theta}_{t_j|t-1} + \tilde{k}_t\big(y_t - A_{t-1}\hat{\theta}_t\big), t > t_j$ (55)

But $A_t = 1$ at $t_j = m+2$ and $t = m+3$ so that (55) becomes

$$\tilde{\theta}_{m+2|m+3} = \tilde{\theta}_{m+2|m+2} + \tilde{k}_{m+3}\,\big(y_{m+3} - \hat{\theta}_{m+3}\big)(56)$$

Thus, $\hat{\theta}_{m+2|m+2} = E\big[\theta_{m+2}|F_{m+2}^y\big] = E\big[(\alpha_{m-4}\theta_{m+1} +$ $\beta m-4 um+2|Fm+2y=\alpha m-4E\theta m+1|Fm+2y=\alpha m-4\theta$ $m+1$ Since we have made an observation up to time $t = m+1$ we have$\hat{\theta}_{m+1} = \hat{X}_{m+1}$ so that, $\hat{\theta}_{m+2|m+2} = \alpha_{m-4}\hat{X}_{m+1}$.

Next, $\hat{\theta}_{m+3} = E\big[\theta_{m+3}|F_{m+2}^y\big] = E\big[(\alpha_{m-3}\theta_{m+2} +$ $\beta m-2 um+3|Fm+2y=\alpha m-3E\theta m+3|Fm+2y=\alpha m-3\theta$ $m+2$ That is, $\hat{\theta}_{m+3} = \alpha_{m-3}\hat{\theta}_{m+2}$and $\hat{\theta}_{m+2} = \alpha_{m-4}\hat{\theta}_{m+1}$combines to give

$$\hat{\theta}_{m+3} = \alpha_{m-3}\big(\alpha_{m-4}\hat{\theta}_{m+1}\big) = \alpha_{m-3}\alpha_{m-4}\hat{X}_{m+2}.$$

Hence (56) gives $\hat{X}_{m+2|m+3} = \alpha_{m-4}\hat{X}_{m+1} + \tilde{k}_{m+3}\big(X_{m+3} - \alpha_{m-3}\alpha_{m-4}\,\hat{X}_{m+1}\big)$.

Substituting for $\tilde{k}_{m+3}$ we have$\tilde{X}_{m+2|m+3} = \alpha_{m-4}\,\hat{X}_{m+1} + \frac{\alpha_{m-3}\,\beta_{m-4}^2}{\beta_{m-3}^2+\alpha_{m-3}^2\beta_{m-4}^2}\big(X_{m+3} -$ $\alpha m-3\alpha m-4\ Xm+1$

This simplifies to give$\tilde{X}_{m+2|m+3} = \frac{\alpha_{m-4}(\beta_{m-3}^2)\,\hat{X}_{m+1}+\alpha_{m-3}\beta_{m-4}^2X_{m+3}}{\beta_{m-3}^2+\alpha_{m-3}^2\beta_{m-4}^2}$ (57)

We now list these models in order, to obtain the general pattern of obtaining estimate of missing observation at any stage. Hence we have;

1. $\tilde{X}_{m|m+1} = \frac{\alpha_{m-2}(\beta_{m-1}^2)\,X_{m-1}+\alpha_{m-1}\beta_{m-2}^2X_{m+1}}{\beta_{m-1}^2+\alpha_{m-1}^2\beta_{m-2}^2}$

2. $\tilde{X}_{m|m+2} = \frac{\alpha_{m-3}(\beta_{m-1}^2\alpha_{m-1}^2\beta_{m-2}^2)\,X_{m-1}+\alpha_{m-1}\alpha_{m-2}^2\beta_{m-3}^2X_{m+2}}{\beta_{m-1}^2+\alpha_{m-1}^2\beta_{m-2}^2+\alpha_{m-1}^2\alpha_{m-2}^2\beta_{m-3}^2}$

$$\tilde{X}_{m|m+2} = \frac{\alpha_{m-3}(\beta_{m-2}^2)\,\hat{X}_m + \alpha_{m-2}\beta_{m-3}^2X_{m+2}}{\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2}$$

3. $\tilde{X}_{m|m+3} = \frac{\alpha_{m-4}(\beta_{m-1}^2\alpha_{m-1}^2\beta_{m-2}^2+\alpha_{m-1}^2\alpha_{m-2}^2\beta_{m-3}^2)\,X_{m-1}+\alpha_{m-1}\alpha_{m-2}\alpha_{m-3}\beta_{m-4}^2X_{m+3}}{\beta_{m-1}^2+\alpha_{m-1}^2\beta_{m-2}^2+\alpha_{m-1}^2\alpha_{m-2}^2\beta_{m-3}^2+\alpha_{m-1}^2\alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2}$

$$\tilde{X}_{m+1|m+3} = \frac{\alpha_{m-4}(\beta_{m-2}^2\alpha_{m-2}^2\beta_{m-3}^2)\,\hat{X}_m + \alpha_{m-2}\alpha_{m-3}\beta_{m-4}^2X_{m+3}}{\beta_{m-2}^2 + \alpha_{m-2}^2\beta_{m-3}^2 + \alpha_{m-2}^2\alpha_{m-3}^2\beta_{m-4}^2}$$

$$\tilde{X}_{m+2|m+3} = \frac{\alpha_{m-4}(\beta_{m-3}^2)\,\hat{X}_{m+1} + \alpha_{m-3}\beta_{m-4}^2X_{m+3}}{\beta_{m-3}^2 + \alpha_{m-3}^2\beta_{m-4}^2}$$

S. $\tilde{X}_{m|m+s} = \frac{\Big[\alpha_{m-(s+1)}(\beta_{m-1}^2+\alpha_{m-1}^2\beta_{m-2}^2+\alpha_{m-1}^2\alpha_{m-2}^2\beta_{m-3}^2+\cdots+\alpha_{m-1}^2\alpha_{m-2}^2\ldots\alpha^2{}_{m-(s+1)}\beta_{m-s}^2)X \;\; \alpha_{m-1}\alpha_{m-2}\ldots\alpha_{m-s}\beta_{m-(s+1)}^2X_{m+s}}{\beta_{m-1}^2+\alpha_{m-1}^2\beta_{m-2}^2+\alpha_{m-1}^2\alpha_{m-2}^2\beta_{m-3}^2+\cdots+\alpha_{m-1}^2\alpha_{m-2}^2\ldots\alpha_{m-s}^2\beta_{m-(s+1)}^2}$

$$\tilde{X}_{m+(s-1)|m+s} = \frac{\alpha_{m-(s+1)}(\beta_{m-s}^2)\,\hat{X}_{m+(s-2)} + \alpha_{m-s}\beta_{m-(s+1)}^2X_{m+s}}{\beta_{m-s}^2 + \alpha_{m-s}^2\beta_{m-(s+1)}^2}$$

The estimate of $\tilde{X}_{m+1|m+2}$ are the same as those obtained by [2] and [19]. The approach in section 3 can only handle some of the nonlinear models mentioned earlier. Let's give an example of AR (1) model with a constant conditional variance, $\beta_t^2 = \sigma^2$ and $\alpha_t = \phi$ (See [1].

In this case the estimates of the missing observations X_m, X_{m+1}, X_{m+2}, $X_{m+3}$for an AR(1) model from 1 up to S missing values will be as follows;

1. $\tilde{X}_{m|m+1} = \frac{\phi X_{m-1}+\phi X_{m+1}}{1+\phi^2}$

2. $\tilde{X}_{m|m+2} = \frac{\phi(1+\phi^2)X_{m-1}+\phi^2X_{m+2}}{1+\phi^2+\phi^4}$

$$\tilde{X}_{m|m+2} = \frac{\phi\hat{X}_m + \phi X_{m+2}}{1+\phi^2}$$

3. $\tilde{X}_{m+2|m+3} = \frac{\phi(1+\phi^2+\phi^4)X_{m-1}+\phi^3X_{m+3}}{1+\phi^2+\phi^4+\phi^6}$

$$\tilde{X}_{m+1|m+3} = \frac{\phi(1+\phi^2)\hat{X}_m + \phi^2X_{m+3}}{1+\phi^2+\phi^4}$$

$$\tilde{X}_{m+2|m+3} = \frac{\emptyset\hat{X}_{m+1} + \emptyset X_{m+3}}{1+\phi^2}$$

S. $\tilde{X}_{m|m+s} = \frac{\emptyset(1+\phi^2+\phi^4+\cdots+\emptyset^{2(s-1)}X_{m+s})+\emptyset^sX_{m+s}}{1+\phi^2+\phi^4+\cdots+\emptyset^{2s}}$

$$\tilde{X}_{m+(s-1)|m+s} = \frac{\emptyset X_{m+(s-2)} + \emptyset X_{m+s}}{1+\phi^2}$$

Hence the general form of these sequences of estimates is given by;

$$\tilde{X}_{m+j|m+i} = \frac{\emptyset\big(\sum_{r=0}^{i-j-1}\emptyset^{2r}\big)X_{m+j-1}+\emptyset^{i-j}X_{m+i}}{\sum_{r=0}^{i-j}\emptyset^{2r}}(58)$$

Where $j = 0,1,2,\ldots,s-1$ and $i = 1,2,3,\ldots,s$for $i > j$.

An empirical study

In this section empirical study is carried out to illustrate the results obtained in section three on simulated AR (1) data. Some values are withheld and then estimated as though they were missing.

Simulated AR (1) data and corresponding estimated missing values generated using formula (77) in section three are compared.

Tables 1 and 2 and tables 4.1 and 4.2 gives actual data and estimates of missing values for different values of the parameter ϕ and initial value X_0.

Estimation of missing observations on simulated AR

(1) data.

AR (1) process is written as $X_t = \phi X_{t-1} + e_t$, where e_t is a purely random process which is normally distributed with mean zero and unit variance (i.e. $e_t \sim N(0,1)$, ϕ is some constant given within the range $|\phi| < 1$and X_0 is the initial observation which we choose to determine the size of the data.

Hence we have,

$$\left.\begin{aligned} X_1 &= \phi X_0 + e_1 \\ X_2 &= \phi X_1 + e_2 \\ &\vdots \\ X_n &= \phi X_{n-1} + e_n \end{aligned}\right\} \quad (59)$$

We generated AR (1) data using a computer programme. The general formula (58) in section three was then applied to the simulated AR (1) data with several consecutive missing observations which were artificially created. Table 1.1 illustrates the missing values and their estimates and Tables 1.2 illustrate the same for different values of ϕ and X_0.

If we set $\phi = 0.86$ and $X_0 = 100$, we obtain the following AR(1) data for the first 20 values.

Table 1. *Eighteen consecutive missing values from position 2 to 19 with their estimates.*

t	Actual AR(1) data(X_t)	Data with missing values	Estimated values
1	86.21399	86.21399	
2	73.03785	-	74.11012
3	63.71109	-	63.69526
4	53.52442	-	54.73207
5	47.14499	-	47.01625
6	38.94666	-	40.37197
7	35.22709	-	34.6478
8	31.41615	-	29.71327
9	25.95318	-	25.45593
10	21.70792	-	21.77875
11	17.5829	-	18.59792
12	15.35208	-	15.84095
13	12.63766	-	13.44501
14	11.30084	-	11.35549
15	9.111297	-	9.524767
16	8.188405	-	7.91112
17	6.679135	-	6.477774
18	5.224529	-	5.192006
19	3.283825	-	4.024678
20	2.94902	2.94902	

Positions of missing values	Calculatedχ^2 value	Table value $\chi^2_{0.05,4}$
2-19	0.53467	27.59

We generate other 20 different values of AR (1) with $\phi = 0.5$ and $X_0 = 1000000$ as illustrated in Table 1.2.

Table 2. *Eighteen consecutive missing values from position 2 to 19 with their estimates.*

t	Actual AR(1) ta(X_t)	Data with missing values	Estimated values
1	500000	500000	
2	249999.8	-	250000
3	125001.2	-	125000
4	62500.23	-	62500
5	31250.01	-	31250
6	15622.97	-	15625
7	7809.437	-	7812.5
8	3905.664	-	3906.25
9	1953.414	-	1953.125
10	977.1117	-	976.5632
11	488.8349	-	488.2828
12	244.358	-	244.1437
13	122.5056	-	122.0765
14	61.77014	-	61.04752
15	33.05295	-	30.54231
16	15.78969	-	15.30825
17	7.215367	-	7.728314
18	3.808213	-	4.012536
19	2.473077	-	2.303026
20	1.745028	1.745028	

Positions of missing values	Calculatedχ^2 value	Table value$\chi^2_{0.05,17}$
2-19	0.27716	27.59

Chi-square goodness of fit statistics was calculated to compare the estimated and actual values. Each of the chi-square values were not significant for any reasonable level of significance which confirms that the estimated values agree well with the actual values.

4. Conclusions

Most methods developed for estimation of missing observations in time series Analysis, have been limited to the case of one or two consecutive missing observations. In this paper, we have employed the state space models which can handle irregularly spaced data. Missing observations in a Time Series can safely be treated as special case of such data. In particular, we have extended the formula derived in [1] to encompass the case where there are several consecutive missing observations. In a special case, we apply the formula on AR (1) model and it performs satisfactorily, since it has only one parameter.

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