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The Uniform Variants of the Glivenko-Cantelli and Donsker Type Theorems for a Sequential Integral Process of Independence

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Abstract: In the analysis of statistical data in biomedical treatments, engineering, insurance, demography, and also in other areas of practical researches, the random variables of interest take their possible values depending on the implementation of certain events. So in tests of physical systems (or individuals) on duration of uptime values of operating systems depend on subsystems failures, in insurance business insurance company payments to its customers depend on insurance claims. In such experimental situations, naturally become problems of studying the dependence of random variables on the corresponding events. The main task of statistics of such incomplete observations is estimating the distribution function or what is the same, the survival function of the tested objects. To date, there are numerous estimates of these characteristics or their functionals in various models of incomplete observations. In this paper investigated the asymptotic properties of sequential processes of independence of the integral structure and uniform versions of the strong law of large numbers and the central limit theorem for integral processes of independence by indexed classes are established. The obtained results can be used to construct statistics of criteria for testing a hypothesis of independence of random variables on the corresponding events.

Keywords: Empirical Processes, Metric Entropy, Glivenko-Cantelli and Donsker Theorems

1. Introduction

The modern asymptotic theory of empirical processes indexed by a class of measurable functions is actively developing and the current results are detailed in monographs [5, 6, 8, 9, 12-14], also in articles [4, 10, 15]. The main results of this theory allow us to establish uniform versions of the laws of large numbers and central limit theorems for empirical measures under the imposition of entropy conditions for a class of measurable functions. These results are essentially generalized analogues of the classical theorems of Glivenko-Cantelli and Donsker. It should be noted the article [15], in which these results are established for a generalized class of random discrete measures under appropriate conditions for uniform entropy numbers. At the same time, such results can be used in applied problems. For example, to generalize Glivenko-Cantelli theorem for a certain class of sets Vapnik and Chervonenkis in 70-s years of the last century made a significant contribution to the development of statistical (machine) learning theory (theory-Vapnik Chervonenkis), which justifies the principle of minimizing empirical risk (for details, see the monograph [14]). In a recently published monography [9] Mason used the main results of the modern theory of empirical processes to study nonparametric the kernel type statistical estimates. In this paper also established properties of empirical processes, which appear in the problems of statistical data analysis.

In papers of the authors [1-3, 7] the limiting properties of generalized empirical processes of independence of random variables and events indexed by a class of measurable functions were investigated. In this paper, we study the asymptotic properties (uniform strong laws of large numbers and central limit theorems) of sequential processes of independence of the integral structure.

Consider the sequence of independent experiments, in which pairs $\{(X_k, A_k), k \geq 1\}$ are observed, where random variables X_k defined on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and take values in a measurable space $(\mathfrak{X}, \mathfrak{B})$, where $\mathfrak{X} \subseteq \mathbb{R}$ and $\mathfrak{B} = \sigma(\mathfrak{X})$-sigma algebra of Borel subsets of $\mathfrak{X}$. Events A_k have a common probability $p = \mathbb{P}(A_k) \in (0,1), k \geq 1$. Indicators of events are denoted by $\delta_k = I(A_k)$. It is observed the repeated sample of size n : $\mathbb{S}^{(n)} = \{(X_k, \delta_k), 1 \leq k \leq n\}$. Each element (X_k, δ_k) of sample induces a statistical model $(\mathfrak{X} \otimes \{0,1\}, \mathfrak{B} \otimes \{0,1\}, \mathcal{P})$, where distribution

$$\{\mathcal{P}(B \otimes D) = \mathbb{P}(X_k \in B, \delta_k \in D), B \in \mathfrak{B}, D \subset \{0,1\}\}$$

for each Borel set B represented through subdistributions: $\mathcal{P}(B \otimes \{0,1\}) = \mathbb{Q}(B) = \mathbb{Q}_0(B) + \mathbb{Q}_1(B)$ and $\mathbb{Q}_m(B) = \mathcal{P}(B \otimes \{m\}), m = 0,1$. Our interest is focused on hypothesis $\mathcal{H}$ of independence of X_k and A_k in each experiment. It's easy to see that under validity of $\mathcal{H}$: $\mathbb{Q}_0(B) = (1-p)\mathbb{Q}(B)$ and $\mathbb{Q}_1(B) = p\mathbb{Q}(B)$ for all $B \in \mathfrak{B}$. Let's introduce the signed measure $\{\Lambda(B) = \mathbb{Q}_1(B) - p\mathbb{Q}(B), B \in \mathfrak{B}\}$, which equal to zero under hypothesis $\mathcal{H}$. Using this measure, we construct an empirical process for testing a hypothesis $\mathcal{H}$. In this regard, we introduce empirical analogues of the above measures by sample $\mathbb{S}^{(n)}$ for $B \in \mathfrak{B}$ as:

$$\mathbb{Q}_{mn}(B) = \frac{1}{n}\sum_{k=1}^{n} I(X_k \in B, \delta_k = m), m = 0,1,$$

$$\mathbb{Q}_n(B) = \mathbb{Q}_{0n}(B) + \mathbb{Q}_{1n}(B) = \frac{1}{n}\sum_{k=1}^{n} I(X_k \in B),$$

$$\Lambda_n(B) = \mathbb{Q}_{1n}(B) - p_n\mathbb{Q}_n(B), p_n = \mathbb{Q}_{1n}(\mathfrak{X}) = \frac{1}{n}\sum_{k=1}^{n}\delta_k.$$

Note that $\Lambda_n(B)$ for all $B \in \mathfrak{B}$ is unbiased estimate of $\Lambda(B)$. Moreover, according to the strong law of large numbers, for each $B \in \mathfrak{B}$ and $n \to \infty$: $\Lambda_n(B) \to \Lambda(B)$, with probability 1. From the theory of empirical processes it is known (see, for example, [5, 6, 8-10, 12, 13]), that such results do not occur uniformly by all elements of $\sigma-$ algebra $\mathfrak{B}$ and can be performed for a special class $\mathcal{J}$ of sets of $\mathfrak{B}$. Consequently, the investigation of the limiting (at $n \to \infty$) properties of $\mathcal{J}-$ indexed processes of form

$$\{\mathbb{G}_n(B) = a_n(\Lambda_n(B) - \Lambda(B)), B \in \mathcal{J}\}, \quad (1)$$

with a possibly random sequence of non-negative normalizing numbers $\{a_n, n \geq 1\}$. It is interesting from the point of view of constructing of statistics for criteria to testing of the hypothesis $\mathcal{H}$. In this paper investigation of papers [1-3], will be advanced and the following wider classes of sequences of empirical processes indexed by the set $\mathcal{F}$ of Borel functions $f : \mathfrak{X} \to \mathbb{R}$ will be considered:

$$\mathbb{G}_n f = \int_{\mathfrak{X}} f d\mathbb{G}_n, f \in \mathcal{F}. \quad (2)$$

Note that, the process (1) is a special case of (2), when $\mathcal{F}$ is a class of indicators $\mathcal{F} = \{I(B), B \in \mathcal{J}\}$. In papers [1-3] the following empirical processes of independence indexed by the class $\mathcal{F}$ were investigated:

$$\left\{\Delta_n f = \left(\frac{n}{p_n(1-p_n)}\right)^{1/2}(\Lambda_n - \Lambda)f, f \in \mathcal{F}\right\}, \quad (3)$$

(see. [1, 2]), and also generalized sequential $\mathcal{D} = T \otimes \mathcal{F}-$ indexed analogue of (3) in [3],

$$\left\{\Delta_n(s;f) = (np_n(1-p_n))^{-1/2}[ns](\Lambda_{[ns]} - \Lambda f), (s;f) \in \mathcal{D}\right\}, \quad (4)$$

where $T = [0,1]$ and $[a]$ the integer part of number a. Obviously, that $\Delta_n(1;f) = \Delta_n(f)$ for all $f \in \mathcal{F}$. Note that, processes (3) and (4) are variants of (2) with a suitable choice of normalizing sequences a_n. In papers [1-3, 7] it was established the uniform (by corresponding indexing classes $\mathcal{F}$ and $\mathcal{D}$) variants of strong laws of large numbers (Glivenko-Cantelli type) and central limit theorems (Donsker type), respectively, for processes (3) and (4). In this paper, we will study another variant of the processes of the form (2) and for it uniform variants of the above limit theorems will be proved.

2. Information from the Theory of Metric Entropy

To prove the uniform variants of Glivenko-Cantelli and Donsker type theorems it necessary entropy properties of the class of measurable functions $\mathcal{F}$. In this regard, we define the space $\mathcal{L}_q(\mathbb{Q})$ of functions $f : \mathfrak{X} \to \mathbb{R}$ with norm

$$\|f\|_{\mathbb{Q},q} = \left(\mathbb{Q}|f|^q\right)^{1/q} = \left\{\int_{\mathfrak{X}} |f|^q d\mathbb{Q}\right\}^{1/q}, \ 1 \leq q < \infty.$$

To determine entropy of class $\mathcal{F}$ introduce $\varepsilon-$brackets in $\mathcal{L}_q(\mathbb{Q})$, which consists of pairs of functions $\varphi, \psi \in \mathcal{L}_q(\mathbb{Q})$, for which $\mathbb{Q}(\varphi(X) \leq \psi(X)) = 1$ and $\|\psi - \varphi\|_{\mathbb{Q},q} \leq \varepsilon$, i.e. $\mathbb{Q}(\psi - \varphi)^q \leq \varepsilon^q$. $\varepsilon-$ brackets (or $\varepsilon-$ balls) are denoted as $[\varphi, \psi]$. We say, that the function $f \in \mathcal{F}$ covered by bracket $[\varphi, \psi]$, if $\mathbb{Q}(\varphi(X) \leq f(X) \leq \psi(X)) = 1$. Moreover, these functions φ and ψ may not belong to the class $\mathcal{F}$, but they must have finite norms. The smallest number $N_{[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q(\mathbb{Q})\right)$ of $\varepsilon-$ brackets in $\mathcal{L}_q(\mathbb{Q})$ needed to cover $\mathcal{F}$, i.e.

$$N_{[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q(\mathbb{Q})\right)=min\{k:\text{for some } f_1,\dots,f_k\in\mathcal{L}_q(\mathbb{Q}),$$

$$\mathcal{F}\subset\bigcup_{i,j}\left[f_i,f_j\right],\left\|f_j-f_i\right\|_{\mathbb{Q},q}\le\varepsilon\} \tag{5}$$

is an important characteristic for determining complexity of the class $\mathcal{F}$. Its logarithm $H_q(\varepsilon)=\log N_{[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q(\mathbb{Q})\right)$ called the metric entropy with bracketing of a class $\mathcal{F}$ and it allows to control the number of sets needed to cover $\mathcal{F}$. Note that this number at $\varepsilon\to 0$ tends to $+\infty$. Growth of metric entropy to $+\infty$ controlled by its integral

$$\mathbb{J}_{[\,]}^{(q)}(\delta)=\mathbb{J}_{[\,]}\left(\delta;\mathcal{F};\mathcal{L}_q(\mathbb{Q})\right)=\int_0^{\delta}[H_q(\varepsilon)]^{1/2}d\varepsilon,\ \text{for }\ 0<\delta\le 1.$$

The convergence of this integral depends on the number $\varepsilon-$ brackets (5). Since the integral $\int_0^1\varepsilon^{-r}d\varepsilon$ converges at $r<1$ and diverges at $r\ge 1$, therefore, metric entropy should not grow faster than ε^{-2}. For example, for Donsker type theorems it necessary that the number (5) tends to $+\infty$ not very fast (see details, for example, [5, 12, 13]). According to the theorem 2.7.5 in [12], for class $\mathcal{F}$ -of monotone functions $f:\mathfrak{X}\to[0,1]$ and each probability measure $\mathbb{Q}$, is true the inequality $H_q(\varepsilon)\le K\cdot\varepsilon^{-1}$, where the constant K depends only from q. In particular, for class $\mathcal{F}=\{(-\infty,t),t\in\mathbb{R}\}$ of intervals $H_1(\varepsilon)\sim|log\varepsilon|$ and at $\varepsilon\downarrow 0$ metric entropy grows slowly.

Through $N_{m[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q(\mathbb{Q}_m)\right),H_{mq}(\varepsilon)$ and $\mathbb{J}_{m[\,]}^{(q)}(\delta)$ denote the numbers of $\varepsilon-$brackets, metric entropies and their integrals corresponding to submeasures $\mathbb{Q}_m, m=0,1$. Let $l^{\infty}(\mathcal{D})$ space of bounded functions on $\mathcal{D}=T\otimes\mathcal{F}$ with supremum norm

$$\|\cdot\|_{\mathcal{D}}=\sup_{(s;f)\in\mathcal{D}}|\cdot|.$$

Through $\|\cdot\|_{\mathcal{D}}^{*}\xrightarrow{\text{a.s.}}0$ denote the convergence to zero almost surely by outer measure $\mathbb{Q}^{*}$ in a Banach space $\left(l^{\infty}(\mathcal{D}),\|\cdot\|_{\mathcal{D}}\right)$ uniformly by the set $\mathcal{D}$ of sequence $\mathcal{D}-$ indexed empirical fields. In [3] the following uniform Glivenko-Cantelli type theorem for processes (4) is proved.

Theorem 1. [3] Let

$$\mathcal{F}\subset\mathcal{L}_2(\mathbb{Q}_m), J_{m[\,]}^{(2)}(1)<\infty,\quad m=0,1. \tag{6}$$

Then $\mathcal{F}$ is a consistent strong Glivenko-Cantelli class, i.e. at $n\to\infty$ almost surely

$$\left\|n^{-1/2}\Delta_n(s;f)\right\|_{\mathcal{D}}^{*}\to 0. \tag{7}$$

The following theorem from [3] is a uniform variant of the central limit theorem for process (4).

Theorem 2. [3] Under conditions (6), at $n\to\infty$

$$\Delta_n(s;f)\Rightarrow\Delta(s;f)\ \text{in}\ l^{\infty}(\mathcal{D}), \tag{8}$$

where $\{\Delta(s;f),(s;f)\in\mathcal{D}\}$ -is a Gaussian random field with zero mean. Under validity of the hypothesis $\mathcal{H}$ this field coincides by distribution with the Kiefer-Muller's random field with covariance at $(s;f),(t;g)\in\mathcal{D}$:

$$cov(\Delta(s;f),\Delta(t;g))=min(s;t)\cdot\{\mathbb{Q}fg-\mathbb{Q}f\mathbb{Q}g\}. \tag{9}$$

Remark 1. Obviously, theorems 1 and 2 contain the corresponding results for processes (3), obtained at $s=1$, however, it should be noted that the result for processes (3) was proved in [1, 2] under validity of weak conditions

$$\mathcal{F}\subset\mathcal{L}_1(\mathbb{Q}_m), N_{m[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_1(\mathbb{Q}_m)\right)<\infty, m=1,2. \tag{10}$$

The result of theorem 2 under hypothesis $\mathcal{H}$ is a generalized uniform analogue of the Donsker theorem, because from (8) at $t=s=1$ obtained covariance of $\mathbb{Q}$ -Brownian bridge.

3. Sequential Integral Empirical Processes of Independence

Now introduce the following sequential $\mathcal{N}=\mathbb{R}\otimes\mathcal{F}-$ indexed processes of structures (3):

$$\left\{\nabla_n(t;f)=\Delta_n f_t,(t;f)\in\mathcal{N}\right\}, \tag{11}$$

where $f_t(x)=f(x)\cdot I(x\le t),\ (t,f)\in\mathcal{N}$. Obviously, that processes (11) also contain processes (3), because $\nabla_n(\infty;f)=\Delta_n f$ for all $f\in\mathcal{F}$ and

$$(\Lambda_n-\Lambda)f=\int_{-\infty}^{t}f\,d(\Lambda_n-\Lambda),(t;f)\in\mathcal{N}. \tag{12}$$

Define the processes in a Banach space $\left(l^{\infty}(\mathcal{N}),\|\cdot\|_{\mathcal{N}}\right)$

$$\xi_i(t;f)=\left[p(1-p)\right]^{-1/2}\cdot\left(\eta_i(t;f)-E\eta_i(t;f)\right),(t;f)\in\mathcal{N}, \tag{13}$$

where $\eta_i(t;f)=f_t(X_i)(\delta_i-p)$. Consequently, for $(t;f),(s,g)\in\mathcal{N}$ we have

$$E\eta_i(t,f)=\int_{-\infty}^{t}f\,d(\mathbb{Q}_1-p\mathbb{Q})=\int_{-\infty}^{t}f\,d\Lambda, \tag{14}$$

and it's easy to calculate

$$\operatorname{cov}\left(\eta_i(t;f),\eta_i(s;g)\right)=\int_{-\infty}^{\min(t;s)} fgd\left(\mathbb{Q}_1-2p\mathbb{Q}_1+p^2\mathbb{Q}\right). \quad (15)$$

In particular, under validity of the hypothesis $\mathcal{H}$, from (14) and (15) we have

$$E\eta_i(t;f)=0, \operatorname{cov}\left(\eta_i(t;f),\eta_i(s;g)\right)=p(1-p)\int_{-\infty}^{\min(t,s)} fgd\mathbb{Q}. \quad (16)$$

For processes (13) taking into account the (14) and (15) at $(t;f),(s;g)\in\mathcal{N}$ we also have

$$E\xi_i(t;f)=0, \operatorname{cov}\left(\xi_i(t;f),\xi_i(s;g)\right)=\left[p(1-p)\right]^{-1}\cdot$$
$$\cdot\left\{\int_{-\infty}^{\min(t,s)} fgd\left(\mathbb{Q}_1-2p\mathbb{Q}_1+p^2\mathbb{Q}\right)-\int_{-\infty}^{t} fd\mathbb{Q}\cdot\int_{-\infty}^{s} gd\mathbb{Q}\right\}. \quad (17)$$

Under validity of the hypothesis $\mathcal{H}$ from (17) we have

$$\operatorname{cov}\left(\xi_i(t;f),\xi_i(s;g)\right)=\int_{-\infty}^{\min(t,s)} fgd\mathbb{Q}, (t;f),(s;g)\in\mathcal{N}. \quad (18)$$

Let $\{W(t;f),(t;f)\in\mathcal{N}\}$ -Gaussian random field with zero mean and covariance structure (18):

$$\operatorname{cov}\left(W(t;f),W(s;g)\right)=\operatorname{cov}\left(\xi_i(t;f),\xi_i(s;g)\right), \quad (19)$$

for all $(t;f),(s;g)\in\mathcal{N}$ and $i\geq 1$. Then it's easy to see, that under validity of the hypothesis $\mathcal{H}$, according to the formulas (18) and (19), Gaussian field by distribution is a Brownian sheet with covariance (18). Let's consider a normalized random field

$$U_n(t;f)=n^{-1/2}\sum_{k=1}^{n}\xi_k(t;f), (t;f)\in\mathcal{N}. \quad (20)$$

To the sequence (20), the uniform central limit theorem is holds.

Theorem 3. Let the conditions (6) are holds. Then at $n\to\infty$

$$U_n(t;f)\Rightarrow W(t;f) \text{ in } l^\infty(\mathcal{N}), \quad (21)$$

where $\{W(t;f),(t;f)\in\mathcal{N}\}$ -Gaussian field with zero mean and covariance, defined by formulas (17)-(19). Under validity of the hypothesis $\mathcal{H}$ it is coincides by distribution with Brownian sheet with covariance

$$cov\left(W(t;f);W(s;g)\right)=\int_{-\infty}^{\min(t,s)} fgd\mathbb{Q}, (t;f),(s;g)\in\mathcal{N}. \quad (22)$$

To prove theorem 3, it is necessary support statement on the limiting Gaussian property of a two-dimensional empirical field

$$\left\{\mathbb{A}_n(t;f),\mathbb{A}_{1n}(s;g)\right\},(t;f),(s;g)\in\mathcal{D}. \quad (23)$$

in the product space $l^\infty(\mathcal{N})\otimes l^\infty(\mathcal{N})$ for each class $\mathcal{F}$ of Donsker functions.

Theorem 4. Let the conditions (6) are holds. Then at $n\to\infty$ the sequence of random fields (23) weak converges in $l^\infty(\mathcal{N})\otimes l^\infty(\mathcal{N})$ to a Gaussian field of Brownian type $\{(\mathbb{A}(t;f),\mathbb{A}_1(s;g)),(t;f),(s;g)\in\mathcal{N}\}$ with zero mean and covariance structure

$$\operatorname{cov}\left(\mathbb{A}(t;f),\mathbb{A}(s;g)\right)=\mathbb{Q}f_tg_s-\mathbb{Q}f_t\mathbb{Q}g_s,$$
$$\operatorname{cov}\left(\mathbb{A}_1(t;f),\mathbb{A}_1(s;g)\right)=\mathbb{Q}_1f_tg_s-\mathbb{Q}_1f_t\mathbb{Q}_1g_s, \quad (24)$$
$$\operatorname{cov}\left(\mathbb{A}(t;f),\mathbb{A}_1(s;g)\right)=\mathbb{Q}_1f_tg_s-\mathbb{Q}f_t\mathbb{Q}_1g_s,$$

where $(t;f),(s,g)\in\mathcal{N}$.

In general, the proof of theorem 4 repeats the proofs of theorems 2 in [2] and 3.1 in [3]. In this case, the vector field (23) is represented by the following sequence of sums of normalized independent and identically distributed random fields with a covariance structure coinciding with (24):

$$\left(\mathbb{A}_n(t;f),\mathbb{A}_{1n}(s;g)\right)=n^{-1/2}\sum_{k=1}^{n}\left(f_t(X_k)-\mathbb{Q}f_t,\delta_kg_s(X_k)-\mathbb{Q}_1g_s\right).$$

Using the corresponding lemmas from [12], we obtain the required result of theorem 4. Details are omitted.

Proof of the theorem 3. For a given $\varepsilon>0$ select split points for a fixed $f\in\mathcal{F}$,

$$-\infty=t_f^{(0)}<t_f^{(1)}<\ldots<t_f^{(k)}=\infty$$

that at each i

$$\int\left[I\left(x\leq t_f^{(i+1)}\right)-I\left(x\leq t_f^{(i)}\right)\right]\mathbb{Q}_m(dx)<\varepsilon, m=0,1.$$

According to the condition (6), $\mathbb{J}_{m[\,]}^{(2)}(\varepsilon)<\infty, m=0,1$ and, consequently, among $\varepsilon-$ brackets $\left\{[\varphi_i,\psi_i], i=0,1,\ldots,N_{m[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_2(\mathbb{Q}_m)\right), m=0,1\right\}$ for a given $f\in\mathcal{F}$ exist $i:0,\ldots,N_{m[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_2(\mathbb{Q}_m)\right)$ and f covered by $\varepsilon-$bracket, i.e. $f\in[\varphi_i,\psi_i]$ so, that $\left\{\mathbb{Q}_m(\psi_i-\varphi_i)^2\right\}^{1/2}\leq\varepsilon$. Consider the system of $\varepsilon-$ brackets $\Gamma_\varepsilon=\left\{\left[\varphi_i(x)I\left(x\leq t_f^{(i)}\right),\psi_i(x)I\left(x\leq t_f^{(i+1)}\right)\right]\right\}$, among which exists is such, that for $f_t\in\Gamma=\left\{f_t:(t;f)\in\mathbb{R}\otimes\mathcal{F}\right\}$ occurs the covering

$$\varphi_i(x)I\left(x\leq t_f^{(i)}\right)\leq f_t^{(x)}\leq\psi_i(x)I\left(x\leq t_f^{(i+1)}\right)$$

and then have estimate

$$\left\{\mathbb{Q}_m\left(\psi_i(\cdot)I\left(\cdot\le t_f^{(i+1)}\right)-\varphi_i(\cdot)I\left(\cdot\le t_f^{(i)}\right)\right)^2\right\}^{1/2}\le$$

$$\le\left\{\mathbb{Q}_m(\psi_i-\varphi_i)^2\right\}^{1/2}+\int I\left(t_f^{(i)}<x\le t_f^{(i+1)}\right)d\mathbb{Q}_m(dx)<2\varepsilon, m=0,1.$$

It follows that

$$\int_0^1\left[\log N_{m[\,]}(2\varepsilon,\Gamma,\rho_m)\right]^{1/2}d\varepsilon<\infty, m=0,1, \quad (25)$$

where $\rho_m((t;f),(s;g))=\max\left\{|t-s|,\left\{\mathbb{Q}_m(f-g)^2\right\}^{1/2}\right\}$ -metric in $\mathcal{N}=\mathbb{R}\otimes\mathcal{F}$ by measure $\mathbb{Q}_m, m=0,1.$ To investigate the random field (20) represent it as a linear combination of empirical field components (23) as

$$U_n(t;f)=[p(1-p)]^{-1/2}\{\mathbb{A}_{1n}(t;f)-p\mathbb{A}_n(t;f)\},(t;f)\in\mathcal{N}. \quad (26)$$

In view of (25), we can use Ossiander's Theorem 3.1 [10] to the terms of representation (26). Then we have

$$U_n(t;f)\Rightarrow W(t;f)=[p(1-p)]^{-1/2}\{\mathbb{A}_1(t;f)-p\mathbb{A}(t;f)\}\text{ in } l^\infty(\mathcal{N}). \quad (27)$$

Process $\{W(t;f),(t;f)\in\mathcal{N}\}$ being a linear combination of two Gaussian fields with zero means and covariance structure, defined by formulas (24) is also a Gaussian field with zero mean. A direct calculation of the covariance of the limiting process in (27) shows, that it exactly coincides with the covariance (17) of the process (13). Consequently, (21) holds. In particular, under validity of the hypothesis $\mathcal{H}$, the limiting process in (27) by distribution is a Brownian sheet with covariance (22). Thus, theorem 3 is completely proved.

Consider a special process $\{U(\infty;f)=U(f),f\in\mathcal{F}\}$, obtained from (20) and the corresponding limit process $\{W(\infty;f)=W(f),f\in\mathcal{F}\}$, which is a Brownian sheet with zero mean and covariance

$$\operatorname{cov}(W(f),W(g))=\int fgd\mathbb{Q}, f,g\in\mathcal{F}. \quad (28)$$

Define a unit ball in a Hilbert space $\mathcal{L}_2(\mathbb{Q})$ through $S=\left\{g\in\mathcal{L}_2(\mathbb{Q}):\|g\|^2=\int g^2d\mathbb{Q}\le 1\right\}$, and also set of functions on $\mathcal{F}$, suppose that

$$\mathcal{U}(\mathcal{F})=\left\{\mathbb{G}:\mathcal{F}\to\mathbb{R}:\mathbb{G}f=\int fgd\mathbb{Q}, f\in\mathcal{F}, g\in S\right\}.$$

Then from theorem 3, taking into account Theorem 4.2 in [10], we will have the functional law of the repeated logarithm for the process $\mathcal{U}_n(f), f\in\mathcal{F}$.

Consequence 1. If conditions (6) are holds, then under the hypothesis $\mathcal{H}$ sequence

$$\left\{(2\log\log n)^{-1/2}\cdot\mathcal{U}_n(f), f\in\mathcal{F}, n\ge 3\right\} \quad (29)$$

relatively compact relatively $\|\cdot\|_{\mathcal{F}}$ and the set of its limiting points coincides with $\mathcal{U}(\mathcal{F})$.

Now investigate the sequence (11). We define a sequence of random processes $\mu_n=\mathbb{A}_{1n}(\infty;1)=n^{1/2}(p_n-p)$ and processes $\mu_n(t;f)=\mu_n\mathbb{Q}f_t$. Then from theorem 4, in particular, it follows, that at $n\to\infty$

$$\mu_n(t;f)\Rightarrow\mu_0(t;f)\text{ in } l^\infty(\mathcal{N}), \quad (30)$$

where $\mu_0(t;f)=\mu_0\mathbb{Q}f_t$ и $\mu_0=\mathbb{A}_1(\infty;1)$ - is random variable with a normal distribution $N(0,p(1-p))$. Consequently, the process $\mu_0(t;f)$ is Gaussian with zero mean and covariance at $(t;f),(s;g)\in\mathcal{N}$

$$\operatorname{cov}(\mu_0(t;f),\mu_0(s;g))=p(1-p)\mathbb{Q}f_t\mathbb{Q}g_s. \quad (31)$$

Theorem 5. Let conditions (6) are holds. Then at $n\to\infty$

$$\nabla_n(t;f)\Rightarrow\nabla(t;f)\text{ in } l^\infty(\mathcal{N}), \quad (32)$$

where $\left\{\nabla(t;f)=W(t;f)+\mu_0(t;f)\cdot[p(1-p)]^{-1/2},(t;f)\in\mathcal{N}\right\}$ is a Gaussian random field with zero mean. Under validity of the hypotheses $\mathcal{H}$ limiting field (32) is $\mathbb{Q}-$ Brownian bridge with covariance at $(t;f),(s;g)\in\mathcal{N}$

$$\operatorname{cov}(\nabla(t;f),\nabla(s;g))=\int_{-\infty}^{\min(t,s)}fgd\mathbb{Q}-\int_{-\infty}^{t}fd\mathbb{Q}\cdot\int_{-\infty}^{s}gd\mathbb{Q}. \quad (33)$$

Proof of the theorem 5. Note that, the statistics p_n is strongly consistent for p and true the following Bernstein's inequality from [11]:

$$\mathcal{P}(|p_n-p|>\varepsilon)\le 2\cdot\exp\left(-\frac{n\varepsilon^2}{4}\right),\varepsilon>0. \quad (34)$$

Consequently, to prove (32) it suffices to establish, that

$$\nabla_n^*(t;f)\Rightarrow\nabla(t;f)\text{ in } l^\infty(\mathcal{N}), \quad (35)$$

where

$$\left\{\nabla_n^*(t;f)=\left(\frac{n}{p(1-p)}\right)^{1/2}\cdot(\Lambda_n-\Lambda)f_t,(t;f)\in\mathcal{N}\right\} \quad (36)$$

asymptotically equivalent to (11) sequence. Process (36) admits a representation

$$\nabla_n^*(t;f)=U_n(t;f)-[\mu_n(t;f)-R_n(t;f)]\cdot[p(1-p)]^{-1/2}, \quad (37)$$

where the remainder term of the representation (37) taking into account (34) at $n \to \infty$ uniformly tends to zero

$$\left\|R_n(t;f)\right\|_{\mathcal{N}} = \left\|n^{-1/2} \cdot \mathbb{A}_{1n}(\infty;1) \cdot \mathbb{A}_n(t;f)\right\|_{\mathcal{N}} = \mathbb{O}_p\left(n^{-1/2}\right). \quad (38)$$

According to Theorem 4, in particular, in view of (27), (30) and (38) we have at $n \to \infty$

$$\nabla_n^*(t;f) \Rightarrow \left[p(1-p)\right]^{-1/2} \cdot \left\{\mathbb{A}_1(t;f) - p\mathbb{A}(t;f) - \mu_n(t;f)\right\} =$$

$$= \nabla(t;f), \text{in } l^\infty(\mathcal{N}), \quad (39)$$

where the limit process being a linear combination of Gaussian processes is Gaussian with zero mean. This process can also be represented as $\nabla(t;f) = W(t;f) + \left[p(1-p)\right]^{-1/2} \cdot$ $\cdot \mu_0(t;f), (t;f) \in \mathcal{N}$. By a direct calculation of covariance according to (24), we obtain at $(t;f),(s;g) \in \mathcal{N}$

$$\operatorname{cov}\left(\nabla(t;f), \nabla(s;g)\right) = \left[p(1-p)\right]^{-1} \cdot \sum_{k=1}^{9} \Psi_k, \quad (40)$$

where

$$\Psi_1 = \mathbb{Q}_1 f_t g_s - \mathbb{Q}_1 f_t \cdot \mathbb{Q}_1 g_s; \ \Psi_2 = -p\left(\mathbb{Q}_1 f_t g_s - \mathbb{Q} f_t \cdot \mathbb{Q}_1 g_s\right);$$

$$\Psi_3 = -(1-p)\mathbb{Q} f_t \cdot \mathbb{Q}_1 g_s; \ \Psi_4 = -p\left(\mathbb{Q} f_t g_s - \mathbb{Q} g_s \cdot \mathbb{Q}_1 f_t\right);$$

$$\Psi_5 = p^2\left(\mathbb{Q} f_t g_s - \mathbb{Q} f_t \cdot \mathbb{Q} g_s\right); \ \Psi_6 = p\mathbb{Q} f_t \cdot \left(\mathbb{Q}_1 g_s - p\mathbb{Q} g_s\right);$$

$$\Psi_7 = -(1-p)\mathbb{Q} g_s \cdot \mathbb{Q}_1 f_t; \ \Psi_8 = p\mathbb{Q} g_s \cdot \left(\mathbb{Q}_1 f_t - p\mathbb{Q} f_t\right);$$

$$\Psi_9 = p(1-p)\mathbb{Q} f_t \cdot \mathbb{Q} g_s;$$

Under validity of the hypothesis $\mathcal{H}$ it is easy to verify, that

$$\sum_{k=1}^{9} \Psi_k = p(1-p)\left(\mathbb{Q} f_t g_s - \mathbb{Q} f_t \mathbb{Q} g_s\right). \quad (41)$$

Now (33) follows from (40) and (41), which completes the proof of theorem 5.

The result of theorem 5 can be used to construct statistics of criteria for testing a hypothesis $\mathcal{H}$.

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