(Alpha, Beta)-Normal and Skew n-Normal Composite Multiplication Operator on Hilbert Spaces

Senthil¹, *, Nithya², Suryadevi³, David Chandrakumar³

¹Department of Economics and Statistics, Government of Tamilnadu, DRDA, Dindigul, India
²Department of Mathematics, Mother Teresa Women’s University, Kodaikanal, India
³Department of Mathematics, Vickram College of Engineering, Enathi, India

Email address: senthilsnc83@gmail.com (Senthil)
*Corresponding author

To cite this article:

Received: February 15, 2019; Accepted: March 19, 2019; Published: May 6, 2019

Abstract: In this paper, the condition under which composite multiplication operators on Hilbert spaces become skew n-normal operators, (Alpha, Beta)-normal, parahyponormal and quasi-parahyponormal have been obtained in terms of radon-nikodym derivative.

Keywords: Composite Multiplication Operator, Conditional Expectation, Aluthge Transformation, Skew n-Normal Operator, Parahyponormal

1. Introduction

Let \((X, \mu)\) be a σ-finite measure space. Then a mapping \(T\) from \(X\) into \(X\) is said to be a measurable transformation if \(T^{-1}(E) \in \mathcal{E}\) for every \(E \in \mathcal{E}\). A measurable transformation \(T\) is said to be non-singular if \(\mu(T^{-1}(E)) = 0\) whenever \(\mu(E) = 0\). If \(T\) is non-singular then the measure \(\mu T^{-1}\) defined as \(\mu T^{-1}(E) = \mu(T^{-1}(E))\) for every \(E \in \mathcal{E}\), is an absolutely continuous measure on \(\mathcal{E}\) with respect to \(\mu\). Since \(\mu\) is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function \(f_0\) in \(L^1(\mu)\) such that \(\mu T^{-1}(E) = \int_E f_0 \, d\mu\) for every \(E \in \mathcal{E}\). The function \(f_0\) is called the Radon-Nikodym derivative of \(\mu T^{-1}\) with respect to \(\mu\).

Every non-singular measurable transformation \(T\) from \(X\) into itself induces a linear transformation \(T^C\) on \(L^p(\mu)\) defined as \(T^C f = f \circ T\) for every \(f\) in \(L^p(\mu)\). In case \(C_T\) is continuous from \(L^p(\mu)\) into itself, then it is called a composition operator on \(L^p(\mu)\) induced by \(T\). We restrict our study of the composition operators on \(L^2(\mu)\) which has Hilbert space structure. If \(u\) is an essentially bounded complex-valued measurable function on \(X\), then the mapping \(M_u\) on \(L^2(\mu)\) defined by \(M_u f = u \cdot f\), is a continuous operator with range in \(L^2(\mu)\). The operator \(M_u\) is known as the multiplication operator induced by \(u\). A composite multiplication operator is linear transformation acting on a set of complex valued \(\sum\) measurable functions \(f\) of the form

\[ M_u \circ T(f) = C_T M_u f = (u \circ T) (f \circ T) \]

where \(u\) is a complex valued, \(\sum\) measurable function. In case \(u = 1\) almost everywhere, \(M_u \circ T\) becomes a composition operator, denoted by \(C_T\).

In the study considered is the using conditional expectation of composite multiplication operator on \(L^2\)-spaces. For each \(f \in L^p(X, \mu)\), \(1 \leq p \leq \infty\), there exists an unique \(T^{-1}(\sum)\)-measurable function \(E(f)\) such that
\[ \int_a^b g \, f \, d \mu = \int_a^b E(f) \, d \mu \]

for every \( T^{-1}(\sum \cdot) \)-measurable function \( g \), for which the left integral exists. The function \( E(f) \) is called the conditional expectation of \( f \) with respect to the subalgebra \( T^{-1}(\sum \cdot) \). As an operator of \( L^p(\mu) \), \( E \) is the projection onto the closure of range of \( T \) and \( E \) is the identity on \( L^p(\mu) \), \( p \geq 1 \) if and only if \( T^{-1}(\sum \cdot) = \sum \cdot \). Detailed discussion of \( E \) is found in [1-4].

1.1. (Alpha, Beta)-Normal Operator [13]

An operator \( T \) is called \((\alpha, \beta)\) -normal operator if \( \alpha^2 \, T^* \, T \leq T \, T^* \leq \beta^2 \, T^* \, T \), \( 0 \leq \alpha \leq 1 \leq \beta \).

1.2. Skew n-Normal Operator [12]

An operator \( T \) is called skew n-normal operator if \( \left( T^* \, T \right)^n \geq \left( T \, T^* \right)^n \), for all natural number \( n \).

1.3. p-Hyponormal Operator [15]

An operator \( T \) is called p-hyponormal operator if \( \left( T^* \, T \right)^p \geq \left( T \, T^* \right)^p \), for \( 0 < p < \infty \).

2. Related Work in the Field

The study of weighted composition operators on \( L^2 \) spaces was initiated by R. K. Singh and D. C. Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in \( L^p(\sum \cdot) \), \( 1 \leq p < \infty \) spaces, where the measure spaces are \( \sigma \)-finite, appeared already in [6]. Also boundedness of weighted operators on \( C(X, E) \) has been studied in [7]. Recently S. Senthil, P. Thangaraju and D. C. Kumar have proved several theorems on \( n \)-normal, \( n \)-quasi-normal, \( k \)-paranormal, and \( (n,k) \) paranormal of composite multiplication operators on \( L^2 \) spaces [8-11]. In this paper we investigate composite multiplication operators of \((\alpha, \beta)\) -normal operator and skew \( n \)-normal operator \( L^2(\mu) \)-spaces.

3. Hyponormality for Composite Multiplication Operator

The results in the following proposition were proved in [12], as part of his doctoral dissertation.

3.1. Proposition [3]

Let \( E = E(\cdot \setminus A) \) and let \( \phi \) be a non-negative \( F \) measurable function.

Define the positive operator \( P_\phi \) by \( P_\phi f = \phi E(f) \).

Let \( \phi = (E(\phi^2))^\frac{1}{2} \). Then \( P_\phi = P_\phi \).

Define the operator \( R_\phi \) by \( R_\phi f = E(\phi \, f) \).

Then \( \| R_\phi f \| = \left\| \sqrt{E(\phi^2)} \right\| \).

In [3], has proved the following lemma, as noted for any non-negative function \( f \),

\[ \text{sup \, port \,} f \, \leq \, \text{sup \, port \,} E(f^r) \text{ \, for any \,} r > 0 \]

3.2. Lemma [14]

Let \( \alpha \) and \( \beta \) be non-negative functions, with \( S = \text{sup \, port \,} \alpha \). Then the following are equivalent:

For every \( f \in L^2(\mu) \)

\[ \int_X \alpha \left| f \right|^2 \, d\mu \geq \int_X \left| E(\beta f \setminus A) \right|^2 \, d\mu \]

\( \text{sup \, port \,} \beta \, \subset \, S \) and \( E \left\{ \beta^2 \, \omega \, X \setminus A \right\} \leq 1 \) almost everywhere.

3.3. Proposition

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then for \( u \geq 0 \)

(i) \( M_{u,T}^* M_{u,T} f = u^2 f_0 \cdot f \)

(ii) \( M_{u,T}^* M_{u,T} f = (u^2 \circ T) \cdot (f_0 \circ T) \cdot E(f) \)

(iii) \( M_{u,T}^n (f) = (C_1 \cdot M_{u,T})^n (f) = u_n \cdot (f \circ T^n) \), \( u_n = (u \circ T) \cdot (u \circ T^2) \cdot (u \circ T^3) \cdots \cdots (u \circ T^n) \)

(iv) \( M_{u,T}^* f = (u, f_0) \cdot E(f) \cdot T^{-1} \)

(v) \( M_{u,T}^n f = u_0 \cdot (E(u f_0) \circ T^{-n-1}) \cdot (E(f) \cdot T^{-n}) \)

where \( E((u f_0) \cdot T^{-n-1}) = (E(u f_0) \circ T^{-1}) \cdot (E(u f_0) \circ T^{-2}) \cdots \cdots (E(u f_0) \circ T^{-n-1}) \)

(vi) \( M_{u,T} = u \sqrt{f_0} \cdot f \)

(vii) \( M_{u,T}^* ((u^2 f_0 \cdot T) \cdot E((u^2 f_0 \circ T) \cdot f)) \)

with the notation from Herron’s proposition.
\[ |M^*_{u,T}| = P_{\sqrt{u^2 f_0 \circ T}} = P_v, \text{ where } v = \frac{\sqrt{(u^2 f_0) \circ T}}{E\left(\frac{(u^2 f_0) \circ T}{\sqrt{f_0}}\right)^2} \]

Theorem 3.1
Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \) with weight \( u \geq 0 \) and \( S \) be the support of \( f_0 \):

\[ |M_{u,T}| \geq |C| \text{ if and only if } u^2 \geq 1 \]

\[ |C| \geq |M^*_{u,T}| \text{ if and only if } \sup \text{port } v = \sup \text{port } \sqrt{f_0} \]

where \( v = \frac{\sqrt{(u^2 f_0) \circ T}}{E\left(\frac{(u^2 f_0) \circ T}{\sqrt{f_0}}\right)^2} \) and \( \chi_S \leq 1 \) almost everywhere.

Proof:
Since \( |M_{u,T}| \) and \( |C| \) are multiplication operators, we need only compare their symbols. After squaring, we get

\[ |M_{u,T}| \geq |C| \text{ if and only if } u \sqrt{f_0} \geq \sqrt{f_0} \]

\[ |M_{u,T}| \geq |C| \text{ if and only if } u^2 f_0 \geq f_0 \text{ because } f_0 > 0 \text{ almost everywhere,} \]

we obtain (i).

As for this \( f, |C| \geq |M^*_{u,T}| \)

\[ \forall f, \int \sqrt{f_0} |f|^2 d\mu \geq \langle |M^*_{u,T}| f, f \rangle \]

\[ |M^*_{u,T}| = P_v, \text{ where } v = \frac{\sqrt{(u^2 f_0) \circ T}}{E\left(\frac{(u^2 f_0) \circ T}{\sqrt{f_0}}\right)^2} = \int v E(v f) \overline{f} d\mu \]

However,

\[ \int v E(v f) \overline{f} d\mu = \langle E(v f), v f \rangle = \|E(v f)\|^2 = \int \|E(v f)\|^2 d\mu \]

Since \( \sup \text{port } v = \sup \text{port } \sqrt{f_0} \), the desired conclusion follows from lemma 3.2.

Theorem 3.5
Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is \( p \)-hyponormal if and only if \( uf_0 > 0, uf_0 \circ T > 0 \)

and \( E\left(\frac{1}{u^{2p} f_0}\right) \leq \frac{1}{u^{2p} f_0 \circ T} \).

Proof:
Here \( \left( M^*_{u,T} M_{u,T} \right)^p f, f \) is \( \int \sqrt{u^{2p} f_0} |f|^2 d\mu \)

and \( \left( M_{u,T} M^*_{u,T} \right)^p f, f \) is \( \int \left| E\left(\frac{u^p f_0^2 \circ T}{f} f\right)\right|^2 d\mu \)

This implies \( \int u^{2p} f_0^p |f|^2 d\mu \geq \int \left| E\left(\frac{u^p f_0^2 \circ T}{f} f\right)\right|^2 d\mu \)

Since by lemma 2.2, for every \( f \in L^2(\mu) \)
$$\Rightarrow \sigma \left( u^p f_0^p \right) \subset \sigma \left( u^p f_0^p \right) \text{ and } E \left[ \frac{u^{2p} f_0^p}{u^{2p} f_0^p} \circ T \right] \leq 1 \Rightarrow E \left[ \frac{1}{u^{2p} f_0^p} \right] \leq \frac{1}{(u^{2p} f_0^p) \circ T} \text{ if } (u^{2p} f_0^p) \circ T > 0 \text{ and } u^{2p} f_0^p > 0.$$

4. Parahyponormal for Composite Multiplication Operator

Mahmoud M. Kutkut [16], has proved that an operator \( A \) is parahyponormal if and only if \( (AA^*)^2 - 2C A^* A + C^2 \geq 0 \) for all real \( C \). In an analogous manner, we derive some characterization of parahyponormal and quasi-parahyponormal composite multiplication operator on \( L^2 \)-spaces.

Theorem 4.1

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is parahyponormal if and only if

\[
 u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) - 2Cu^2 f_0 f + C^2 \geq 0 \quad \text{almost everywhere, for all } C \geq 0.
\]

Proof:

Suppose \( M_{u,T} \) is parahyponormal. Then \( (M_{u,T} M_{u,T}^*)^2 - 2C M_{u,T}^* M_{u,T} + C^2 \geq 0 \) for all \( C \geq 0 \).

This implies that

\[
 \left\langle \left( (M_{u,T} M_{u,T}^*)^2 - 2C M_{u,T}^* M_{u,T} + C^2 \right) f, f \right\rangle \geq 0 \quad \text{for all } f \in L^2(\mu)
\]

By the proposition 3.3 we get,

\[
 \left\{ \left( (u^2 \circ T \cdot f_0 \circ T \cdot E(f))^2 - 2Cu^2 f_0 f + C^2 \right) \right\} \geq 0 \quad \text{for every } E \in \sum.
\]

\[
 \Rightarrow u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) - 2Cu^2 f_0 f + C^2 \geq 0 \quad \text{almost everywhere, for all } C \geq 0.
\]

Theorem 4.2

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is M-parahyponormal if and only if

\[
 m^2 u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) - 2C u^2 f_0 f + C^2 \geq 0 \quad \text{almost everywhere, for all } C \geq 0.
\]

Proof:

Suppose \( M_{u,T} \) is M-parahyponormal.

Then \( m^2 (M_{u,T} M_{u,T}^*)^2 - 2C M_{u,T}^* M_{u,T} + C^2 \geq 0 \) for all \( C \geq 0 \).

This implies that

\[
 \left\langle \left( m^2 (M_{u,T} M_{u,T}^*)^2 - 2C M_{u,T}^* M_{u,T} + C^2 \right) f, f \right\rangle \geq 0 \quad \text{for all } f \in L^2(\mu)
\]

By the proposition 3.3 we get,

\[
 \left\{ m^2 (u^2 \circ T \cdot f_0 \circ T \cdot E(f))^2 - 2Cu^2 f_0 f + C^2 \right\} \geq 0 \quad \text{for every } E \in \sum.
\]

\[
 \Rightarrow m^2 u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) - 2Cu^2 f_0 f + C^2 \geq 0 \quad \text{almost everywhere, for all } C \geq 0.
\]

Theorem 4.3

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is \( M^* \)-parahyponormal if and only if

\[
 m^2 u^2 f_0 E(u^2 f_0) \circ T^{-1} f - 2Cu^2 \circ T \cdot f_0 \circ T E(f) + C^2 \geq 0 \quad \text{almost everywhere, for all } C \geq 0.
\]

Proof:

Suppose \( M_{u,T} \) is \( M^* \)-parahyponormal.

Then \( m^2 M_{u,T}^2 M_{u,T}^2 - 2C M_{u,T}^* M_{u,T}^* + C^2 \geq 0 \) for all \( C \geq 0 \).

This implies that

\[
 \left\langle \left( m^2 M_{u,T}^2 M_{u,T}^2 - 2C M_{u,T}^* M_{u,T}^* + C^2 \right) f, f \right\rangle \geq 0 \quad \text{for all } f \in L^2(\mu)
\]

By the proposition 3.3 and \( M_{u,T}^2 M_{u,T}^2 = u^2 f_0 E(u^2 f_0) \circ T^{-1} f \geq 0 \) we get,

\[
 \left\{ m^2 u^2 f_0 E(u^2 f_0) \circ T^{-1} f - 2Cu^2 \circ T \cdot f_0 \circ T E(f) + C^2 \right\} \geq 0 \quad \text{for every } E \in \sum.
\]
\[ m^2 u^2 f_0 E(u^2 f_0) \circ T^{-1} f - 2 C u^2 \circ T \cdot f_0 \circ T E(f) + C^2 \geq 0 \text{ almost everywhere, for all } C \geq 0 \]

### 5. Quasi-Parahyponormal for Composite Multiplication Operator

By result of [16] an operators \( A \) on \( H \) is quasi-para hyponormal if and only if
\[ (A^2 A^* - 2 C (A A^*)^2 + C^2 \geq 0 \text{ for all } C. \]
In an analogous manner, we derive the characterization of quasi-parahyponormal composite multiplication operator on \( L^2 \)-spaces.

**Theorem 5.1**

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is quasi-para hyponormal if and only if
\[ u^2 \circ T \cdot u^2 \circ T \cdot f_0 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) - 2 C u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) + C^2 \geq 0 \text{ almost everywhere, for all } C \geq 0 \]

**Proof:**
Suppose \( M_{u,T} \) is quasi-para hyponormal.

Then \( (M_{u,T}^2)^2 - 2 C (M_{u,T} M_{u,T}^*)^2 + C^2 \geq 0 \text{ for all } C \geq 0 \).

This implies that
\[ \left\langle \left((M_{u,T}^2)^2 - 2 C (M_{u,T} M_{u,T}^*)^2 + C^2 \right) f, f \right\rangle \geq 0 \text{ for all } f \in L^2(\mu) \]

By the proposition 3.3 and \( M_{u,T}^2 f = u \circ T \cdot u^2 \circ T \cdot f_0 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) \)
\[ \left\{ \left(u \circ T \cdot u^2 \circ T \cdot f_0 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) - 2 C u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) + C^2 \right) \right\} \text{ almost everywhere, for all } C \geq 0 \]

**Theorem 5.2**

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is M-quasi-para hyponormal if and only if
\[ m^2 u^2 \circ T \cdot u^2 \circ T \cdot f_0^2 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) - 2 C u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) + C^2 \geq 0 \text{ almost everywhere, for all } C \geq 0 \]

**Proof:**
Suppose \( M_{u,T} \) is M-quasi-para hyponormal.

Then \( m^2 (M_{u,T}^2)^2 - 2 C (M_{u,T} M_{u,T}^*)^2 + C^2 \geq 0 \text{ for all } C \geq 0 \).

This implies that
\[ \left\langle \left(m^2 (M_{u,T}^2)^2 - 2 C (M_{u,T} M_{u,T}^*)^2 + C^2 \right) f, f \right\rangle \geq 0 \text{ for all } f \in L^2(\mu) \]

By the proposition 3.3 and \( M_{u,T}^2 f = u \circ T \cdot u^2 \circ T \cdot f_0 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) \)
\[ \left\{ m^2 (u \circ T \cdot u^2 \circ T \cdot f_0 \circ T^2 \cdot (E(u f_0))^2 \circ T \cdot E(f) - 2 C u^4 \circ T \cdot f_0^2 \circ T \cdot E(f) + C^2 \right\} \text{ almost everywhere, for all } C \geq 0 \]

### 6. Skew n-normal and (Alpha, Beta)-normal Composite Multiplication Operator

**Theorem 6.1**

Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is skew n-normal if and only if
\[ u_n (u^2 \circ T^n) (f_0 \circ T^n) = (u^2 \circ T) (f_0 \circ T) (E(u_n) \circ T) \text{ almost everywhere.} \]

**Proof:**
Suppose \( M_{u,T} \) is skew n-normal. Then
\[ (M_{u,T}^n M_{u,T}^*) M_{u,T} f = (M_{u,T}^n M_{u,T}^*) (u f) \circ T \]
\[ = M_{u,T}^n (u f) E((u f) \circ T) \circ T^{-1} \]

\[ M^\alpha_{u,T} u^2 f_0 f = u_n (u^2 f_0 f) \circ T^n \]
\[ = u_n (u^2 \circ T^n) (f_0 \circ T^n) (f \circ T^n) \]

Also,
\[ M_{u,T} (M^\alpha_{u,T} M^\alpha_{u,T}) f = M_{u,T} M^\alpha_{u,T} (u_n (f \circ T^n)) \]
\[ = M_{u,T} u f_0 E(u_n (f \circ T^n)) \circ T^{-1} \]
\[ = M_{u,T} u f_0 E(u_n (f \circ T^{-n-1})) \]
\[ = u (u f_0 E(u_n (f \circ T^{-n-1}))) \circ T \]
\[ = (u^2 \circ T) (f_0 \circ T) (E(u_n) \circ T) (f \circ T^n) \]
\[ \Leftrightarrow u_n (u^2 \circ T^n) (f_0 \circ T^n) = (u^2 \circ T) (f_0 \circ T) (E(u_n) \circ T) \text{ almost everywhere.} \]

Theorem 6.2
Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M^\alpha_{u,T} \) is skew n-normal if and only if
\[ u^2 f_0^2 = (u^2 \circ T^{-(n+1)}) (f_0 \circ T^{-(n+1)}) \text{ almost everywhere.} \]

Proof:
\( M^\alpha_{u,T} \) is skew n-normal. Then
\[ M^\alpha_{u,T} (M^\alpha_{u,T} M^\alpha_{u,T}) f = M^\alpha_{u,T} M^\alpha_{u,T} \left( u f_0 \left( E(u f_0) \circ T^{-(n+1)} \right) \left( E(f) \circ T^{-n} \right) \right) \]
\[ = M^\alpha_{u,T} \left( u \left( u f_0 \left( E(u f_0) \circ T^{-(n+1)} \right) \left( E(f) \circ T^{-n} \right) \right) \right) \circ T \]
\[ = M^\alpha_{u,T} \left( (u^2 \circ T) (f_0 \circ T) (E(u f_0) \circ T^{-(n+1)}) (E(f) \circ T^{-(n+1)}) \right) \]
\[ = u f_0 E \left[ ((u^2 \circ T) (f_0 \circ T) (E(u f_0) \circ T^{-(n+1)}) (E(f) \circ T^{-(n+1)}) \right] \circ T^{-1} \]
\[ = u^3 f_0^2 (E(u f_0) \circ T^{-(n+1)}) (E(f) \circ T^{-n}) \]

Also,
\[ (M^\alpha_{u,T} M_{u,T}) M^\alpha_{u,T} f = M^\alpha_{u,T} M_{u,T} \left( u f_0 E(f) \circ T^{-1} \right) \]
\[ = M^\alpha_{u,T} \left( u^2 \circ T \right) (f_0 \circ T) (E(f)) \]
\[ = u f_0 E(u f_0) \circ T^{-(n+1)} (u^2 \circ T^{-(n+1)}) (f_0 \circ T^{-(n+1)}) (E(f) \circ T^{-n}) \]
\[ \Leftrightarrow u^2 f_0^2 = (u^2 \circ T^{-(n+1)}) (f_0 \circ T^{-(n+1)}) \text{ almost everywhere.} \]

Theorem 6.2
Let the composite multiplication operator \( M_{u,T} \in B(L^2(\mu)) \). Then \( M_{u,T} \) is \((\alpha, \beta)\) -normal if and only if
\[ \alpha^2 u^2 f_0 f \leq (u^2 f_0) \circ T E(f) \leq \beta^2 u^2 f_0 f \text{ almost everywhere.} \]

Proof:
\( M_{u,T} \) is \((\alpha, \beta)\) -normal. Then it easy to check,
\[
M^*_{u,T} M_{u,T} f = u^2 f_0 f, \quad M^*_{u,T} M_{u,T} f = (u^2 \circ T)(f_0 \circ T) E(f)
\]

By definition,
\[
\alpha^2 u^2 f_0 f \leq (u^2 f_0) \circ T E(f) \leq \beta^2 u^2 f_0 f \text{ almost everywhere.}
\]

8. Conclusion

In the study of p-hyponormal operator, the Aluthge transform is a very useful tool. We investigate some basic properties of such operators and study the relation among skew n-normal operators, \((\alpha, \beta)\) -normal operator, parahyponormal and quasi-parahyponormal composite multiplication operators on \(L^2(\mu)\) -space. In future try to generalize the composite multiplication operator on Poisson weighted sequence spaces.

Acknowledgements

We would like to thank the reviewers for carefully reading manuscript and for their constructive comments. I want to thank Professor Dr. R. David Chandarakumar, Department of Mathematics, Vickram College of Engineering for his support and encouragement during preparation of the paper.

References


