Bayesian Prediction Based on Type-I Hybrid Censored Data from a General Class of Distributions

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Abstract: One and two-sample Bayesian prediction intervals based on Type-I hybrid censored for a general class of distribution

\[ F(x) = \left[ ah(x) + b \right] c, \]

are obtained. For the illustration of the developed results, the inverse Weibull distribution with two unknown parameters and the inverted exponential distribution are used as examples. Using the importance sampling technique and Markov Chain Monte Carlo (MCMC) to compute the approximation predictive survival functions. Finally, a real life data set and a generated data set are used to illustrate the results derived here.

Keywords: Bayesian Prediction, Type-I Hybrid Censored, General Class, Markov Chain Monte Carlo, Importance Sampling Technique

1. Introduction

Prediction plays an important role in different areas of applied statistics such as medical sciences and reliability analysis. Bayesian prediction have more attention among other issues of prediction. Discussion of the prediction intervals (one-sample and two-sample prediction) for a future sample is valuable in lifetime studies. Bayesian prediction intervals for future observations have been discussed by several authors, including Howlader [1], Geisser [2], Raqab and Nagaraja [3], Al-Hussaini and Jaheen ([4]; [5]), Abdel-Aty et al. [6], Kundu and Howlader [7], Mohie El-Din et al. ([8]; [9]), Shafay and Balakrishnan [10], Mohie El-Din and Shafay [11] and Shafay et al. [12]. In this article, we use a general class of distribution (see; Khan and Abu-Salih [13], Athar and Islam [14]) to derive general procedure for determining the one- and two-sample Bayesian prediction intervals based on Type-I hybrid censored data. In the rest of this section, we derive the likelihood function and the conditional density functions of \( Xs:n \) given the Type-I hybrid censored data. In Section 2, we derive the one-sample Bayesian predictive survival function and the one-sample Bayesian predictions bounds for the \( s-th \) ordered lifetime from a future independent sample. In Section 3, special cases of this general class such as the inverse Weibull distribution when the two parameters are unknown and the inverted exponential distribution are considered as illustrative examples, wherein we adopt the importance sampling technique to compute the approximation predictive survival function in the one-sample case and the Markov Chain Monte Carlo (MCMC) method to compute the approximation predictive survival function in the two-sample case. Finally, some numerical examples are conducted to illustrate the prediction procedures.

Let the general form of distributions be

\[ 1 - F(x) = \left[ ah(x) + b \right]^c, \quad \alpha \leq x \leq \beta, \quad (1) \]

where \( a, b \) and \( c \) are constants \( (a; c \neq 0) \) s.t \( \frac{F(\alpha)}{F(\beta)} = 1 \) and \( h(x) = h(x; \theta) \) is a monotonic and differentiable function of \( x \) in the interval \([\alpha, \beta]\) and the parameter \( \theta \in \Theta \) may be a real vector, then

\[ f(x) = -cah'(x)\left[ ah(x) + b \right]^{-1}, \quad (2) \]

where \( h'(x) = \frac{d}{dx} h(x) \). The following table gives some distributions with proper choice of \( a; b; c \) and \( h(x) \) as examples of the general class.
Table 1. Some distributions derived from the general class.

<table>
<thead>
<tr>
<th>Distributions</th>
<th>( F(x) )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>h(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>( e^{-\lambda x} )</td>
<td>( x &gt; 0, \lambda &gt; 0 )</td>
<td>1</td>
<td>0</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Generalized Exponential</td>
<td>( 1 - \left(1 - e^{-\alpha x}\right)\beta )</td>
<td>( x &gt; \alpha, \beta &gt; 0 )</td>
<td>1</td>
<td>1</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Generalized Inverted Exponential</td>
<td>( \left(1 - e^{-\alpha x}\right)^{\alpha} )</td>
<td>( x &gt; 0, \alpha &gt; 0 )</td>
<td>1</td>
<td>1</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Weibull</td>
<td>( e^{-\beta x} )</td>
<td>( x \geq \mu )</td>
<td>1</td>
<td>0</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Inverse Weibull</td>
<td>( 1 - e^{-\beta x} )</td>
<td>( 0 \leq x &lt; \infty )</td>
<td>1</td>
<td>1</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Pareto</td>
<td>( \alpha x^{-\alpha} )</td>
<td>( \alpha \leq x &lt; \infty )</td>
<td>( \alpha )</td>
<td>0</td>
<td>( \beta )</td>
</tr>
<tr>
<td>Gumbel</td>
<td>( 1 - e^{-\beta x} )</td>
<td>( \beta &gt; 0 )</td>
<td>1</td>
<td>1</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Burr type XII</td>
<td>( \left(1 + \frac{x}{c}\right)^{-k} )</td>
<td>( x &gt; 0, c &gt; 0, k &gt; 0 )</td>
<td>1</td>
<td>1</td>
<td>( \kappa )</td>
</tr>
</tbody>
</table>

Let \( X_{1n} < X_{2n} < \cdots < X_{kn} \) be the order statistics from a random sample of size \( n \) from a distribution function \( F(x) \) given in (1) with density function \( f(x) \) given in (2).

Let \( K \) denote the number of \( X_{kn} \)'s that are at most \( T \). Then \( K \) is a discrete random variable with support \( \{0, 1, \ldots, n\} \) and probability density function as

\[
P(K = k) = \binom{n}{k} p^{k} q^{n-k}, \quad k = 0, 1, \ldots, n, \quad (3)
\]

where \( p = F(T) \) and \( q = 1 - p = 1 - F(T) \).

We have one of the two following types of observations:

Case I. \( X_{1n} < X_{2n} < \cdots < X_{kn} \) if \( X_{rn} \leq T \) with \( r \leq K \leq n \);

Case II: \( X_{1n} < X_{2n} < \cdots < X_{kn} \) if \( T < X_{rn} \) with \( 0 \leq K \leq r-1 \).

The likelihood function of a Type-I hybrid censored sample is as follows:

Case I.

\[
L(\theta, X_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} c a h(x_i+b)^{-[ah(x_i+b)]^{(r-s)-1}}, \quad (4)
\]

where \( x_r = (x_1, \ldots, x_r) \) and \( x_i < \cdots < x_r \leq T \).

Case II:

\[
L(\theta, X_k) = \frac{n!}{(n-k)!} \prod_{i=1}^{k} c a h(x_i+b)^{-[ah(x_i+b)]^{(k-s)-1}}, \quad (5)
\]

where \( x_k = (x_1, \ldots, x_k) \) and \( x_i < \cdots < x_k \leq T < x_{k+1} \).

When \( r < s \leq n \), the conditional density function of \( X_{rn} \), given the Type-I hybrid censored data, is obtained as follows:

Case I.

\[
f_1(x_i \mid X_r) = \begin{cases} f_{11}(x_i \mid X_r), & x_i < x_r \leq T, \\ f_{12}(x_i \mid X_r), & x_i > T, \end{cases} \quad (6)
\]

where

\[
f_{11}(x_i \mid X_r) = \frac{1}{P(r \leq K \leq n)} \sum_{K=r}^{n} f(x_i \mid x_i, K = k) P(K = k)
\]

\[
= \sum_{r=1}^{n} (n-r) P(K = k) \frac{[F(x_r) - F(x_i)]^{r-1} [F(T) - F(x_i)]^{s-r} f(x_i)}{[F(T) - F(x_i)]^{s}}
\]

\[
= \sum_{r=1}^{n} \sum_{s=1}^{n-r+1} \sum_{k=1}^{n} A_1 \phi(x_r, T) ah(x_i+b)^{(s-r-1)} c a h(x_i+b)^{(r+s-k-1)} \left[ ah(T)+b \right]^{(r+s-k-1)}, \quad (7)
\]

and

\[
f_{12}(x_i \mid X_r) = \frac{1}{P(r \leq K \leq n)} \sum_{K=r}^{n} f(x_i \mid x_i, K = k) P(K = k)
\]

\[
= \sum_{r=1}^{n} (n-k) P(K = k) \frac{[F(x_r) - F(T)]^{r-k} [1 - F(x_i)]^{s-r} f(x_i)}{[1 - F(T)]^{s}}
\]

\[
= \sum_{r=1}^{n} \sum_{s=1}^{r-1} \sum_{k=1}^{n} A_2 \phi(x_r, T) c a h(x_i+b)^{(r+s-k-1)} \left[ ah(T)+b \right]^{(r+s-k-1)}, \quad (8)
\]

with
\[A_1 = \frac{(-1)^{q-s}(k-r)!}{q!(s-r-1-q)! (k-s-r)!},\]
\[A_2 = \frac{(-1)^{q-s}(n-k)!}{q!(n-s)! (k-s-r)!},\]

and
\[\phi(x_i, T) = \frac{\sum_{i=0}^{n} \binom{n}{k} \binom{k}{i} (-1)^i \binom{n-k}{i} \left[ a(T) + b \right]^{i(n-k)} \left[ a(T) + b \right]^{n-k-i}}{\sum_{j=0}^{n} \sum_{s=0}^{j} \binom{n}{j} \binom{j}{s} (-1)^s \binom{n-j}{s} \left[ a(T) + b \right]^{s(n-j)} \left[ a(T) + b \right]^{j-n+s}}.\]

Case II:
\[f_2(x_i | X_i) = \frac{1}{P(0 \leq k \leq r-1)} \sum_{k=0}^{r-1} f(x_i | x_s, K = k) P(K = k) = \frac{1}{(n-k)!} P(K = k) \sum_{k=0}^{r-1} P(K = k) \binom{n-k}{r-k} \left[ a(T) + b \right]^{r-k} \left[ a(T) + b \right]^{k} \left[ 1 - F(T) \right]^{r} \left[ 1 - F(T) \right]^{k},\]

\[= \sum_{k=0}^{r-1} \sum_{j=0}^{r-k} \psi(x_i, T) c a k \binom{n}{k} \binom{k}{i} (-1)^i \binom{n-k}{i} \left[ a(T) + b \right]^{i(n-k)} \left[ a(T) + b \right]^{n-k-i},\]

where
\[\psi(x_i, T) = \sum_{j=0}^{r-k} \sum_{s=0}^{j} \binom{n}{j} \binom{j}{s} (-1)^s \binom{n-j}{s} \left[ a(T) + b \right]^{s(n-j)} \left[ a(T) + b \right]^{j-n+s} \binom{n-k}{r-k} \left[ a(T) + b \right]^{r-k} \left[ a(T) + b \right]^{k} \left[ 1 - F(T) \right]^{r} \left[ 1 - F(T) \right]^{k}.

2. Bayesian Analysis

Bayesian approach has received a lot of attention for estimating the parameters of statistical distributions and for predicting samples. It makes use of ones prior knowledge about the parameters and also takes into consideration the data available. If ones prior knowledge about the parameter is available, it is suitable to make use of an informative prior but in a situation where one does not have any prior knowledge about the parameter and cannot obtain vital information from experts to this regard, then a non-informative prior will be a suitable alternative to use, Guure et al. [15].

Let the prior distribution denoted by \( \pi(\theta, \delta) \), where \( \theta \in \Theta \) is the vector of parameters of the distribution under consideration and \( \delta \) is the vector of prior parameters. Then the posterior density function of \( \theta \), can be written as:

Case I:
\[\pi_1^*(\theta | X_i) = C_1^{-1} L_1(\theta; X_i) \pi(\theta, \delta),\]

where \( X_i = (x_i, \ldots, x_i) \) and \( x_i < \cdots < x_i \leq T \); and
\[C_1 = \int L_1(\theta; X_i) \pi(\theta, \delta) d\theta.\]

Case II:
\[\pi_2^*(\theta | X_i) = C_2^{-1} L_2(\theta; X_i) \pi(\theta, \delta),\]

where \( X_i = (x_i, \ldots, x_i) \) and \( x_i < \cdots < x_i \leq T < x_{i+1} \), and
\[C_2 = \int L_2(\theta; X_i) \pi(\theta, \delta) d\theta.\]

2.1. One-Sample Bayesian Prediction Intervals

We simply obtain the predictive survival function of \( X_{x_n} \) as follows:

Case I.
\[\bar{F}_1(x_i | X_i) = \begin{cases} \bar{F}_{11}(x_i | X_i), & x_i < x_i \leq T, \\ \bar{F}_{12}(x_i | X_i), & x_i > T, \end{cases}\]

where
\[\bar{F}_{11}(t | X_i) = \int_{x_i}^{t} \int_{x_i}^{x_i} f_{11}(x_i | X_i) \pi_1^*(\theta | x_i) d\theta dx_i + \int_{x_i}^{t} \int_{x_i}^{x_i} f_{12}(x_i | X_i) \pi_1^*(\theta | x_i) d\theta dx_i,\]

and
\[\bar{F}_{12}(t | X_i) = \int_{x_i}^{t} \int_{x_i}^{x_i} f_{12}(x_i | X_i) \pi_1^*(\theta | x_i) d\theta dx_i.\]

Case II.
\[\bar{F}_2(t | X_i) = \int_{x_i}^{t} \int_{x_i}^{x_i} f_{2}(x_i | X_i) \pi_2^*(\theta | x_i) d\theta dx_i.\]

The Bayesian predictive 100(1 - \( \gamma \))% interval for \( X_{x_n}, r < s \leq n \), can be obtained by solving the following two equations:

\[\bar{F}_1(t | X_i) = 1 - \frac{\gamma}{2}, \text{ and } \bar{F}_2(t | X_i) = \frac{\gamma}{2},\]

and \( L_{x_n} \) and \( U_{x_n} \) denote the lower and upper bounds, respectively.
2.2. Two-Sample Bayesian Prediction Intervals

Let us consider a future sample \( \{Y_1, Y_2, \ldots, Y_m\} \) of size \( m \), independent of the informative sample \( \{X_1, X_2, \ldots, X_n\} \) and let \( Y_1 \leq Y_2 \leq \cdots \leq Y_m \) be the order statistics of the future sample. Suppose we are interested in the predictive density of the order statistic \( Y_{r,m} \) of the future sample, given the informative data set \( \{X_1, X_2, \ldots, X_n\} \). The probability density function of the \( s \)-th order statistic of the future sample of size \( m \) from a continuous distribution with the distribution function \( F(x) \) and the probability density function \( f(x) \) is given by

\[
f_{r,m}(y | \theta) = \frac{m!}{(s-1)!(m-s)!} [F(y)]^{s-1} [1 - F(y)]^{m-s} f(y),
\]

where \( 1 \leq s \leq m \); (see, Arnold et al. [16]).

From (1) and (2), we simply obtain the probability density function of the \( s \)-th order statistic from a general class as follows:

\[
f_{r,m}(y | \theta) = \sum_{l=0}^{s-1} C_l c a h(y) (ah(y) + b)^{(m-s-l+1)-1},
\]

where \( C_l = \frac{(-1)^{s-l} m!}{l!(m-s)!((s-l)!)} \), and we simply obtain the Bayesian predictive density function of \( Y_{r,m} \) as follows:

Case I.

\[
f_{r,m}^1(y | X_r) = \sum_{l=0}^{s-1} C_l c a h(y) (ah(y) + b)^{(m-s-l+1)-1} \pi_1^*(\theta | X_r) d\theta,
\]

Case II.

\[
f_{r,m}^2(y | X_r) = \sum_{l=0}^{s-1} C_l c a h(y) (ah(y) + b)^{(m-s-l+1)-1} \pi_2^*(\theta | X_r) d\theta.
\]

From (17) and (18), we simply obtain the predictive survival function of \( Y_{r,m} \) as follows:

Case I.

\[
\bar{F}_{r,m}^1(t | X_r) = \int_t^\infty f_{r,m}^1(y | X_r) dy
\]

\[
= \frac{1}{\theta} \sum_{l=0}^{s-1} \frac{C_l}{(m+l-s+1)!} \int_{t}^{\infty} (ah(t) + b)^{(m-s-l)+1} \pi_1^*(\theta | X_r) d\theta.
\]

Case II.

\[
\bar{F}_{r,m}^2(t | X_r) = \int_t^\infty f_{r,m}^2(y | X_r) dy
\]

\[
= \frac{1}{\theta} \sum_{l=0}^{s-1} \frac{C_l}{(m+l-s+1)!} \int_{t}^{\infty} (ah(t) + b)^{(m-s-l)+1} \pi_2^*(\theta | X_r) d\theta.
\]

Then, the Bayesian predictive 100(1 - \( \gamma \))% interval for \( Y_{r,m} \), \( 1 \leq s \leq m \), can be obtained by solving the following two equations:

\[
\frac{\bar{F}_{r,m}^1(L_{r,m} | X)}{2} = \frac{1 - \gamma}{2} \quad \text{and} \quad \frac{\bar{F}_{r,m}^2(U_{r,m} | X)}{2} = \frac{1 - \gamma}{2},
\]

where

\[
\bar{F}_{r,m}^1(t | X_r) = \begin{cases} \bar{F}_{r,m}^1(t | X_r), & \text{Case I}, \\ \bar{F}_{r,m}^2(t | X_r), & \text{Case II}, \end{cases}
\]

and \( L_{r,m} \) and \( U_{r,m} \) denote the lower and upper bounds, respectively.

3. Examples

In this section, we discuss the Bayesian prediction of observations from the inverse Weibull distribution when both parameters are unknown and from the inverted exponential distribution. To our knowledge, no one study these distributions for determining the Bayesian prediction intervals for future lifetimes based on an observed Type-I hybrid censored data.

3.1. Inverse Weibull Distribution

In this subsection, we take a special case from this general class, the inverse Weibull distribution, when \( h(x) = e^{-\theta x^{-p}}, a = -1, b = 1, \) and \( c = 1 \), we provide the posterior density function depend on the maximum likelihood distribution given in (4) and (5). Here, we assumed that the model parameters \( \theta \) and \( p \) follow the independent gamma prior density of the following forms:

\[
\pi_1^*(\theta | u_1, v_1) \propto \theta^{u_1-1} e^{-v_1\theta}, \quad \theta > 0,
\]

\[
\pi_2^*(p | u_2, v_2) \propto p^{u_2-1} e^{-v_2 p}, \quad p > 0,
\]

where \( u_1, v_1, u_2, \) and \( v_2 \) are the hyper-parameters. Then, the joint posterior density function of \( \theta \) and \( p \), given the Type-I hybrid censored data, can be written as:

Case I.

\[
\pi_1^*(\theta, p | x_i) \propto \theta^{u_1-1} e^{-v_1\theta} \left(\sum_{i=1}^{n} e^{-\theta x_i^{p}}\right)^{n-ne^{-\theta x_i^{p}}} \left[1-\theta x_i^{p}\right]^{-n-1},
\]

where \( X_i = (x_1, \cdots, x_n) \) and \( x_i < \cdots < x_n \leq T \).

Therefore, the posterior density function of \( \theta \) and \( p \) given \( X_i \), can be written as

\[
\pi_2^*(\theta | p, x_i) \propto \pi_2^*(\theta | p, X_i) \pi_2^*(p | X_i) h_1(\theta, p | X_i),
\]

where \( \pi_2^*(\theta | p, X_i) \) is a gamma density function with the shape and scale parameters as \( r+u_1 \) and \( \left(\sum_{i=1}^{n} x_i^{-p} + v_1\right) \), respectively, \( \pi_2^*(p | X_i) \) is a proper density function given by...
\[ \pi_{12}(p \mid X_r) \propto p^{n-r-1} \left[ \prod_i x_i^{-\gamma} \right] e^{-\sum_i x_i^{-\gamma} + v_1} X_r^{\gamma - n}, \quad (24) \]

and

\[ h_i(\theta, p \mid X_r) = [1 - e^{-\theta x_i^{-\gamma}}]^{r}. \quad (25) \]

**Case II.**

\[ \pi_2(\theta, p \mid X_r) \propto \theta^{n+r-1} e^{\theta r} \prod_i x_i^{-\gamma}, \quad (26) \]

where \( X_r = (x_1, \ldots, x_k) \) and \( x_1 < \cdots < x_k \leq T < x_{k+1} \).

Similarly as above, we can write the posterior density function of \( \theta \) and \( p \) given \( X_r \) as

\[ \pi_2(\theta, p \mid X_r) \propto \pi_2(\theta \mid p, X_r) \pi_{2p}(p \mid X_r) h_2(\theta, p \mid X_r), \quad (27) \]

where \( \pi_{2p}(p \mid X_r) \) is a gamma density function with the shape and scale parameters as \( k + u_i \) and \( \sum_i x_i^{-\gamma} + v_i \), respectively, \( \pi_2(\theta \mid p, X_r) \), is proper density function given by

\[ \pi_{2p}(p \mid X_r) \propto p^{n+r-1} \left[ \prod_i x_i^{-\gamma} \right] e^{-\sum_i x_i^{-\gamma} + v_1} X_r^{\gamma - n}, \quad (28) \]

and

Similarly, from Eq. (8), we can obtain

\[ f_{12}(x_r \mid X_r) = \sum_{k=0}^{r-1} \frac{(n-k)! B_1(T)}{(s-k-1)!(n-s)!} (e^{-\theta x_r^{-\gamma}} - e^{-\theta r^{-\gamma}})^{k-1} (1 - e^{-\theta x_r^{-\gamma}})^{n-k} p \theta x_r^{-\gamma} e^{\theta x_r^{-\gamma}} (1 - e^{\theta r^{-\gamma}})^{-s} \]

\[ = \sum_{k=0}^{r-1} \frac{(n-k)! B_1(T)}{(s-k-1)!(n-s)!} (e^{-\theta x_r^{-\gamma}} - e^{-\theta r^{-\gamma}})^{k-1} \frac{e^{-\theta r^{-\gamma}}}{p \theta x_r^{-\gamma}} (e^{-\theta x_r^{-\gamma}})^{n-k-1} \frac{e^{-\theta r^{-\gamma}}}{(e^{-\theta r^{-\gamma}})^{-k-1}}. \quad (31) \]

**Case II.**

The conditional density function of \( X_{ra} \) given the Type-I hybrid censored data, in this case, can be written as:

\[ f_2(x_r \mid X_r) = \sum_{k=0}^{r-1} \frac{(n-k)! B_2(T)}{(s-k-1)!(n-s)!} (e^{-\theta x_r^{-\gamma}} - e^{-\theta r^{-\gamma}})^{k-1} (1 - e^{-\theta x_r^{-\gamma}})^{n-k} p \theta x_r^{-\gamma} e^{\theta x_r^{-\gamma}} (1 - e^{\theta r^{-\gamma}})^{-s} \]

\[ = \sum_{k=0}^{r-1} \frac{(n-k)! B_2(T)}{(s-k-1)!(n-s)!} (e^{-\theta x_r^{-\gamma}} - e^{-\theta r^{-\gamma}})^{k-1} \frac{e^{-\theta r^{-\gamma}}}{p \theta x_r^{-\gamma}} (e^{-\theta x_r^{-\gamma}})^{n-k-1} \frac{e^{-\theta r^{-\gamma}}}{(e^{-\theta r^{-\gamma}})^{-k-1}}. \quad (32) \]

where

\[ B_2(T) = \frac{n!}{k^j} (e^{-\theta r^{-\gamma}})^j (1 - e^{-\theta r^{-\gamma}})^{-s}. \]

Then, we obtain the predictive survival function of \( X_{ra} \) as follows:

\[ \overline{F}_{11}(t \mid X_r) = \int_{0}^{t} \int_{0}^{t} f_{11}(x; \theta, p) dx \pi_{1}(\theta, p \mid X_r) d\theta dp + \int_{t}^{\infty} \int_{t}^{\infty} f_{11}(x; \theta, p) dx \pi_{1}(\theta, p \mid X_r) d\theta dp \]
It does not seem to be possible to compute the probabilities in Eqs (33), (34), (35) analytically. Hence, we use the importance sampling technique, (see, Geweke [17]; Chen and Shao ([18]; [19])) to construct the Bayesian prediction interval. The details are explained below.

Importance Sampling technique

Firstly, we need to prove that the \( \pi_{\theta}(\theta | p_{data}) \) has a log-concave density function: From (24),

\[
\ln \pi_{\theta}(\theta | X_r) = \ln e + (r + u_k - 1) \ln p
\]

\[
- v_k - p - (p + 1) \sum_{j=1}^{k} \ln x_j - (r + u_k) \ln \sum_{j=1}^{k} x_j^{\nu_j} + v_k,
\]

it is easy to show that \( \frac{d^2}{dp^2} \ln \pi_{\theta}(\theta | X_r) < 0 \), which implies that \( \pi_{\theta}(\theta | p_{data}) \) has a log-concave density function.

Since \( \pi_{\theta}(\theta | p_{data}) \) has a log-concave density, using the idea of Devroye [20], it is possible to generate sample from \( \pi_{\theta}(\theta | p_{data}) \). Moreover, since \( \pi_{\theta}(\theta | p_{data}) \) follows gamma, it is quite simple to generate sample from \( \pi_{\theta}(\theta | p_{data}) \). Now we would like to provide the importance sampling procedure to compute the probabilities in Eqs (33), (34), (35).

Algorithm:

- **Step1:** Generate \( p \) from \( \pi_{\theta}(p | \text{data}) \) using the method developed by Devroye [20].
- **Step2:** Generate \( \theta \) from \( \pi_{\theta}(\theta | p_{data}) \).
- **Step3:** Repeat Step 1 and Step 2 and obtain \( (p_1; \theta_1), (p_2; \theta_2), \ldots, (p_M; \theta_M) \).
- **Step4:** The approximate value of

\[
\int_0^1 f(\theta, p) \pi_{\theta}(\theta, p | X_r) d\theta dp
\]

can be obtained as

\[
\int_0^1 f(\theta, p) \pi_{\theta}(\theta, p | X_r) d\theta dp = \frac{1}{M} \sum_{i=1}^{M} f(\theta_i, p_i) h_i(\theta_i, p_i | X_r).
\]

Similarly, we can use the above algorithm to compute

\[
\int_0^1 f(\theta, p) \pi_{\theta}(\theta, p | X_r) d\theta dp.
\]

The Bayesian predictive 100(1 - \( \gamma \))% interval for \( X_s, r < s \leq n \) can be obtained by solving the two equations given in (16).

### 3.1.2. Two-Sample Bayesian Prediction

The predictive survival function of \( Y_{rs} \), in this special case is obtained as follows:

Case 1.

\[
\overline{F}^{*}_{Y_{rs}}(y | X_r) = \int_0^y f_{Y_{rs}}(y | X_r) dy
\]
\[ E_{Y_{x,s}}(r | X_s) = \int_0^y f(r | X_s) dy \]

\[ \frac{C_i}{(m+l-s+1)} \sum_{i=0}^{s} \left( 1 - e^{-m(r+1)} \right) \pi_i(\theta | X_s) d\theta dp. \]  

Case II.

\[ F_{Y_{x,s}}(r | X_s) = \int_0^y f(r | X_s) dy \]

\[ \frac{C_i}{(m+l-s+1)} \sum_{i=0}^{s} \left( 1 - e^{-m(r+1)} \right) \pi_i(\theta | X_s) d\theta dp. \]  

where \( C_i = \frac{(-1)^{s+i} m!}{l(m-s)![(s-l-1)!]} \).

By the same importance sample technique, the Bayesian predictive 100(1-\( \gamma \))% interval for \( Y_{x,s} ; r < s \leq m \), can be obtained by solving the two equations in (21).

### 3.2. Inverted Exponential Distribution

The inverted exponential distribution is a special case from the inverse Weibull distribution when the shape parameter is known (\( p = 1 \)). We provide the posterior density function depend on the maximum likelihood distribution given in (4) and (5), when the shape parameter is \( p = 1 \). It is assumed that the scale parameter has a gamma prior distribution with the shape and scale parameters as \( u \) and \( v \), respectively and it has the probability density function

\[ \pi(\theta | u, v) \propto \theta^{-1} e^{-\theta}, \quad \theta > 0. \]

The posterior density function of \( \theta \), given the Type-I hybrid censored data, can be written as:

**Case I.**

\[ \pi_i(\theta | x) = \frac{\sum_{i=1}^{n-r} \left( -1 \right)^{i+r} e^{-\left( \frac{1}{x_i} + \sum_{j=i+1}^{n} \frac{1}{x_j} \right)} \Gamma(r+u)}{\sum_{i=1}^{n-r} \left( -1 \right)^{i+r} \Gamma(r+u)} \]

where \( x_i = (x_1, \ldots, x_i) \) and \( x_i < \ldots < x_i \leq T \).

**Case II.**

\[ \pi_i(\theta | x) = \frac{\sum_{i=1}^{n-k} \left( -1 \right)^{i+k} e^{-\left( \frac{1}{x_i} + \sum_{j=i+1}^{n} \frac{1}{x_j} \right)} \Gamma(k+u)}{\sum_{i=1}^{n-k} \left( -1 \right)^{i+k} \Gamma(k+u)} \]

where \( x_i = (x_1, \ldots, x_i) \) and \( x_i < \ldots < x_i \leq T < x_{i+1} \).

In this case the predictive survival function of \( Y_{x,s} \) is the same as above when we put \( p = 1 \).

#### 3.2.1. One-Sample Bayesian Prediction

To compute \( \int_0^\gamma f(\theta) \pi_i(\theta | X_s) d\theta \) by using the MCMC technique, we use the following procedure:

- Step 1. Generate \( \theta \) from \( \pi_i(\theta | X_s) \);
- Step 2. Repeat Step 1 and obtain \( \theta_1, \theta_2, \ldots, \theta_N \);
- Step 3. The approximate value of \( \int_0^\gamma f(\theta) \pi_i(\theta | X_s) d\theta \) is then obtained as

\[ \int_0^\gamma f(\theta) \pi_i(\theta | X_s) d\theta = \frac{1}{N} \sum_{i=1}^{N} f(\theta_i). \]

Similarly, we can use the above algorithm to compute \( \int_0^\gamma g(\theta) \pi_i(\theta | X_s) d\theta \).

The Bayesian predictive 100(1-\( \gamma \))% interval for \( Y_{x,s} ; r < s \leq m \), can be obtained by solving the two equations given in (16).

#### 3.2.2. Two-Sample Bayesian Prediction

The predictive survival function of \( Y_{x,s} \) in this special case is obtained as follows:

**Case I.**

\[ F_{Y_{x,s}}(r | X_s) = \int_0^y f(r | X_s) dy \]

\[ \frac{C_i}{(m+l-s+1)} \sum_{i=0}^{s} \left( 1 - e^{-m(r+1)} \right) \pi_i(\theta | X_s) d\theta dp. \]

**Case II.**

\[ F_{Y_{x,s}}(r | X_s) = \int_0^y f(r | X_s) dy \]

\[ \frac{C_i}{(m+l-s+1)} \sum_{i=0}^{s} \left( 1 - e^{-m(r+1)} \right) \pi_i(\theta | X_s) d\theta dp. \]

Then, the Bayesian predictive interval for \( Y_{x,s} ; 1 \leq s \leq m \), can be obtained by the same manner.

### 4. Numerical Results

In this section we consider a real life data set which the inverse Weibull distribution fits it well and another generated data set from the inverted exponential distribution to illustrate the methods proposed in the previous sections.

The real data set is given by Dumonceaux and Antle [21], and it represents the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265. Maswadah [22] and Singh et al. [23] checked the suitability of the inverse Weibull distribution to this real data set and concluded that the inverse Weibull distribution fits the data very well.

We consider two different Type-I hybrid censoring schemes:

1. When \( n = 20, r = 15 \) and \( T = 0.43 \). In this case, the life-test would have terminated at \( T < x_{15;20} \), and we have obtained the following data: 0.265, 0.269, 0.297, 0.315, 0.324,
0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418 and 0.423;
2. When \( n = 20, r = 15 \) and \( T = 0.47 \). In this case, the
life-test would have terminated at \( x_{15,20} = 0.449 \), and we have
obtained the following data: 0.265, 0.269, 0.297, 0.315, 0.324,
0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423
and 0.449.
By using the procedures presented earlier, we construct
95% one-sample Bayesian prediction intervals for order statistics
\( X_{s,s} = s = 16, \ldots, 20 \) as well as 95% two-sample
Bayesian prediction intervals for order statistics
\( Y_{m,s} = 1,5,10,20 \) from a future sample of size \( m = 20 \). To
explore the sensitivity of the predictors with respect to the
hyperparameters \((s_1,v_1,u_1,v_2)\), we have considered the
following four hyperparameters: \((1,7,3,1),(1,7,3,2),\)
\((1,8,3,1),(1,8,3,2)\). Table 2 presents the lower and upper
95% one-sample Bayesian prediction bounds for
\( X_{s,s} = s = 16, \ldots, 20 \) for these four choices of the
\[0.45175, 0.84893, 0.95595, 1.33698, 1.38886, 1.42612, 2.91384, 3.34748, 3.39207, 3.49797, 0.41589, 4.56534,\]
\[0.0716, 11.7103, 13.3331, 78.4651, 135.999, 848.432\]

The corresponding results for the one-sample and
two-sample prediction intervals are represented in Tables 4
and 5, respectively.

5. Concluding Remarks
In this paper, we obtained one and two sample prediction
bounds based on Type-I hybrid censored data under the
general class of distributions. We introduced two examples,
the inverse Weibull distribution with unknown two parameters
and the inverted exponential distribution, to illustrate the
developed results. Bayesian predictive survival function can
not be obtained in closed form and so importance sampling
technique and Markov Chain Monte Carlo samples, which are
then used to compute the approximate predictive survival
function. Finally, some numerical results are presented to
illustrate the results and we observe the following remarks:
1. From Tables 2-5, we notice that the lengths of the
Bayesian prediction intervals are short when there are a large
number of observed values. It is clear that when we use the
same value of \( r \) but larger value of \( T \), the Bayesian
prediction intervals become tighter.
2. It is observed that the prediction intervals tend be wider
when \( s \) increase. This is a natural, since the prediction of the
future order statistic that is far away from the last observed
value has less accuracy than that of other future order
statistics.
3. It is evident from Tables 2 and 3 that the lower bounds of
Bayesian prediction are relatively insensitive to the
specification of the hyperparameters \((s_1,v_1,u_1,v_2)\) while the
upper bounds are somewhat sensitive.

Table 2. 95% one-sample Bayesian prediction bounds for \( X_{s,s} = s = 16, \ldots, 20 \) from inverse Weibull distribution.

<table>
<thead>
<tr>
<th>((u_1,v_1,u_2,v_2))</th>
<th>((1,7,3,1))</th>
<th>((1,7,3,2))</th>
<th>((1,8,3,1))</th>
<th>((1,8,3,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S)</td>
<td>(L_{x_s})</td>
<td>(U_{x_s})</td>
<td>(L_{x_s})</td>
<td>(U_{x_s})</td>
</tr>
<tr>
<td>16</td>
<td>0.43907</td>
<td>0.63118</td>
<td>0.44275</td>
<td>0.68220</td>
</tr>
<tr>
<td>17</td>
<td>0.45561</td>
<td>0.78217</td>
<td>0.45847</td>
<td>0.78082</td>
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<tr>
<td>18</td>
<td>0.47628</td>
<td>0.96853</td>
<td>0.47105</td>
<td>0.81186</td>
</tr>
<tr>
<td>19</td>
<td>0.49904</td>
<td>1.19722</td>
<td>0.51241</td>
<td>1.34441</td>
</tr>
<tr>
<td>20</td>
<td>0.53634</td>
<td>1.80118</td>
<td>0.55875</td>
<td>2.22601</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((u_1,v_1,u_2,v_2))</th>
<th>((1,7,3,1))</th>
<th>((1,7,3,2))</th>
<th>((1,8,3,1))</th>
<th>((1,8,3,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S)</td>
<td>(L_{x_s})</td>
<td>(U_{x_s})</td>
<td>(L_{x_s})</td>
<td>(U_{x_s})</td>
</tr>
<tr>
<td>16</td>
<td>0.44205</td>
<td>0.53420</td>
<td>0.46177</td>
<td>0.57326</td>
</tr>
<tr>
<td>17</td>
<td>0.44273</td>
<td>0.54657</td>
<td>0.45013</td>
<td>0.65100</td>
</tr>
<tr>
<td>18</td>
<td>0.45028</td>
<td>0.67311</td>
<td>0.45330</td>
<td>0.77419</td>
</tr>
<tr>
<td>19</td>
<td>0.46014</td>
<td>0.84750</td>
<td>0.47350</td>
<td>1.10977</td>
</tr>
<tr>
<td>20</td>
<td>0.50359</td>
<td>1.70414</td>
<td>0.51569</td>
<td>1.9332</td>
</tr>
</tbody>
</table>

5. Concluding Remarks
Table 3. 95% two-sample Bayesian prediction bounds for $Y_{s,m}$, $s=1,5,10,15,20$ and $m=20$ from inverse Weibull distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$(1,7,3,1)$</th>
<th>$(1,7,3,2)$</th>
<th>$(1,8,3,1)$</th>
<th>$(1,8,3,2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20628</td>
<td>0.32424</td>
<td>0.18440</td>
<td>0.31540</td>
</tr>
<tr>
<td>5</td>
<td>0.27089</td>
<td>0.40667</td>
<td>0.26681</td>
<td>0.39713</td>
</tr>
<tr>
<td>10</td>
<td>0.32206</td>
<td>0.48557</td>
<td>0.32817</td>
<td>0.50129</td>
</tr>
<tr>
<td>15</td>
<td>0.37542</td>
<td>0.59968</td>
<td>0.38940</td>
<td>0.71546</td>
</tr>
<tr>
<td>20</td>
<td>0.51570</td>
<td>2.12839</td>
<td>0.53647</td>
<td>2.34615</td>
</tr>
</tbody>
</table>

Table 4. 95% one-sample Bayesian prediction bounds for $X_{s,m}$, $s=26,30,30$ from inverted exponential distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$(r=25$ and $T=10.6)$</th>
<th>$(r=25$ and $T=11.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>11.6371</td>
<td>25.0702</td>
</tr>
<tr>
<td>27</td>
<td>13.3179</td>
<td>30.9066</td>
</tr>
<tr>
<td>28</td>
<td>15.4181</td>
<td>31.8399</td>
</tr>
<tr>
<td>29</td>
<td>19.5173</td>
<td>31.2495</td>
</tr>
<tr>
<td>30</td>
<td>28.6901</td>
<td>36.8573</td>
</tr>
</tbody>
</table>

Table 5. 95% two-sample Bayesian prediction bounds for $Y_{s,m}$, $s=1,5,10,20,30$ and $m=30$ from inverted exponential distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$(r=25$ and $T=10.6)$</th>
<th>$(r=25$ and $T=11.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.35180</td>
<td>1.28963</td>
</tr>
<tr>
<td>5</td>
<td>0.84871</td>
<td>2.38517</td>
</tr>
<tr>
<td>10</td>
<td>1.37635</td>
<td>3.97899</td>
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<td>20</td>
<td>3.28013</td>
<td>12.5664</td>
</tr>
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<td>30</td>
<td>20.5459</td>
<td>30.8374</td>
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</table>

References


