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Solvability of Some Nonlinear Integral Functional Equations

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Abstract: This paper discussed some existence theorems for nonlinear functional integral equations in the space L^1 of Lebesgue integrable functions,by using the Darbo fixed point theorem associated with the Hausdorff measure of noncompactness. Also, as an application, we discuss the existence of solutions for some nonlinear integral equations with fractional order.

Keywords: Superposition Operator, Carathe'odory Conditions, Measure of Noncompactness, Fixed Point Theorem

1. Introduction

The subject of nonlinear integral equations considered as an important branch of mathematics because it is used for solving many problems such as physics, chemistry [4, 20].

In this paper we will use the technique of measures of noncompactness and Darbo fixed point theorem to prove the existence theorem for a nonlinear integral equation in the spaces $L^1(R_+)$. Also, as an application, we discuss the existence of solutions for some nonlinear integral equations with fractional order, which extends to some previous results in literature [2].

2. Notation and Auxiliary Facts

Let R be the field of real numbers, R_+ be the interval $[0,\infty)$ and L^1, be the space of Lebesgue integerable functions on a measureable subset $[0,\infty)$ of R, with the standard norm.

$$\|x\|_{L^1(R_+)} = \int_0^\infty |x(t)|\, dt$$

One of the most important operators studied in nonlinear functional analysis is the so-called superposition operator [1].

Assume that a function $f(t,x) = f: I \times R \to R$ satisfies Carath$\acute{e}$odory conditions, i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$.

Then to every function $x(t)$ being measurable on I, we may assign the function

$$(Fx)(t) = f\big(t, x(t)\big), t \in I.$$

The operator F in such a way is called the superposition operator generated by the function f.

We have the following theorem due to Appell and Zabrejko [1].

Theorem 2.1

The superposition operator F generated by the function f maps continuously the space L^1 *into* L^1 if and only if $|f(t,x)| \le a_1(t) + b|x|\ \forall\, t \in I$ and $x \in R$, where $a(t) \in L^1$ and $b \ge 0$.

Next, we will mention a desired theorems concerning the compactness in measure of a subset X

Of $L^1(R_+)$ [2].

Theorem 2.2

Let X be a bounded subset of $L^1[0,\infty)$ consisting of functions which are almost everywhere nondecreasing (or nonincreasing) on the interval $[0,\infty)$. Then X is compact in measure.

Furthermore, we recall few facts about the convolution operator [19].

Let $k \in L^1(R)$ be a given function. Then for any function $x \in L^1$, the integral

$$(Kx)(t) = \int_0^\infty k(t-s)\, x(s)\, ds,$$

exists for almost every $t \in R_+$. Moreover, the function $(Kx)(t)$ belongs to the space L^1.

Thus K is a linear operator which maps the space L^1 into L_1 and K is also bounded since $\|Kx\|_{L^1(R)} \leq \|K\|_{L^1(R)}\|x\|$, for every $x \in L^1$; so, it will be continuous.

Hence the norm $\|K\|$ of the convolution operator is majored by $\|K\|_{L^1(R)}$.

In the sequel, we have the following theorem due to Krzyz [18].

Theorem 2.3

Assume that $k(t,s) = k: R_+^2 \to R$ is measurable on R_+ such that the integral operator,

$$(Kx)(t) = \int_0^\infty k(t,s)x(s)ds, t \geq 0,$$

maps L^1 into itself, then K transforms the set of nonincreasing functions from L^1 into itself if and only if for any $A > 0$, the following implication is true.

$$t_1 < t_2 \to \int_0^A k\,(t_1,s)ds \geq \int_0^A k\,(t_2,s)ds.$$

In the case space $L^1(0,1)$ we will use the following corollary

Corollary 2.1

Let $k_i: (0,1)^2 \to R_+$ be ameasurable function generated the Fredholm operator K acting from $L^1(0,1)$ into $L^1 1(0,1)$, if for every $p \in (0,1)$ and for all $t_1, t_2 \in (0,1)$ the implication holds,

$$t_1 < t_2 \to \int_0^p k_i\,(t_1,s)ds \geq \int_0^p k_i\,(t_2,s)ds.$$

Finally, we give a short note on measures of noncompactness and fixed point theorem.

Let E be an arbitrary Banach space with $\|.\|$ and the zero element θ.

Let also X be a nonempty and bounded subset of E and B_r be a closed ball in E centered at $\theta$and radius r.

The Hausdorff measure of noncompactness $\chi(X)$ [3] is defined as

$$\chi(X) = inf\,\{r > 0: there\ exists\ a\ finite\ subset\ Y\ of\ E\ such\ that\ X \subset Y + B_r\}.$$

The De Blasi measure of weak noncompactness [2] is defined as

$$\beta(X) = inf \begin{Bmatrix} r > 0: there\ exists\ a\ weakly\ compact\ subset\ W\ of\ E\ such\ that \\ X \subset W + B_r \end{Bmatrix}$$

De Blasi measure can be expressed in the space $L^1(0,1)$ in a very useful formula, given by Appell and De Pascale [2]

$$\beta(X) = \lim_{\varepsilon \to 0}\left\{\sup_{x \in X}\left\{sup\left[\int_D |x(t)|dt: D \subset (0,1), meas\ D \leq \varepsilon\right]\right\}\right\}$$

Another measure was defined in the space L_1 [2]. For any $\varepsilon > 0$, let

$$c(X) = lim_{\varepsilon \to 0}\{\sup_{x \in X}\{\sup\left[\int_D |x(t)|dt, D \subset R_+, meas\ (D) \leq \varepsilon\right]\}\},$$

and

$$d(X) = \lim_{T \to \infty}\{\sup\left[\int_T^\infty |x(t)|dt, x \in X\right]\}$$

Where meas D denotes the lebesgue measure of sub set D. put

$$\gamma(X) = c(X) + d(X).$$

Then we have the following theorem [2], which connects between the two measures $\chi(X)$ and $\gamma(X)$.

Theorem 2.4

Let X be a nonempty, bounded and compact in measure subset of $L^1(R_+)$, then

$$\chi(X) \leq \gamma(X) \leq 2\chi(X).$$

In the case space $L^1(0,1)$ we have the following theorems [2].

Theorem 2.5

Let X be abounded subset of $L^1(0,1)$ and suppose that there is a family of measurable subset $\{\Omega_c\}_{0 \leq c \leq mea\ I}$, of the interval I such that $meas\Omega_c = c\ for\ evrey\ c \in [0, meas\ I]$

And for every $x \in X$: $cx(t_1) \leq x(t_2), (t_1 \in \Omega_c, t_2 \notin \Omega_c)$

Then the set X is compact in measure.

Theorem 2.6

Let X be an arbitrary nonempty and bounded subset of $L^1(0,1)$. If X is compact in measure then $\beta(x) = \chi\,(X)$.

As an application of measures of noncompactness, we recall the fixed point theorem due to Darbo [5].

Theorem 2.7

Let Q be a non-empty, bounded, closed and convex subset of E and let $A: Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness, i.e there exist $k \in [0,1)$ such that $\mu(A(X) \leq k\mu(X)$ for any nonempty subset X of, then A has at least one fixed point$Q$.

3. Existence Solutions in $L^1(R_+)$

Now we will discuss the solvability for the following nonlinear integral equation

$$x(t) = g(t) + \int_0^\infty k_1(t-s)f_1(s, \int_0^s k_2(s,\tau)f_2(\tau, x(\tau))d\tau)ds, t > 0 \quad (1)$$

in the space $L^1(R_+)$.

We shall treat equation (3.1) under the following assumptions which are listed below:

(i) $g \in L^1(R_+)$, almost everywhere positive and nonincreasing in $L^1(R_+)$.

(ii) $f_i: R_+ \times R \to R, i = 1,2$ are nonincreasing functions on R_+ with respect to t and x, satisfy

Caratheodory conditions and there are two functions $a_i \in L^1(R_+)$ and two Constants $b_i \geq 0$, such that

$|f_i(t,x)| \leq a_i(t) + b_i|x|$, for all $t \in R_+, x \in R$ and $f_i(t,x) \geq 0, \forall x \geq 0, i = 1,2$

(iii) $k_i: R_+ \to R, i = 1,2$ are measurable with respect to t and s, $K_i: L^1 \to L^1$, and for all $A > 0$ and $t_1, t_2 \in R_+$, the following condition is satisfied

$$t_1 < t_2 \Longrightarrow \int_0^A k_i(t_1 - s)ds \geq \int_0^A k_i(t_2 - s)ds\,, i = 1,2$$

Note that, from the assumption (iii) we deduce that the operator K is bounded with norm $\|K\|$.

(iv) $b_1 b_2 \|K_1\| \|K_2\| < 1$.

Then we can prove the following theorem

Theorem 3.1

Let the assumptions (i) $-$ (iv) be satisfied, then the equation (1) has at least one solution, $x \in L^1(R_+)$ being almost everywhere non increasing on R_+.

Proof

Consider the operator H

$$Hx(t) = g(t) + \int_0^\infty k_1(t-s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) ds$$

Then, the equation (3.1) takes the form

$$x(t) = Hx(t)$$

First, let $x \in L^1(R_+)$

Then using our assumption (i)→(iii), we have

$$|Hx(t)| \leq |g(t)| + \left| \int_0^\infty k_1(t-s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| ds$$

$$\int_0^\infty |Hx(t)|\, dt \leq \|g\| + \|K_1 F_1 K_2 F_2 x\|$$

$$\leq \|g\| + \|K_1\| \|F_1 K_2 F_2 x\|$$

$$\leq \|g\| + \|K_1\| \int_0^\infty |f_1(s, \int_0^t k_2(s,t) f_2(t, x(t)) ds)|\, dt$$

$$\leq \|g\| + \|K_1\| \int_0^\infty [a_1(s) + b_1 | \int_0^t k_2(s,t) f_2(t, x(t)) ds |]\, dt$$

$$\leq \|g\| + \|K_1\| [\|a_1\| + b_1 \int_0^\infty \int_0^t |k_2(s,t) f_2(t, x(t))| ds\,] dt$$

$$\leq \|g\| + \|K_1\| \left[\|a_1\| + b_1 \|K_2\| \int_0^\infty [a_2(t) + b_2 |x(t)|] \right] dt$$

$$\leq \|g\| + \|K_1\| \big[\|a_1\| + b_1 \|K_2\| [\|a_2\| + b_2 \|x\|] \big] \to (1)$$

From the last estimate, the space L^1 into itself using theorem (2.1)

Moreover, using the estimate (1), we see that the operator H transforms the ball B_r into itself, where:

$$r = \frac{\|g\| + \|K_1\| \|a_1\| + b_1 \|K_1\| \|K_2\| \|a_2\|}{1 - b_1 b_2 \|K_1\| \|K_2\|}$$

Let Q_r be subset of B_r consisting of all functions being are almost everywhere positive and non-increasing on R_+.

Note that Q_r is non-empty, bounded, closed, convex subset of $L^1(R_+)$.

Moreover, in view of Theorem (2.2) the set Q_r is compact in measure.

Next, by taking $x \in Q_r$,

Then $x(t)$ is almost everywhere positive and non-decreasing on R_+, and consequently $K_i x(t)$ is also of the same type (in virtue of the assumption (iii) and theorem (2.3))

Further, the assumption (ii) permits us to deduce that,

$$Hx(t) = g(t) + K_1F_1K_2F_2x(t),$$

is almost everywhere positive and non-decreasing on R_+, this fact together with assertion $H: B_r \to B_r$, gives that self-mapping of the set Q_r, since the Operator K is continuous and F is continuous in view theorem (2.1), we conclude that H maps continuously Q_r into Q_r.

Finally, assume that X is non-empty subset of Q_r and $\epsilon > 0$ is fixed, then for an arbitrary $x \in X$ and for a set $D \subset R_+$, meas $D \leq \epsilon$, we obtain

$$\int_D |(Hx)(t)|\,dt = \int_D \left| g(t) + \int_0^\infty k_1(t-s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| dsdt$$

$$\leq \int_D |g(t)|\,dt + \int_D \left| \int_0^\infty k_1(t-s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) ds \right| dt$$

$$\leq \int_D |g(t)|\,dt + \int_D \int_0^\infty |k_1(t-s)| \left| \left[a_1(s) + b_1 \left| \int_0^s k_2(s,\tau) f_2(\tau, x(\tau) \right) d\tau \right| \right] dsdt$$

$$\leq \|g\|_{L^1(D)} + \int_D \int_0^\infty |k_1(t-s)| \left[a_1(s) + b_1 \int_0^s |k_2(s,\tau)| \,[a_2(\tau) + b_2|x(\tau)|] d\tau ds \right] dt$$

$$\leq \|g\|_{L^1(D)} + \|K_1\|_D \|a_1\|_{L^1(D)} + b_1 \|K_1\|_D \|K_2\|_D \|a_2\|_{L^1(D)} + b_1 b_2 \|K_1\|_D \|K_2\|_D \int_D |x(s)|\,ds$$

Where, $K: L^1(D) \to L^1(D)$, since

$$\lim_{\epsilon \to 0} \{ \sup [\int_D |g(t)| dt + \|K_1\| \int_D |a_1(t)| dt + b_1 \|K_1\|_D \|K_2\|_D \int_D |a_2(t)| dt :$$

$D \subset R_+$, meas $D \leq \epsilon]\} = 0$

Then, the last inequality gives

$$c(HX) \leq b_1 b_2 \|K_1\| \|K_2\| c(X) \tag{2}$$

Further, more fixing $T > 0$ we arrive at the following estimate

$$\int_T^\infty |Hx(t)| dt \leq \int_T^\infty |g(t)| dt + \|K_1\| \int_T^\infty |a_1(t)| dt + b_1 \|K_1\|_D \|K_2\|_D \int_T^\infty |a_2(t)|\,dt + \; + b_2 b_2 \|K_1\|_D \|K_2\|_D \int_T^\infty |x(t)| dt$$

Since $\lim_{T\to\infty} T = \infty$, the above inequality gives

$$d(HX) \leq b_1 b_2 \|K_1\|_D \|K_2\|_D \tag{3}$$

Hence, combining (2) and (3) we get

$$\gamma(HX) \leq b_1 b_2 \|K_1\| \|K_2\| \gamma(X)$$

Where γ denotes the measure of noncompactness, since $Q_r$is compact in measure, then by using Theorem (2.4) the last inequality together with the assumption (iv), enable us to apply Theorem (2.7), which proves the existence of a fixed point for the operator H in Q_r. ■

4. Nonlinear Integral Equations with Fractional Order

In this section we will discuss solvability for the following nonlinear integral equation with fractional order in $L^1(R_+)$.

$$x(t) = g(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)\,ds, t > 0 \tag{4}$$

We shall treat equation (4.1) under the following assumptions which are listed below

(i) $g \in L^1(R_+)$, and almost everywhere positive and nondecreasing in $L^1(R_+)$.

(ii) $f_i: R_+ \times R \to R, i = 1,2$ are nondecreasing functions on $R_+$with respect to t and x satisfy Caratheodory conditions, there are two functions $a_i \in L^1(R_+)$ and two constants $b_i \geq 0$ such that $|f_i(t,x)| \leq a_i(t) + b_i|x|$, for all $t \in R_+, x \in R$ and $f_i(t,x) \geq 0, \forall x \geq 0, i = 1,2$

(iii) $k: R_+ \times R_+ \to R$, is a measurable with respect to t and s and $K: L^1 \to L^1$ is bounded with norm $\|K\|$

Note that, for all $A > 0$ and $t_1, t_2 \in R_+$, we have

$$t_1 < t_2 \Rightarrow \int_0^A e^{-(t_1-s)}(t_1 - s)^{\alpha-1}ds \geq \int_0^A e^{-(t_1-s)}(t_2 - s)^{\alpha-1}ds$$

(iv) $b_1 b_2 \|K\| < 1$

Then we can prove the following theorem,

Theorem 4.1

Let the assumptions (i)-(iv) are satisfied, then the equation (4.1) has at least one solution, $x \in L^1(R_+)$ being almost everywhere non-decreasing on R_+.

Proof

Consider the operator H:

$$Hx(t) = g(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\propto-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)\, ds, o < \alpha \leq 1, t > 0$$

Then, the equation (4.1) takes the form

$$x(t) = Hx(t)$$

First, let $x \in L^1(R_+)$

Then using our assumption (i)-(iii) we have,

$$|Hx(t)| \leq |g(t)| + \int_0^t \left| \frac{e^{-(t-s)}(t-s)^{\propto-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| ds$$

$$\int_0^\infty |Hx(t)|dt \leq \int_0^\infty |g(t)|dt + \int_0^\infty \int_0^t |\frac{e^{-(t-s)}(t-s)^{\propto-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)|dsdt$$

$$\leq \|g\| + \int_{s=0}^\infty \int_{t=s}^\infty |\frac{e^{-(t-s)}(t-s)^{\propto-1}}{\Gamma(\alpha)} ||f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)|dtds$$

$$\leq \|g\| + \int_{s=o}^\infty |f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)|ds$$

$$\leq \|g\| + \|F_1 K F_2\|$$

$$\leq \|g\| + \int_0^\infty [a_1(s) + b_1| \int_0^s k(s,\tau) f_2(\tau, x(\tau))|d\tau)]ds$$

$$\leq \|g\| + [\|a_1\| + b_1 \int_0^\infty \int_0^s |k(s,\tau) f_2(\tau, x(\tau))|d\tau]ds$$

$$\leq \|g\| + [\|a_1\| + b_1\|K\| \int_0^\infty [a_2(\tau) + b_2|x(\tau)|]ds\ ]$$

$$\leq \|g\| + [\|a_1\| + b_1\|K\|[\|a_2\| + b_2\|x\|]] \to \tag{5}$$

From the last estimate we deduce that the operator H maps continuously, the space L^1 into itself using theorem (2.1).

Moreover, using the estimate (1), we see that the operator H transforms the ball B_r into itself where:

$r = \frac{\|g\|+[\|a_1\|+b_1\|K\|\|a_2\|]}{(1-b_1 b_2\|K\|)}$

Let Q_r be subset of B_r consisting of all functions being are almost everywhere positive and non-increasing on R_+.

Note that Q_r is nonempty, bounded, closed, convex subset of $L^1(R_+)$.

Moreover, in view of Theorem (2.2) the set Q_r is compact in measure.

Next, by taking $x \in Q_r$, then $x(t)$ is almost everywhere positive and non-increasing on $L^1(R_+)$. and consequently $Kx(t)$ is also of the same type (in virtue of the assumption (iii) and Theorem (2.3).

Further, the assumption (ii) permits us to deduce that:

$$Hx(t) = g(t) + F_1 K F_2 x(t)$$

Is also almost everywhere positive and non-decreasing on R_+, this fact together with assertion, $H: B_r \to B_r$ gives that self – mapping of the set Q_r.

Since the operator K is continuous and F is continuous in view theorem (2.1), we conclude that H mapps continuous Q_r into Q_r.

Note, that

$$k(t,s) = \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ (Kx)(t) = \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

$$\|Kx\| = \int_{t=0}^{\infty}\int_{s=0}^{t}\left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| |x(s)|dsdt$$

$$= \int_{s=0}^{\infty}\int_{t=s}^{\infty}\left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| |x(s)|dtds$$

$$\text{Let } J=\int_{t=s}^{\infty}\left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| dt$$

$$=\int_{t-s=0}^{\infty}\left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| d(t-s) = 1$$

Then

$$\|Kx\| = \int_0^{\infty}|x(s)|\, ds = \|x\|, \text{ then } \|K\| = 1$$

Finally, assume that X is nonempty subset of Q_r and $\epsilon > 0$ is fixed, then for an arbitrary $x \in X$ and for a set $D \subset R_+$, meas $D \le \epsilon$, we obtain

$$\int_D |(Hx)(t)|\, dt = \int_D \left| g(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau))d\tau) \right| dsdt$$

$$\le \int_D |g(t)|\, dt + \int_D \left| \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau))d\tau)ds \right| dt$$

$$\le \int_D |g(t)|\, dt + \int_D \int_{t=s}^{\infty} \left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| [a_1(s) + b_1 \left|\int_0^s k(s,\tau) f_2(\tau, x(\tau))d\tau\right|]dtds$$

$$\le \|g\|_D + [\|a_1\| + b_1\|K\| \int_D [a_2(\tau) + b_2|x(\tau)|]ds]$$

$$\le \|g\|_D + \|a_1\| + b_1\|K\|\|a_2\| + b_1 b_2 \|K\| \int_D |x(s)|ds]]$$

Where $K: L^1(D) \to L^1(D)$, since

$$\lim_{\epsilon\to 0}\{\sup[\int_D |g(t)|dt + \int_D |a_1(t)|dt + +b_1||K||_D \int_D |a_2(t)|dt:$$

$D \subset R_+$, meas $D \leq \epsilon]\} = 0$

Then, the above inequality gives

$$c(HX) \leq b_1 b_2 ||K|| c(X) \tag{6}$$

Further, more fixing $T > 0$ we arrive at the following estimate

$$\int_T^\infty |Hx(t)|dt \leq \int_T^\infty |g(t)|dt + \int_T^\infty |a_1(t)|dt + b_1\|K\|_D \int_T^\infty |a_2(t)|\,dt + b_1 b_2 \|K\|_D \int_T^\infty |x(t)|dt$$

Since $\lim_{T\to\infty} T = \infty$, the above in quality gives

$$d(HX) \leq b_1 b_2 \|K\|_D \tag{7}$$

Hence, combining (2) and (3) we get

$$\gamma(HX) \leq b_1 b_2 \|K\| \gamma(X)$$

Where γ denotes the measure of noncompactness, since Q_r is compact in measure, then by using Theorem (2.4), The last inequality together with the assumption (iv), enable us to apply Theorem (2.7), Which proves the existence of a fixed point for the operator H in Q_r. ■

In the same way, we will discuss the solvability of the nonlinear integral equation with fractional order

$$x(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau))d\tau)\,ds, t \in [0,1] \tag{8}$$

in the space $L^1(0,1)$.

We shall treat the equation (4.2) under the following assumptions which are listed below

(i) $g \in L^1(0,1)$, almost everywhere positive and nondecreasing in (0,1),

(ii) f_i: $(0,1) \times R \to R, i = 1,2$, are nondecreasing functions with respect to t and x, satisfy Caratheodory conditions and there are two functions $a_i \in L^1(0,1)$ and two constants $b_i \geq 0$ such that $|f_i(t,x)| \leq a_i(t) + b_i|x|$, for all $t \in (0,1), x \in R$ and $f_i(t,x) \geq 0, \forall x \geq 0, i = 1,2$

(iii) k: $(0,1) \times (0,1) \to R_+$, is a measurable with respect to t and s and K: $L^1 \to L^1$ (From assumption (iii), we see that K is continuous and so it is bounded with norm $\|K\|$).

Also, for all $A > 0$ and $t_1, t_2 \in (0,1)$, we have

$$t_1 < t_2 \to \int_0^A (t_1 - s)^{\alpha-1} ds \geq \int_0^A (t_2 - s)^{\alpha-1} ds$$

(iv) $\frac{b_1 b_2 \|K\|}{\Gamma(\alpha+1)} < 1$

Then we can prove the following theorem,

Theorem 4.2

Let the assumptions (i)-(iv) are satisfied, then the equation (4) has at least one solution, $x \in L^1(0,1)$ being almost everywhere non-decreasing on (0,1).

Proof

Consider the operator H

$$Hx(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau))d\tau)\,ds, o < \alpha \leq 1, t \in [0,1]$$

Then, the equation (4.2) takes the form

$$x(t) = Hx(t)$$

First, let $x \in L^1[0,1]$

Then using our assumption (i)-(iii) we have,

$$|Hx(t)| \le |g(t)| + \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| ds$$

$$\int_0^1 |Hx(t)| dt \le \int_0^1 |g(t)| dt + \int_0^1 \int_0^t |\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds dt$$

$$\le \|g\| + \int_{s=0}^1 \int_{t=s}^1 |\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}||f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| dt ds$$

$$\le \|g\| + \int_{s=o}^1 |\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(s-s)^{\alpha}}{\Gamma(\alpha+1)}||f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$$

$$\le \|g\| + \int_{s=o}^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} |f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$$

$$\le \|g\| + \int_{s=o}^1 \frac{1}{\Gamma(\alpha+1)} |f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$$

$$\le \|g\| + \frac{1}{\Gamma(\alpha+1)} \int_0^1 |f_1(s, \int_0^\tau k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$$

$$\le \|g\| + \frac{1}{\Gamma(\alpha+1)} ||F_1 K F_2||$$

$$\le \|g\| + \frac{1}{\Gamma(\alpha+1)} \int_0^1 [a_1(s) + b_1 | \int_0^s k(s,\tau) f_2(\tau, x(\tau)) | d\tau] ds$$

$$\le \|g\| + \frac{1}{\Gamma(\alpha+1)} [\|a_1\| + b_1 \int_0^1 \int_0^s |k(s,\tau) f_2(\tau, x(\tau))| d\tau] ds$$

$$\le \|g\| + \frac{1}{\Gamma(\alpha+1)} [\|a_1\| + b_1 ||K|| [\|a_2\| + b_2 \|x\|] \to (1)$$

From the last estimate we deduce that the operator H maps continuously, the space L^1 into itself using theorem (2.1).

Moreover, using the estimate (1), we see that the operator H transforms the ball B_r into itself where: $r = \frac{\|g\| + \frac{1}{\Gamma(\alpha+1)}[\|a_1\| + b_1\|K\|\|a_2\|]}{(1 - \frac{b_1 b_2 \|K\|}{\Gamma(\alpha+1)})}$

Let Q_r be subset of B_r consisting of all functions being are almost everywhere positive and non-increasing on $(0,1)$.

Note that Q_r is nonempty, bounded, closed, convex subset of $L^1(0,1)$.

Moreover, in view of Theorem (2.5) the set Q_r is compact in measure.

Next, by taking $x \in Q_r$, then $x(t)$ is almost everywhere positive and nonincreasing on (0, 1) and consequently $Kx(t)$ is also of the same type (in virtue of the assumption (iii) and Theorem (2.1).

Further, the assumption (ii) permits us to deduce that the operator

$$Hx(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F_1 K F_2 x(t) ds$$

is also almost everywhere positive and nondecreasing on (0, 1), this fact together with assertion, $H: B_r \to B_r$ gives that self – mapping of the set Q_r.

Since the operator K is continuous and F is continuous in view Theorem (2.1), we conclude that H mapps continuous Q_r into Q_r.

Note, that $k(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$, $Kx(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$ then

$$\|Kx\| = \int_{t=0}^{1}\int_{s=0}^{t} \frac{|t-s|^{\alpha-1}}{\Gamma(\alpha)} |x(s)|dsdt$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_{s=0}^{1} |x(s)|ds$$

$\|Kx\| = \frac{\|x\|}{\Gamma(\alpha+1)}$, then $\|K\| = \frac{1}{\Gamma(\alpha+1)}$

Finally, assume that X is nonempty subset of Q_r and $\epsilon > 0$ is fixed, then for an arbitrary $x \in X$ and for a set $D \subset (0,1)$, meas $D \leq \epsilon$, we obtain

$$\int_D |(Hx)(t)|\, dt = \int_D \left| g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| dsdt$$

$$\leq \int_D |g(t)|\, dt + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau) ds \right| dt$$

$$\leq \int_D |g(t)|\, dt + \int_D \int_{t=s}^{1} \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right| [a_1(s) + b_1 \left| \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau \right|] dtds$$

$$\leq \|g\|_D + \frac{1}{\Gamma(\alpha+1)} [\|a_1\| + b_1 \|K\| \int_D [a_2(\tau) + b_2|x(\tau)|] ds]$$

$$\leq \|g\|_D + \frac{1}{\Gamma(\alpha+1)} \|a_1\| + \frac{1}{\Gamma(\alpha+1)} b_1 \|K\| \|a_2\| + \frac{b_1 b_2 \|K\|}{\Gamma(\alpha+1)} \int_D |x(s)|ds$$

Where $K: L^1(D) \to L^1(D)$, since

$$\lim_{\epsilon\to 0}\{ \sup[\int_D |g(t)|dt + \frac{1}{\Gamma(\alpha+1)} \int_D |a_1(t)|dt + + \frac{1}{\Gamma(\alpha+1)} b_1 \left||K|\right|_D \int_D |a_2(t)|dt:$$

$D \subset (0,1)$, meas $D \leq \epsilon]\} = 0$

Then the above inequality gives

$$\beta(Hx(t)) \leq \frac{b_1 b_2 \|K\|}{\Gamma(\alpha+1)} \beta(X)$$

Where β is the De Blasi measure of noncompactness:

Since Q_r is compact in measure, then by using Theorem (2.6), we can write the last inequality in the form

$$\chi(HX) \leq \frac{b_1 b_2 \left||K|\right|}{\Gamma(\alpha+1)} \chi(X)$$

This inequality together with the assumption (vi) enables us to apply Theorem (2.8), which proves the existence of a fixed point for the operator H in Q_r. ■

5. Conclusion

In this work, we determined the sufficient conditions under which the existence theorem of a nonlinear integral equation with convolution kernel is proved in the space $L^1(R_+)$. Also, the same situation is proved for a nonlinear integral equation with fractional order in the spaces $L^1(R_+)$ and $L^1[0,1]$.

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