Parameters estimation based on progressively censored data from inverse Weibull distribution

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Abstract: In this article, our main aim is to investigate the parameters estimation of inverse Weibull distribution in the frame work of progressively type II. We consider the censored sample from a two parameters inverse Weibull. The point estimators of the parameters derived by using the maximum likelihood method. The exact joint confidence region and confidence interval for the parameters are obtained. A numerical example is provided to illustrate the proposed estimation methods developed here.

Keywords: Joint Confidence Region, Maximum Likelihood Estimator, Progressively Type II Censored Sample, Confidence Interval

1. Introduction

Censoring is very common in life tests. It usually applies when exact lifetimes are known for only a portion of the products and the remainder of the lifetimes are known only to exceed certain values under a life test. In the study, we consider a censoring scheme called progressive type II censoring. Under this scheme, n units are placed on a test at time zero, with m failures to be observed. When the first failure is observed, \( r_1 \) of the surviving units are randomly selected and removed. At the second observed failure, \( r_2 \) of the surviving units are randomly selected and removed. This experiment stops at the time when the \( m \)-th failure is observed and the remaining \( r_m=n-r_1-r_2-\ldots-r_{m-1}-m \) surviving units are all removed.

The statistical inference on the parameters of some distributions under progressive type-II censoring has been investigated by several authors such as [1,2,3,4,5]. The Inverse Weibull distribution plays an important role in many applications, including the dynamic components of diesel engine and several data set such as the times to breakdown of an insulating fluid subject to the action of a constant tension, see [6]. Calabria and Pulcimina in [7] provided an interpretation of the inverse Weibull distribution in the context of the load strength relationship for a component. Maswedah, [8] has fitted the inverse Weibull distribution to the flood data reported in [9]. For more details on the inverse Weibull distribution, see, for example Johnson et al. [10], Murthy et al. [11] and Mohie El-Din et al [12].

The inverse Weibull model was developed by Erto [13]. The probability density function \((pdf)\) of the random variable \(X\) having a three-parameter inverse Weibull distribution with location parameter \(\alpha \geq 0\), scale parameter \(\eta > 0\) and shape parameter \(\beta > 0\) is given by Erto [13], and Maru et al. [14]. The probability density function \((pdf)\) of a two parameter inverse Weibull distribution has the form:

\[
f(x|\alpha, \beta, \eta) = \begin{cases} \frac{\beta (\frac{x}{\eta})^{\alpha \beta - 1}}{\eta^\beta} e^{-\left(\frac{x}{\eta}\right)^\beta}, & x > \alpha, \quad \eta, \beta > 0, \\ 0, & x \leq \alpha, \end{cases} \tag{1}
\]

If \(\alpha = 0\), the resulting distribution is called the two-parameter inverse Weibull distribution.

In this article, we consider progressively type II censored data from a two parameters inverse Weibull distribution. We obtain the maximum likelihood estimators of the parameters in Section 2. In Section 3, we derive an exact confidence interval for the parameter \(\beta\) and an exact joint confidence region for the parameters \(\beta\) and \(\eta\). A numerical example...
is presented for illustration in Section 4.

2. Point Estimations of Parameters

In this section, the maximum likelihood estimators (MLEs) for the parameters of the inverse Weibull distribution based on progressive type II censoring are derived. The cumulative distribution (cdf) of the inverse Weibull distribution is given by:

\[
F(x; \alpha, \beta, \eta) = e^{-\left(\frac{x-\alpha}{\eta}\right)^\beta}, \ x > \alpha, \ \eta, \beta > 0.
\] (2)

Let \(X_1, X_2, \ldots, X_m\) be a progressively type II censored sample from a three parameter inverse Weibull distribution, with censoring scheme \(r = (r_1, r_2, \ldots, r_n)\). The likelihood function is given by:

\[
L(\beta, \alpha, \eta) = k^n \prod_{i=1}^{m} f(x_i; \alpha, \beta, \eta) [1 - F(x_i; \alpha, \beta, \eta)]
\]

where \(k = n! (n-1-r_1)! (n-2-r_1-r_2)! \cdots (n-m+1-r_1-\cdots-r_m)\).

The MLEs \(\hat{\eta}\) and \(\hat{\beta}\) can be obtained by:

\[
m + \sum_{i=1}^{m} \left( \frac{x_i - \alpha}{\eta} \right)^{-\beta} + \sum_{i=1}^{n} \left( \frac{x_{(i-m)} - \alpha}{\eta} \right)^{-\beta} e^{-\left(\frac{x_{(i-m)} - \alpha}{\eta}\right)^\beta} = 0.
\] (7)

since it can solve the equations (7) and (8) for \(\hat{\beta}\) and \(\hat{\eta}\) by numerical solution.

3. Interval Estimations of Parameters

In this section, an exact confidence interval for \(\beta\) and \(\eta\) an exact joint confidence region for \(\beta\) and \(\eta\) are investigated. Let \(X_1 < X_2 < \cdots < X_{m-1} < X_m\) denote a progressively type II censored sample from a three parameter inverse Weibull distribution, with censoring scheme \(r = (r_1, \cdots, r_n)\). Further, let \(Z_i = \left(\frac{x_i - \alpha}{\eta}\right)^{\beta}, i = 1, \cdots, m\). It can be seen that \(Z_1 < Z_2 < \cdots < Z_{m-1} < Z_m\) is a progressively type II censored sample from an exponential distribution with mean 1. Let us consider the following transformation:
\[
\begin{aligned}
S_1 &= nZ_1 \\
S_2(n - r_1 - 1)(Z_2 - Z_1) \\
&\vdots \\
S_m &= (n - r_1 - \cdots - r_{m-1} - m + 1)(Z_m - Z_{m-1}).
\end{aligned}
\]

Thomas and Wilson [15] proved that the generalized spacing \(S_1, S_2, \ldots, S_m\) as defined in (9), are independent and identically distributed as an exponential distribution with mean 1. Hence,

\[V = 2S_i = 2nZ_i\]

has a chi-square distribution with 2 degrees of freedom and

\[U = 2 \sum_{i=1}^{n} \left[ \sum_{i=1}^{n} (r_i + 1)Z_i - nZ_i \right]\]

has a chi-square distribution with \(2m - 2\) degrees of freedom. We can also find that \(U\) and \(V\) are independent random variables. Let

\[T_1 = \frac{U}{(m - 1)V} = \frac{\sum_{i=1}^{n} (r_i + 1)Z_i - nZ_i}{n(m - 1)Z_i},\]

and

\[T_2 = U + V = 2 \sum_{i=1}^{n} (r_i + 1)Z_i = 2\eta \sum_{i=1}^{n} (r_i + 1)(x_i - \alpha)^\beta.\]

It is easy to show that \(T_1\) has an \(F\) distribution with \(2m - 2\) and 2 degrees of freedom and \(T_2\) has a chi-square distribution with \(2m\) degrees of freedom.

Furthermore, by Johnson et al. [10], \(T_1\) and \(T_2\) are independent.

To obtain the confidence interval for \(\beta\) and the joint confidence region for \(\beta\) and \(\eta\), we use the following lemma.

**Lemma**

Suppose that \(w_1 < w_2 < \cdots < w_m, w > \alpha\). Let

\[T_i(\beta) = \frac{\sum_{i=1}^{n} (r_i + 1)(w_i - \alpha)^\beta - n(w_i - \alpha)^\beta}{n(m - 1)(w_i - \alpha)^\beta}\]

Then, \(T_i(\beta)\) is strictly increasing in \(\beta\) for any \(\beta > 0\). Furthermore, if \(t > 0\), the equation \(T_i(\beta) = t\) has unique solution for any \(\beta > 0\).

**Theorem 1.**

Suppose that \(X_{(i)}, i = 1, \ldots, m\), are the order statistics of a progressively type-II censored sample of size \(n\) from a three parameter inverse Weibull distribution, with censoring scheme \((r_1, \ldots, r_m)\). Then a \((1 - \phi)\)% confidence interval for \(\beta\) is:

\[
\Psi(X_1, \ldots, X_m, F_{(1 - \phi)(2m - 2)}) < \beta < \Psi(X_1, \ldots, X_m, F_{\phi(2m - 2)}).
\]

**Proof.** From (10), we obtain

\[
T_i = \frac{\sum_{i=1}^{n} (r_i + 1)Z_i - nZ_i}{n(m - 1)Z_i} = \frac{\sum_{i=1}^{n} (r_i + 1)(x_i - \alpha)^\beta - n(x_i - \alpha)^\beta}{n(m - 1)(x_i - \alpha)^\beta}
\]

hence, \(T_i\) has an \(F\) -distribution with \(2m - 2\) and 2 degrees of freedom. Hence, for \(0 < \phi < 1\), the event

\[
1 - \phi = \int_{F_{\phi}(2m - 2)}^{1} \frac{\Psi(X_1, \ldots, X_m, F_{(1 - \phi)(2m - 2)}) < \beta}{F_{\phi}(2m - 2)}\, dt = \int_{F_{\phi}(2m - 2)}^{1} \frac{[\Psi(X_1, \ldots, X_m, F_{(1 - \phi)(2m - 2)}) < \beta]}{F_{(1 - \phi)(2m - 2)}(t)}\, dt.
\]

is equivalent to the event

\[
\Psi(X_1, \ldots, X_m, F_{(1 - \phi)(2m - 2)}) < \beta < \Psi(X_1, \ldots, X_m, F_{\phi(2m - 2)}).
\]

Let us discuss the joint confidence region for the parameters \(\beta\) and \(\eta\). A naive \((1 - \phi)\)% joint confidence may be constructed by using \(B_\phi\) for \((T_1, T_2)\) such that

\[B_\phi = \{(t_1, t_2) : g_1(t_1)g_2(t_2) > \beta_\phi\}, 0 < \phi < 1\]

where \(g_1\) is the density function of \(F\) distribution, \(g_2\) is the density function of chi-square distribution, and \(\beta_\phi\) is a constant given by

\[1 - \phi = \int_{B_\phi} g_1(t_1)g_2(t_2)\, dt_1\, dt_2 = \int_{B_\phi} g_1(t_1)\int_{\beta_\phi(t_1)}^{\infty} g_2(t_2)\, dt_2\, dt_1.
\]
Since $T_1$ and $T_2$ are independent, a simpler way is to choose $\xi$ and $\lambda$ depending only upon $\phi$ such that

$$\int_{t_1}^{\xi(t)} g_1(t) \, dt = 1 - \phi$$

and

$$\int_{t_1}^{\lambda(t)} g_2(t) \, dt = 1 - \phi.$$ 

Let $X^2_{\alpha,\phi}$ denote the percentile of chi-square distribution with right-tail probability $\phi$ and $\delta$ degrees of freedom. An exact joint confidence region for the parameters $c$ and $\xi$ is given in the following theorem.

**Theorem 2.** Suppose that $X_i, i = 1, \cdots, m$, are the order statistics of a progressively type-II censored sample of size $n$ from a three parameter inverse Weibull distribution, with censoring scheme $(r_1, \cdots, r_m)$. Then a $100(1 - \phi)$% joint confidence region for $\xi$ and $\lambda$ is determined by the following inequalities:

$$\Psi(X_1, \cdots, X_m, F_{\alpha,\phi}(2m - 2), \lambda) < \beta < \Psi(X_1, \cdots, X_m, F_{\alpha,\phi}(2m - 2), \lambda)$$

$$\frac{X^2_{\alpha,\phi}(2m)}{2 \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2} < \eta < \frac{X^2_{\alpha,\phi}(2m)}{2 \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2}$$

where $0 < \phi < 1$ and $\Psi(X_1, \cdots, X_m, \lambda, t)$ is the solution of $\beta$ for the equation

$$\sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2 - n(x_i - \alpha)^2 = t$$

**Proof:** From (1e10) and (1e11) we obtain

$$T_1 = \sum_{i=1}^{m}(r_i + 1)Z_i - nz_1$$

$$= \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2 - n(x_i - \alpha)^2$$

and

$$T_2 = 2 \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2$$

hence, $T_1$ has a $F$ distribution with $2m - 2$ and 2 degrees of freedom and $T_2$ has a chi-square distribution with $2m$ degrees of freedom, $T_1$ and $T_2$ are independent. Then, for $0 < \phi < 1$,

$$1 - \phi = \sqrt{1 - \phi} \sqrt{1 - \phi}$$

$$= P\left( F_{\alpha,\phi}(2m - 2, 2) < T_1 < F_{\alpha,\phi}(2m - 2, 2) \right)$$

$$\times P\left( X^2_{\alpha,\phi}(2m) < T_2 < X^2_{\alpha,\phi}(2m) \right)$$

$$= \Psi\left( X_1, \cdots, X_m, F_{\alpha,\phi}(2m - 2, 2) \right) < \beta$$

$$< \Psi\left( X_1, \cdots, X_m, F_{\alpha,\phi}(2m - 2, 2) \right)$$

$$\frac{X^2_{\alpha,\phi}(2m)}{2 \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2} < \eta < \frac{X^2_{\alpha,\phi}(2m)}{2 \sum_{i=1}^{m}(r_i + 1)(x_i - \alpha)^2}$$

4. **Illustrative Example**

To illustrate the use of the estimation methods discussed in the article, the following example is discussed.

Example. Consider a progressively type-II censored sample of size $m = 8$ from a sample of size $n = 20$ from the inverse Weibull distribution with $\beta = 0.5$, $\eta = 2$ and $\alpha = 1$ was simulated, with censoring scheme $r = (0, 0, 3, 0, 4, 0, 0, 5)$. The simulated progressively type-II censored sample is in the table.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
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<td>$x_i$</td>
<td>1.7096</td>
<td>1.7401</td>
<td>2.3352</td>
<td>2.3352</td>
</tr>
<tr>
<td>$r_i$</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$i$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$x_i$</td>
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<td>3.2457</td>
<td>4.5363</td>
<td>9.4191</td>
</tr>
<tr>
<td>$r_i$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

Using the iterative formula presented in section 2, the $MLE$s of $\beta$ and $\eta$ are $\hat{\beta} = 0.9047$ and $\hat{\eta} = 1.7665$, respectively. To find a 95% confidence interval for $\eta$, we need the percentiles
By Theorem 1 and using the Mathematica nonlinear equation solver, the 95% confidence interval for $\beta$ is $(0.6053, 2.7472)$. Furthermore, to obtain a 95% joint confidence region for $\beta$ and $\eta$, we need the percentiles $F_{0.027}(14, 2) = 78.4147$, $F_{0.973}(14, 2) = 0.1648$, $\chi^2_{0.0127}(16) = 31.2069$, and $\chi^2_{0.9873}(16) = 6.0684$.

By Theorem 2, a 95% joint confidence region for $\beta$ and $\eta$ is determined by the following inequalities:

$$
0.1118 < \beta < 3.0276 \\
\frac{6.0684}{\sum_{i=1}^{m}(\bar{r}_i + 1)(X_i - \bar{r})^{-\beta}} < \eta < \frac{31.2069}{\sum_{i=1}^{m}(\bar{r}_i + 1)(X_i - \bar{r})^{-\beta}}.
$$

5. Conclusion

We use maximum likelihood method to obtain the point estimators of the parameters of three parameter inverse Weibull distribution based on progressive Type II censoring. We provide two pivotal quantities to construct an exact confidence interval and an exact joint confidence region for the parameters, respectively. A numerical data set is analyzed in section 4 to illustrate our approach.

References


