Performance analysis of powers of skewness and kurtosis based multivariate normality tests and use of extended Monte Carlo simulation for proposed novelty algorithm

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Abstract: An ample study of the comparative powers of a number of omnibus multivariate normality tests is main object in this paper. Since testing for multivariate normality tests is considerably more challenging process than for testing of univariate one and therefore, study of testing for multivariate normality tests has its increasing demand. Through this paper, we have explored several techniques for assessing multivariate normality (MVN) and as well as comparative analysis for their competence have also been demonstrated. The results of extensive Monte Carlo simulation study of the size corrected power of various tests of multivariate normality for drawn samples from contaminated normal distributions have been explored as well. Moreover, a novel algorithm has been proposed in order to evaluate the size corrected powers for testing multivariate normality. The algorithm proposed herein is a fast easily implementable algorithm and it can be applied for both types of univariate and multivariate normality tests. Using Different omnibus tests for sample size 50 and 200, graphs for empirical powers of multivariate normal data with lower and upper contamination have been presented. Finally, some significant conclusions of our present study have been drawn.

Keywords: Multivariate Normality Tests, Goodness-of-Fit Tests, Correlation Coefficient, Skewness, Kurtosis, Monte Carlo Simulation Technique

1. Introduction

In statistics, multivariate normality tests are used for checking a given set of data for similarity to the multivariate normal distribution. Multivariate normality test is a fundamental predicament in statistics. The null hypothesis is that the data set is similar to the normal distribution; therefore a sufficiently small p-value indicates non-normal data. Though, the rigorous study of literature shows that since last six decades number of researchers contributed for various methods for assessing multivariate normality (MVN). Among these, here some noteworthy researchers Anscombe and Glynn [1], Bera [2], Bowman and Shenton [3], D’Agostino [7], Grianadesikan [11], Inhof [12], Kaziol [17], Ozturk and Romeu [37], Romeu and Ozturk [40], Roy [41], Shenton and Bowman [42] are worth mentioning. Moreover, some recent research works of Doornik and Hansen[8], Enomoto et al. [9], Nakagawa et al.[35], Rencher [39], and Thode[46] in this direction may also be referred here. Thus, there are various methods for assessing multivariate normality (MVN) yet in some cases, statisticians and researchers face challenging task for assessing multivariate normality. Multivariate normality tests include tests explored by Cox and Small [4], Smith and Jain [44] and Friedman-Rafsky [10]. Despite availability of the large amount of methods, Rencher [39] commented in 2002 that checking for multivariate normality is conceptually not as simple as assessing univariate normality and consequently the state of
the art is not developed as well. Most of the multivariate normality test procedures are extensions of univariate normality tests. The accessible tests of multivariate normality can be categorically described as following:

- Procedures based on graphical plots and correlation coefficients
- Goodness-of-fit tests
- Tests based on measures of skewness and kurtosis
- Consistent procedures based on the empirical characteristic function.

In this paper, our main objective is to develop an implementable algorithm to calculate size corrected powers of competitive MVN tests that can be used to compare the efficiency of various multivariate normality tests. The organization of the paper is as follows. Section 2 discusses our emphasized tests for assessing MVN. In section 3, we describe comparative study of various multivariate normality tests. Section 4 concerns with a novel algorithm for multivariate normality tests and its empirical results have been explored by using an extended Monte Carlo simulation. Moreover, purpose of present study and its future scope is highlighted in section 5. By the end of paper, some valuable conclusive remarks have been drawn in section 6.

2. Literature Review

Rigorous study of literature shows that there is no shortage of methods for assessing multivariate normality (MVN). Among abundant available tests for testing MVN, here we stress only those multivariate normality tests which are mainly based on skewness and kurtosis. Numerous omnibus tests based on skewness, kurtosis are proposed by D’Agostino & Pearson [5], Bowman and Shenton [3], Pearson et al. [38], D’Agostino et al. [6], Loony [18], and the coordinate-dependent and invariant procedures described by Cox & Small [4] and Small [43] offers an overall test of multivariate normality that are quadratic forms involving multivariate normality tests explored by Mardia and Foster [20], Mardia [21,22] and Johnson’s (1949) $S_u$ transformation on a vector of the marginal skewness or kurtosis statistics. Mardia [22] developed multivariate extensions of multivariate skewness and kurtosis. The measures derived by Mardia [22] are affine invariant. The sample statistic for multivariate skewness is given as follows:

$$b_{1,p} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ (y_i - \bar{y}) S^{-1} (y_j - \bar{y}) \right]$$

And the corresponding sample statistic for kurtosis is as following:

$$b_{2,p} = \frac{1}{n} \sum_{i=1}^{n} \left[ (y_i - \bar{y}) S^{-1} (y_i - \bar{y}) \right]$$

Here, we remark that skewness and kurtosis functions explored by Mardia [22] are functions of the squared Mahalanobis distances. This fact makes Mardia’s measures, particularly the multivariate kurtosis measure, useful in multivariate outlier detection. We further remark that Mardia [21,22] determined the asymptotic distributions of the multivariate skewness and kurtosis statistics. However, Mardia’s kurtosis statistic is skewed and converges very slowly to the limiting normal distribution. For medium size samples, the parameters of the asymptotic distribution of the kurtosis statistic are modified. For small sample tests empirical critical values are used. Tables of critical values for both statistics are given by Rencher [39] for $k = 2,3,4$. It can be further examined that Mardia’s [22] multivariate normality tests are affine invariant but not consistent. For example, the multivariate skewness test is not consistent against symmetric non-normal alternatives.

If we consider $A = nb_{1,p}/6$, it can be shown that $A$ is asymptotically distributed as a chi-square random variable with $p(p+1)(p+2)/6$ degrees of freedom. Similarly, the statistic $B = b_{2,p}/\sqrt{bp(p+2)/n}$ is asymptotically distributed as a standard normal random variable. The asymptotic distributions of functions of the multivariate skewness and kurtosis statistics were exploited by Mardia [20,21,22] to develop two tests for the multivariate normality.

Moreover, some other tests of MVN based on $W$-statistic were proposed by Malkovich and Afifi [19] presented a test based on Roy’s [41] union-intersection principle that utilized generalized measures of skewness and kurtosis. Subsequently, Isogai [13] introduced another measure of multivariate skewness to test of MVN and Isogai [14] used influence functions to develop two test statistics for MVN that was mostly related to Mardia’s measure of multivariate skewness.

Sample measures of multivariate skewness and kurtosis proposed by Srivastava [45] based on principal components have been discussed by number of previous researchers and using the measures of multivariate skewness and kurtosis calculated with the principal components method., they succeeded to find their research works in this direction which were very similar to Mardia [21,22]. Okamoto and Seo [36] derived the exact expectation and variance of Srivastava’s skewness and improved $X^2$-statistic defined by Srivastava [45] for assessing multivariate normality. Mardia and Foster [20] constructed six possible omnibus test statistics, three of which did consider the covariance between skewness and kurtosis. D’Agostino and Pearson [5] proposed an omnibus test based on the distributions of the standardized third and fourth moments, $\sqrt{b_1} = m_3/m_2^{1.5}$, $b_2 = m_4/m_2^2$ in random samples of $n$ observations from a univariate normal population. Here for a sample having its $n$-sample points $X_1, X_2, ..., X_n$, we define as following:

$$\bar{X} = (1/n) \sum_{i=1}^{n} X_i, m_r = (1/n) \sum_{i=1}^{n} (X_i - \bar{X})^r,$$

$r=2,3,4$.

Jarque and Bera’s [15] test statistics using Srivastava’s [45] sample skewness and kurtosis which are asymptotically distributed as $X^2$-distribution were proposed by Koizumi et al. [16] as well as an improved tests of Jarque and Bera [15]
have been discussed by many authors. A popular test of
normality based on D’Agostino-Pearson $K^2$ statistic that
combines these two statistics $\sqrt{b_1}$ and $b_2$, discussed in
D’Agostino, et al. [6]. Normalizing transformation for the
skewness $\sqrt{b_1}$ into $z_1$ is as in D’Agostino [7] and another
transformation for the kurtosis $b_2$ into $z_2$ based on the
Wilson–Hilferty transformation as described in Doornik and
Hansen [8]. The improved omnibus test was recently
presented in Nakagawa et al. [35]. The above mentioned test
statistics are all based on third and fourth sample moments.

3. Compilation of Multivariate Normality Tests

Numerous procedures have been proposed for assessing
multivariate normality, some of them are discussed in the
following paragraphs.

3.1. The Multivariate Jarque-Bera Test

In view of Enomoto et al. [9], let $X_1, X_2, \ldots, X_p$ be samples
of size $N$ from a univariate population. Now, if we consider
that $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$, and $S = n^{-1}\sum_{i=1}^{n} (X_i - \bar{X})^2$ be the
sample mean and the sample covariance respectively. Then
the univariate sample skewness is given by following equations:

$$\sqrt{b_1} = m_3 / m_2^{3/2}, \quad b_2 = m_4 / m_2^{2}$$

Where, $m_j = n^{-1}\sum_{i=1}^{n} (X_i - \bar{X})^j$.

Then Jarque and Bera [15] proposed the test statistic using
univariate sample skewness and kurtosis for normality test is
given by

$$JB = n \left( \frac{\sqrt{b_1}^2}{6} + \frac{(b_2 - 3)^2}{24} \right).$$

It holds that JB statistic is asymptotically distributed as $\chi^2$
distribution under normality.

On the other hand, Koizumi et al. [16] proposed multivariate
Jarque and Bera test statistics as follows:

$$MJB = np \left(\frac{b_{1,p}}{6} + \frac{(b_{2,p} - 3)^2}{24}\right) \rightarrow \chi^2_{p+1} \quad (N \rightarrow \infty),$$

$$MJB^* = E[b_{1,p}] + \frac{E[b_{2,p}]}{Var[b_{2,p}]} \rightarrow \chi^2_{p+1} \quad (N \rightarrow \infty),$$

respectively. $MJB^*$ statistic is improved so that accuracy of
upper percentile for approximate test statistic is better than
that of $MJB$ statistic for small $N$. Where for large $N$, the
expectation of $b_{1,p}$ and variance of $b_{2,p}$ when the population is

$$N_p(\mu, \Sigma)$$ are given by

$$E[b_{1,p}^2] = \frac{6(N - 2)}{(N + 1)(N + 3)},$$

$$E[b_{2,p}^2] = \frac{3(N - 1)}{(N + 1)},$$

$$Var[b_{2,p}] = \frac{24 N(N - 2)(N - 3)}{p(N + 1)^2(N + 3)(N + 5)}.$$
\[ V = \text{diag} \left( \sigma_1^2, \ldots, \sigma_p^2 \right) \] and form the correlation matrix
\[ C = V^{-1/2} S V^{-1/2}. \] Define the \( p \times n \) matrix \( y' = (y_1, \ldots, y_p) \) of transformed observations:
\[ y_i = H N^{-1/2} H' V^{-1/2} (x_i - \mu), \]
with \( \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_p) \), the matrix with the eigen values of \( C \) on the diagonal. The columns of \( H \) are the corresponding eigenvectors, such that \( H' H = I_p \) and \( \Lambda = H' C H \). Consequently \( n^{-1} Y Y' = I_p \). Using the population values for \( C \) and \( V \), a multivariate normal can thus be transformed into independent standard normal; using sample values this is only approximately so.

We may now compute univariate skewness and kurtosis, defining \( b_i' = (\sqrt{b_{1i}}, \ldots, \sqrt{b_{pi}}) \), \( b_i = (b_{1i}, \ldots, b_{pi}) \) and \( l \) as a \( p \) – vector of ones, the test statistic:
\[ \frac{n b_i' b_i}{6} + \frac{n (b_i - 3l)' (b_i - 3l)}{24} \sim \chi^2(2p) \]
will again require large samples.

### 3.4. Extension of Jarque–Bera Test

An extension of Jarque and Bera’s [15] test was proposed by Nakagawa et al. [35], wherein they proposed a new test statistic based on the Jarque–Bera’s [15] test. Let \( X_1, X_2, \ldots, X_p \) be a sample drawn from a normal population. Using \( \sqrt{b_i} \) and \( b_i \), the test statistic is given by
\[ T = \frac{(\sqrt{b_1})^2}{6} + \frac{b_2^2}{24} \]

It is clearly indicated that \( T \) is invariant under origin and scale changes. We only consider the case of a sample from a standard normal population as a null hypothesis.

### 3.5. Omnibus \( K^2 \) Statistic Test

This test was explored by D’Agostino et al. [6]. In this multivariate normality test, statistic \( z(\sqrt{b_1}) \) and \( z(b_1) \) can be combined to produce an omnibus test of normality. By omnibus, we mean it is able to detect deviations from normality due to either skewness or kurtosis. The test statistics is given as following:
\[ K^2 = z^2(\sqrt{b_1}) + z^2(b_1) \]
where \( z(\sqrt{b_1}) \) and \( z(b_1) \) are the normal approximation to \( \sqrt{b_1} \) and \( b_1 \).

### 3.6. Transformed Skewness and Kurtosis Test

The transformed skewness and kurtosis test was proposed by Doornik and Hansen [8]. Another omnibus test of normality is presented combining statistic \( z_1 \) and \( z_2 \). Let \( z_1 \) and \( z_2 \) denote the transformed skewness and kurtosis, respectively, where they are transformed in a way that makes their distribution as close to standard normal as possible. The test statistic is (‘\( \alpha \bar{p} \)’ denotes ‘approximately distributed as’):
\[ E_p = z_1^2 + z_2^2 \sim \chi^2(2) \]

The transformation for the skewness is based on D’Agostino [7], who uses Johnson \( S_u \) to approximate the distribution of \( \sqrt{b_1} \). The kurtosis is transformed from a gamma distribution to a \( \chi^2 \) distribution with non-integer degrees of freedom, which is then translated into standard normal using the Wilson–Hilferty cubed root transformation.

### 3.7. Modified Multivariate Jarque-Bera Test

Multivariate Jarque-Bera test was modified by Koizumi et al. [16]. Let \( b_{M,1} \) and \( b_{M,2} \) be the sample measures of multivariate skewness and kurtosis, respectively, on the basis of a random sample of size \( N \) drawn from \( N_p(\mu, \Sigma), \Sigma > 0 \).

Then, it is fairly easy to get following expression:
\[ z_{M,1} = \frac{N}{6} b_{M,1} \]
is asymptotically distributed as \( \chi^2 \) distribution with \( f = (p+1)(p+2)/6 \) degrees of freedom, and
\[ z_{M,2} = \sqrt{\frac{N}{8p(p+2)}} (b_{M,2} - p(p+1)) \]
is asymptotically distributed as \( \chi^2 \) distribution.

By making reference to moments of \( b_{M,1} \) and \( b_{M,2} \), Mardia [21] considered the following approximate test statistics as competitors of \( z_{M,1} \) and \( z_{M,2} \): A modified is given by equation:
\[ MJB_{M} = z_{M,1}^* + z_{M,2}^* \]
\[ MJB_{M} \] is distributed as \( \chi^2 \) distribution asymptotically.

Where,
\[ z_{M,1}^* = \frac{N}{6} b_{M,1} \left( \frac{(p+1)(N+1)(N+3)}{N(N+1)(p+1) - 6} \right) \sim \chi^2_{p(p+1)(p+2)/6} \]
\[ z_{M,2}^* = \sqrt{\frac{1}{8p(p+2)}(N+3)(N+5)(N+1)b_{M,2} - p(p+2)(N-1)} \sim \chi^2_{N(0,1)} \]

### 4. Proposed Algorithm and Key Features

Here, we propose a fast easily implementable algorithm for determining size corrected power. Power calculation for MVN tests are considered by many authors and in most of
the cases their suggested approach was either based on the percentage of rejection or location and scale parameter contamination whether by increasing or decreasing the parameter value but in practice these types of contaminations cannot make data non-normal. That’s why we are considering the characteristics of normal distribution i.e. skewness and kurtosis. By contaminating the upper and lower percentages of data, say 10%, 20% or more, we are making them highly skewed or asymmetric by multiplying with a increasing constant \( c \) where \( c = 1, 2, 3, \ldots \) when \( c = 1 \) then it will calculate the power of null hypothesis as the value of \( c \) will increase it will go far from the null which expresses the departure from normality.

4.1. Algorithm and Hypothesis Testing

In power study, a common practise is to choose the one with the highest empirical power, when several testing procedures are considered. However, this usage is difficult in few cases, so a widely employed practice is to report what is called size corrected power that is computing the empirical power with simulated critical values. In this work, we are also paying attention in evaluating size corrected power. To calculate the size corrected power of multivariate normality tests, we propose the following algorithm:

i. Suppose \( x_1, x_2, \ldots, x_n \) is a random sample from a \( p \)-variates multivariate normal population.

ii. Sort each variable \( x_{(i)} \), \( x_{(i+1)} \), \ldots, \( x_{(n)} \) where \( i = 1, 2, \ldots, p \) in ascending order of magnitude.

iii. Multiply the upper \( k \% \) of data or lower \( k \% \) of data; say 5%, 10% by a positive constant \( c \geq 1 \).

iv. Calculate the power on the basis of the hypothesis. The hypothesis can be stated as \( H_0 : c = 1 \) (i.e., the distribution is normal) against \( H_1 : c > 1 \) (i.e., the distribution is non-normal).

4.2. Simulation for the Proposed Algorithm

Simulation on which this study based is enumerated below. The considered null hypothesis is as follows-

\( H_0 : \text{Observation are normally distributed} \)

\( H_1 : \text{Observation are not normally distributed} \)

To evaluate whether size or level of test achieves advertised \( \alpha \), generate data under normality assumption and calculate proportion of rejections of \( H_0 \). To calculate power, we follow the 4 steps proposed above.

4.3. Monte Carlo Simulation Assessing Power

This section demonstrates powers of different omnibus multivariate normality tests using Monte Carlo simulation with contaminations. We remark here that simulation techniques play a key role in exploring results in Mathematical and Computational Sciences. An extended Monte Carlo simulation with contaminations is one of the best techniques among different simulation techniques. Many more researchers including Maurya et al. [23-34] confined their attention to explore significant results using different simulation techniques. For present purpose of using an extended Monte Carlo simulation here, we generate data for different sample sizes under the null hypothesis and carry out 10,000 repetitions to calculate size corrected powers with upper and lower contamination of a certain percentage say 10%, 20% or more and obtained powers are presented through power curves to emphasize on the comparative performances of the tests.

Using different omnibus tests for sample size \( n = 50 \) and \( n = 200 \), the corresponding graphs of empirical powers of multivariate normal data with upper contamination are demonstrated in figures 1-2 respectively. Moreover, empirical powers of multivariate normal data with lower contamination are demonstrated in figures 3-4 respectively.


5. Purpose of Study and Future Scope

The purpose of the study was to provide general indication of the comparative effectiveness of the different multivariate normality test procedures. The omnibus measured tests in this study have optimum asymptotic power properties and good finite sample performance. Due to their simplicity, all of the omnibus tests should prove to be useful tools in multivariate statistical analysis. However, some general conclusions can be gleaned from the results. We observed that above power curves of the different omnibus multivariate tests show good power properties and their powers vary for different sample sizes and for unlike contaminations. In all cases, the omnibus test of transformed skewness and kurtosis shows utmost power. For the samples of lower contamination, though MMJB (modified multivariate Jarque and Bera) test have highest power but exhibits reverse power for upper contamination both for small and large samples. MJB (Multivariate Jarque and Bera) and IMJB (Improved multivariate Jarque and Bera) test have almost same power in most of the cases. Remaining omnibus multivariate tests have good and moderate powers for simulated samples.

6. Conclusions

Here, comparative powers of skewness and kurtosis based multivariate normality tests have been demonstrated and a novel implementable algorithm for MVN has been proposed. In addition to this, powers of different omnibus multivariate normality tests using Monte Carlo simulation with contaminations have also been successfully explored. Basically all the simulated results of this paper originated from empirical sampling studies and the information are intended to be broadly analytical. We propose an efficient algorithm for calculating size-corrected powers of multivariate normality tests and obtained powers of multivariate omnibus tests through the proposed algorithm are superior. In general, this algorithm is applicable to all tests of normality for the calculation of size-corrected power. Using different omnibus tests for sample size $n = 50$, and $n = 200$, the corresponding graphs of empirical powers of multivariate normal data with lower and upper contamination are demonstrated.


