Fixed Point Theorem in Cone Metric Space

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1. Introduction and Preliminaries

In 2007, Huang and Zhang [7] introduced the concept of a cone metric space, and they replacing the set of real numbers by an ordered Banach space and proved some fixed point theorems for mappings satisfying the different contractive conditions with using the normality condition in cone metric spaces. Subsequently many authors were inspired with these results and they (see, e.g., [2-4, 6, 8, 9, 10]) extended, improved and generalized the fixed point theorems of Huang and Zhang [7]. In 2008, Abbas and Jungck [1] obtained some fixed point theorems in cone metric spaces. In this paper, we proved a unique common fixed point theorem for four self-mappings in cone metric spaces using the continuity and commuting mappings. This result extends, improves and generalizes the results of Abbas and Jungck [1].

The following definitions and properties are in [7].

Definition 1.1 [7]. Let B be a real Banach space and P be a subset of B. The set P is called a cone if and only if:
(a). P is closed, non–empty and P ≠ {0};
(b). a, b ∈ ℝ, a, b ≥ 0 , x, y ∈ P implies ax + by ∈ P;
(c). x ∈ P and -x ∈ P implies x = 0.

Definition 1.2 [7]. Let P be a cone in a Banach space B, define partial ordering '≤' with respect to P by x ≤ y if and only if y - x ∈ P. We shall write x < y to indicate x ≤ y but x ≠ y while x << y will stand for y - x ∈ int P, where int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3 [7]. Let B be a cone in a Banach space B, define partial ordering '≤' with respect to P by x ≤ y if and only if y - x ∈ P. We shall write x < y to indicate x ≤ y but x ≠ y while x << y will stand for y - x ∈ int P, where int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.4 [7]. Let X be a nonempty set of Banach space B. Suppose that the map d: X × X → B satisfies:
(a). 0 ≤ d(x, y) for all x, y ∈ X and d(x,y) = 0 if and only if x = y;
(b). d(x, y) = d(y, x) for all x, y ∈ X;
(c). d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

Example 1.5 [7]. Let B = ℝ², P = {(x, y) ∈ B such that : x, y ≥ 0} ⊂ ℝ², X = ℝ and d: X × X → B such that d(x, y) = (√x² + y²), where α ≥ 0 is a constant. Then (X, d) is a cone metric space.

Definition 1.6 [7]. Let (X, d) be a cone metric space. We say that {xₙ} is

(i) a Cauchy sequence if for every c in B with c >> 0, there is a natural number N such that for all n, m > N, d(xₙ, xₘ) << c;
(ii) a convergent sequence if for any c >> 0, there is a natural number N such that for all n > N, d(xₙ, x) << c, for some fixed x in X. We denote this xₙ → x as n → ∞.

Lemma 1.7 [7]. Let (X, d) be a cone metric space. Let
{xₙ} be a sequence in X. Then {xₙ} is a Cauchy sequence if and only if d(xₙ, xₙ₋₁)→0 as n, m→∞.

In 2008 Abbas and Jungck [1] proved the following Theorem without continuity.

**Theorem 1.8. (Theorem 2.1 of [1])** Let (X, d) be a complete cone metric space and P be normal cone with normal constant K. Suppose mappings f,g:X→X satisfy
\[ d(fx, fy) ≤ Kd(gx, gy) \]
for all x, y∈X, where K∈(0,1) is a constant. If the range of f contains the range of g(X), then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

We have extended and generalized the above Theorem for four self-mappings using the continuity and commuting mappings.

### 2. Main Result

In this section, we prove a unique common fixed point theorem for four self-mappings in cone metric spaces by using the continuity and commuting mappings.

The following theorem is extended and generalized the Theorem 2.1 of [1].

**Theorem 2.1.** Let (X, d) be a complete cone metric space. And P be normal cone with normal constant M. Suppose that the four self-mappings S, T, I and J are commuting mappings satisfying the following
\[ d(Sx, Ty) ≤ \lambda d(Ix, Jy) \]
(1)
for all x, y∈X, where 0<\lambda<1. If S(X)⊂J(X) and T(X)⊂I(X) and if I and J are continuous, then all S, T, I and J have a unique common fixed point.

**Proof:** Let x₀∈X be arbitrary. Since S(X)⊂J(X), let x₁∈X be such that Jx₁=Sx₀. also, as Tx₁∈I(X), letx₂∈X be such that Ix₂=Tx₁. In general, x₂ₙ₊₁∈X is chosen such that Jx₂ₙ₊₁=Sx₂ₙ andx₂ₙ₊₂∈X is chosen such that Ix₂ₙ₊₂=Tx₂ₙ₊₁. n = 0, 1, 2, 3, ... we denote
\[ y₀=Ix₂ₙ, \]
\[ y₂ₙ₊₁=Ix₂ₙ₊₂ = Tx₂ₙ₊₁, n = 0, 1, 2, 3,... \]
We shall show that {yₙ} is a Cauchy sequence. For this we have
\[ d(y₂ₙ, y₂ₙ₊₁) = d(Sx₂ₙ, Tx₂ₙ₊₁) ≤ \lambda d(Ix₂ₙ, Jx₂ₙ₊₁) = \lambda d(y₂ₙ, y₂ₙ₊₁) \]
\[ d(y₂ₙ₊₁, y₂ₙ+2) = d(Sx₂ₙ₊₁, Tx₂ₙ+₁) ≤ \lambda d(Ix₂ₙ₊₁, Jx₂ₙ₊₂) = \lambda d(y₂ₙ₊₁, y₂ₙ+2) \]
that is, for n ≥ 2
\[ d(y₂ₙ₊₁, y₂ₙ₊₂) ≤ \lambda d(y₂ₙ, y₂ₙ₊₁) ≤ … ≤ \lambda^n d(y₀, y₁). \]

Hence, for n ≥ 2 it follows that
\[ d(y₀, y₁) ≤ \lambda d(y₀, y₁). \]
By the triangle inequality, for n>m we have
\[ d(yₙ, yₙ₊₁) ≤ \lambda d(yₙ₋₁, yₙ). \]
By (Definition 1.3) we have
\[ \|d(yₙ₋₁, yₙ)\| ≤ M(\lambda^n / 1-\lambda) (\|d(y₁, y₀)\|) \] (as m→∞)
By the Lemma 1.7, then {yₙ} is a Cauchy sequence in X. Let ϵ>0 be such that limₙ→∞ Sx₂ₙ = limₙ→∞ Jx₂ₙ₊₁ = limₙ→∞ Tx₂ₙ₊₁ = limₙ→∞ lx₂ₙ+₁ = z.
Since, I is continuous and S and T commute, it follows that \[ limₙ→∞ I^2x₂ₙ₊₁ = Ix₂ₙ₊₁ \] limₙ→∞ STx₂ₙ = limₙ→∞ ISx₂ₙ = Iz.
From (1), it follows that
\[ d(Sx₂ₙ, Tx₂ₙ₊₁) ≤ \lambda d(Ix₂ₙ, Jx₂ₙ₊₁). \]
Letting n→∞, we get that
\[ d(Iz, z) ≤ \lambda d(Iz, z). \]
That is, as 0<\lambda<1 it follows that Iz = z.
Similarly, since J is continuous and T and J commute, it follows that
\[ limₙ→∞ J^2x₂ₙ₊₁ = Jx₂ₙ₊₁ \] limₙ→∞ TJx₂ₙ₊₁ = limₙ→∞ JTx₂ₙ₊₁ = Iz.
From (1), it follows that
\[ d(Sx₂ₙ, Tx₂ₙ₊₁) ≤ \lambda d(Ix₂ₙ, Jx₂ₙ₊₁). \]
Letting n→∞, we get that
\[ d(z, Jz) ≤ \lambda d(z, Jz). \]
that is again it follows that z = Tz.
From (1) we have Iz = z.
\[ d(Sz, Tz) = 0. \] That is, Sz = z.
Again (1) we have d(Sz, Tz) = 0, that is Sz = Tz.
Thus, we proved that Sz = Tz = Iz = Jz = z.
Let w be another common fixed point in X of S, T, I and J, then
\[ d(z, w) = d(Sz, Tw) ≤ \lambda d(z, w). \]
Since 0<\lambda<1, it follows that d(z, w) = 0.That is z = w.
Therefore, z is a unique common fixed point of S, T, I and J. This completes the proof.

**Remark 2.2.** If we choose S = T = f and I = J = g and without continuity, non-commuting mappings in the above Theorem 2.1, then we get the Theorem 2.1 of [1].

### 3. Conclusion

In this paper, the author extended, improved and generalized the Theorem 2.1 of [M. Abbas and G. Jungck, Common fixed point results for non commuting mappings]

References


